Compact operator synthesis and spectral synthesis in harmonic analysis

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Essential spectral synthesis for compact abelian groups

Let G be a locally compact abelian group, $\widehat{G}$ its dual.
Let $A(G)$ be the Fourier algebra of G, $A(G) = \mathcal{F}L^1(\widehat{G})$, $\mathcal{F}(f)(g) = \int_{\widehat{G}} f(\chi)\overline{\chi(g)}d\chi$.
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For a closed subset $E \subset G$ let

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E is a set of spectral synthesis if $N(E) = PM(E)$.

Definition

We call a closed set $E \subseteq G$ a set of essential spectral synthesis if

$$PM(E) \cap PF(G) = N(E) \cap PF(G),$$

equivalently, $\langle F, \varphi \rangle = 0$ if $F \in PF(E)$ and $\varphi \in A(G)$, $\varphi = 0$ on E.

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There exist sets of essential spectral synthesis that are not of spectral synthesis:

1. There exist sets of uniqueness $E \subset \mathbb{T}$, (i.e. $PF(E) = \{0\}$) that are not sets of spectral synthesis.
2. Let $G = \mathbb{R}^n$, $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$.
   - S^{n-1} is not a set of spectral synthesis iff $n \geq 3$ (Herz, Schwartz, Varopoulos).
   - S^2 is a set of essential spectral synthesis (Varopoulos).

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- S^{n-1} is a not a set of essential spectral synthesis if $n \geq 4$: Let μ be the normalized surface area measure on S^{n-1}.
  - $\hat{\mu}(t) = O(\frac{1}{|t|^{(n-1)/2}})$, as $|t| \to \infty$ and hence $t_1\hat{\mu}(t) \in C_0(\mathbb{R}^n)$ if $n \geq 4$ and $Q := \frac{\partial \mu}{\partial x_1}$ is a pseudofunction.
  - Q is supported in S^{n-1}.
  - Let $f(x) = x_1[exp(-|x|^2 + 1) - exp(-2|x|^2 + 2)]$. Then $f \in A(\mathbb{R}^n)$ and f vanishes on S^{n-1}. Moreover

$$\langle Q, f \rangle = -\langle \mu, \frac{\partial f}{\partial x_1} \rangle = -\int 2x_1^2 d\mu \neq 0.$$

Generalized Fuglede-Putnam theorem

Fuglede-Putnam Theorem: If $A \in \mathcal{B}(H)$, $B \in \mathcal{B}(K)$ are normal operators and $X \in \mathcal{B}(K, H)$ then

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Let $\{A_i\} \in \mathcal{B}(H)$, $\{B_i\} \in \mathcal{B}(K)$ be commuting families of normal operators. Is it true that

$$\sum_i A_i X B_i = 0 \text{ if and only if } \sum_i A_i^* X B_i^* = 0 \tag{1}$$

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Answer: No. (V.Shulman, 1991)

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Does (1) hold for compact operators X? or $X \in \mathcal{S}_p$?

Answer: No if $p > 2$. If $p = 2$ then $\Delta : X \in \mathcal{S}_2 \mapsto \sum_i A_i X B_i$ is a bounded linear operator on the Hilbert space $\mathcal{S}_2$ and $\tilde{\Delta} : X \mapsto \sum A_i^* X B_i^*$ is its adjoint. Then $\ker \Delta = \ker \Delta^*$ giving that (1) holds in $\mathcal{S}_2$ and hence in $\mathcal{S}_p$, $p < 2$.

Let $p(x_1,\dots,x_{2n}) = \sum_{i=1}^{n} x_i^2 - 1 + i(\sum_{i=n+1}^{2n} x_i^2 - 1)$ and let s_i, $r_i = 1,\dots,m$ be polynomials such that

$$p(x-y) = \sum_{i=1}^{m} s_i(x) r_i(y), \quad x, y \in \mathbb{R}^{2n}.$$

Let u, $v \in C_c^\infty(\mathbb{R}^{2n})$, and $a_i = us_i$, $b_i = vr_i$. Consider $A_i = M_{a_i}$, $B_i = M_{b_i}$ in $\mathcal{B}(L^2(\mathbb{R}^{2n}))$

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$$p(x - y) = \sum_{i=1}^{m} s_i(x) r_i(y), \quad x, y \in \mathbb{R}^{2n}.$$

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Let $m = \mu \times \mu$, μ is the normalized surface measure of S^{n-1}. As $\hat{m}(t) = O(\frac{1}{|t|^{(n-1)/2}})$, $\frac{\partial m}{\partial x_i} \in PF(S^{n-1} \times S^{n-1})$.

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$$L = (1+i) x_{n+1} \frac{\partial}{\partial x_1} - (1-i) x_1 \frac{\partial}{\partial x_{n+1}},$$

$Lm \in PF(S^{n-1} \times S^{n-1})$ and $X = M_a \mathcal{F}^{-1} M_{\mathcal{F}(Lm)} \mathcal{F} M_b$ is compact on $L^2(\mathbb{R}^{2n})$ for any $a, b \in C_0(\mathbb{R}^{2n})$.

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$$\sum_i A_i X B_i = 0 \text{ and } \sum_i A_i^* X B_i^* \neq 0.$$

We have

$$\langle pLm, \varphi \rangle = \langle m, L(p\varphi) \rangle = \langle m, pL(\varphi) \rangle + \langle m, L(p)\varphi \rangle = 0,$$

$$\langle \bar{p}Lm, \varphi \rangle = \langle m, L(\bar{p})\varphi \rangle + \langle m, \bar{p}L(\varphi) \rangle = 4(1+i)\langle x_1 x_{n+1} m, \varphi \rangle$$

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and hence $pLm = 0$ and $\bar{p}Lm \neq 0$. A direct calculation shows that

$$\left(\sum_{i=1}^{m} M_{b_i} X M_{a_i} \varphi, \psi \right) = \langle pLm, ua\varphi * \widetilde{\overline{vb}\psi} \rangle = 0$$

and

$$\left(\sum_{i=1}^{m} M_{b_i}^{*} X M_{a_i}^{*} \varphi, \psi \right) = \langle \bar{p}Lm, ua\varphi * \widetilde{\overline{vb}\psi} \rangle \neq 0$$

showing the statement.

Essential (compact) operator synthesis

$H_1 = L^2(X,\mu), H_2 = L^2(Y,\nu)$ separable Hilbert spaces.
$\mathcal{B}(H_1,H_2)$, $\mathcal{K}(H_1,H_2)$ and $\mathcal{C}_1(H_1,H_2)$ are the spaces of all bounded resp. compact and nuclear linear operators from H_1 into H_2.

$$(\mathcal{K}(H_1,H_2))^* = \mathcal{C}_1(H_2,H_1),\ (\mathcal{C}_1(H_2,H_1))^* = \mathcal{B}(H_1,H_2),$$

where the duality is given by the map $(T,S) \mapsto \langle T,S\rangle = \operatorname{tr}(TS)$.

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The space $\mathcal{C}_1(H_2, H_1)$ can be identified with the space $L_2(X)\hat{\otimes}L_2(Y)$ of all $F : X \times Y \to \mathbb{C}$ s.t.

$$F(x, y) = \sum_{i=1}^{\infty} f_i(x)g_i(y),$$

$f_i \in L^2(X)$, $g_i \in L^2(Y)$, $\sum_{i=1}^{\infty} \|f_i\|_2^2 < \infty$, $\sum_{i=1}^{\infty} \|g_i\|_2^2 < \infty$,

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$$F \in L^2(X)\hat{\otimes}L^2(Y) \mapsto I_F \in C_1(H_1, H_2), (I_F\xi)(y) = \int_X F(x,y)\xi(x)d\mu(x)$$

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The duality between $\mathcal{B}(H_1,H_2)$ and $L^2(X)\hat{\otimes}L^2(Y)$ is given by

$$\langle T, f\otimes g\rangle = (Tf,\bar{g}),\ \ T \in \mathcal{B}(H_1,H_2), f \in L^2(X), g \in L^2(Y)$$

- ω-**topology**: $E \subseteq X \times Y$ is marginally null if $E \subseteq (X_0 \times Y) \cup (X \times Y_0)$, $\mu(X_0) = \nu(Y_0) = 0$. E is ω-open if $E \simeq \cup_{n=1}^{\infty} \alpha_n \times \beta_n$, $(\omega\text{-open})^c = \omega$-closed.

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- ω-closed $\kappa \subseteq X \times Y$ **supports** $T \in \mathcal{B}(H_1, H_2)$ if $M_{\chi_\beta} T M_{\chi_\alpha} = 0$ whenever $\alpha \times \beta \cap \kappa \simeq \emptyset$.

  For any $\mathcal{M} \subseteq \mathcal{B}(H_1, H_2)$, $\exists$ a smallest (up to marginal equivalence) ω-closed set $\operatorname{supp} \mathcal{M}$ which supports $\forall T \in \mathcal{M}$.

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  For any $\mathcal{M} \subseteq \mathcal{B}(H_1, H_2)$, $\exists$ a smallest (up to marginal equivalence) ω-closed set $\operatorname{supp} \mathcal{M}$ which supports $\forall T \in \mathcal{M}$.
- For any ω-closed set κ $\exists$ a smallest (resp. largest) w*-closed $L^\infty(X)$-$L^\infty(Y)$-bimodule $\mathfrak{M}_{\min}(\kappa)$ (resp. $\mathfrak{M}_{\max}(\kappa)$) with support κ, i.e. if $\mathfrak{M} \subseteq \mathcal{B}(H_1, H_2)$ is a w*-closed bimodule with supp $\mathfrak{M} = \kappa$ then

$$\mathfrak{M}_{\min}(\kappa) \subseteq \mathfrak{M} \subseteq \mathfrak{M}_{\max}(\kappa).$$

Definition

Let (X, μ) and (Y, ν) be standard measure spaces. An ω-closed set $\kappa \subseteq X \times Y$ is called a set of essential (compact) synthesis if

$$\mathfrak{M}_{\min}(\kappa) \cap \mathcal{K}(H_1, H_2) = \mathfrak{M}_{\max}(\kappa) \cap \mathcal{K}(H_1, H_2)$$

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- For all $p \geq 1$ let $\mathfrak{M}_p(E) = \mathfrak{M}_{max}(E) \cap \mathcal{S}_p$ and let $\mathfrak{M}_f(E)$ be the space of all finite rank operators supported in E.

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Proof.

It is enough to see that $\mathfrak{M}_f(E) \subset \mathfrak{M}_{min}(E)$. $T = I_K \in \mathcal{S}_2$ is supported in E iff K vanishes $\mu \times \nu$ a.e. outside E and therefore $\langle T, \Phi \rangle = \int K(x,y)\Phi(x,y)d\mu(x)d\nu(y) = 0$ whenever $\Phi \in \Gamma(X,Y)$ vanishes m.a.e. on E. The statement follows from

$$\mathfrak{M}_{\min}(E) = \{h \in L^2(X)\hat{\otimes}L^2(Y) : h\chi_E \simeq 0\}^\perp.$$

Quasi-diagonal sets and sets of finite width

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A set $D = \{\Pi_j = \alpha_j \times \beta_j \subset X \times Y : 1 \leq j \leq J\}$ is called a *diagonal system* if $\alpha_i \cap \alpha_j = \beta_i \cap \beta_j = \emptyset$, $i \neq j$.
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A subset $E \subset X \times Y$ is called n-**quasi-diagonal** if for each $\varepsilon > 0$ there is an n-diagonal system $D = \{\Pi_j\}_{j=1}^{J}$ with $E \subset \cup_j \Pi_j$ and $r(D) < \varepsilon$.

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Theorem

Let $E \subset X \times Y$ be ω-closed.

1. *If E is n-quasi-diagonal then $\mathfrak{M}_f(E) = \mathfrak{M}_{max}(E) \cap \mathcal{K} = \{0\}$*
2. *If $E = E_1 \cup E_2$, where E_1 is n-quasi-diagonal and E_2 is ω-open then*

$$\overline{\mathfrak{M}_f(E)}^{||\cdot||} = \mathfrak{M}_{max}(E) \cap \mathcal{K}.$$

Idea of the proof.

1. For $D = \{\Pi_j = \alpha_j \times \beta_j : 1 \leq j \leq J\}$ let

$$\pi_D(T) = \sum_j^J M_{\chi_{\beta_j}} T M_{\chi_{\alpha_j}}, \; T \in \mathcal{B}(H_1, H_2).$$

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If E is n-quasi-diagonal and $\{D^{(k)}\}_k$ is n-diagonal systems such that $r(D^{(k)}) \to 0$ then for rank one operator $T = u \otimes v$

$$||\pi_{D^{(k)}}(T)|| \le ||\pi_{D^{(k)}}(T)||_2 \le C(T) r(D^{(k)})^{1/2} \to 0$$

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giving that for $T \in \mathfrak{M}_{max}(E) \cap \mathcal{K}$, $T = \lim \pi_{D^{(k)}}(T) = 0$.
2. Let $D = \{\Pi_j\}_{j \in J}$, $E_1 \subset \cup_j \Pi_j$, and let $\tilde{D}$ be a complementing system. Then
$$T = \pi_D(T) + \pi_{\tilde{D}}(T).$$
For $T \in \mathcal{K}$, $\pi_D(T) \to 0$ when $r(D) \to 0$.

$\pi_{\tilde{D}}(T) \in \overline{\mathfrak{M}_f(E)}^{||\cdot||}$: We can think that rectangles in $\tilde{D}$ are unions of rectangles that are either in E^c or in E_2.
If $\Pi \in E^c$ then $\pi_\Pi(T) = 0$ by the definition of the support;
if $\Pi \in E_2$ then $\pi_\Pi(T) \in \mathfrak{M}_{max}(\Pi) \cap \mathcal{K} \subset \overline{\mathfrak{M}_f(\Pi)}^{||\cdot||} \subset \overline{\mathfrak{M}_f(E)}^{||\cdot||}$
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Corollary

Let $E = \{(x, y) : f_j(x) \leq g_j(y), j = 1, .., n\}$, $f_j, g_j : X \to \mathbb{R}$. *Then* $\overline{\mathfrak{M}_f(E)}^{||\cdot||} = \mathfrak{M}_{max}(E) \cap \mathcal{K}$ *and hence* E *is a set of essential synthesis.*

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Proposition

If $E = E_1 \cup E_2$, where E_1 is n-quasi-diagonal and E_2 is ω-open, and Λ is a set of essential synthesis then $E \cup \Lambda$ and $E \cap \Lambda$ are sets of essential synthesis.

Connection between essential operator synthesis and essential spectral synthesis

Let G be a locally compact group, $A(G)$, $\mathrm{VN}(G)$, $C_r^*(G)$ be the Fourier algebra, the von Neumann algebra and the reduced C^*-algebra of G.
For a closed $E \subseteq G$ let

$$PM(E) = \{T \in \mathrm{VN}(G) : \operatorname*{supp}_{\mathrm{VN}(G)} T \subset E\}$$

$$N(E) = \overline{\{\lambda(s) : s \in E\}}^{w^*}$$

E is said to a *set of essential spectral synthesis* if

$$PM(E) \cap C_r^*(G) = N(E) \cap C_r^*(G).$$

Theorem

Let G be a second countable locally compact group. Then a closed set $E \subset G$ is a set of essential spectral synthesis iff $E^* = \{(s,t) : ts^{-1} \in E\}$ is a set of essential operator synthesis.

Idea of the proof

- $T \in VN(G)$, $\mathrm{supp}_{VN(G)}\, T \subset E \Rightarrow \mathrm{supp}\, T \subset E^*$
- $T \in VN(G)$, $u \in A(G) \Rightarrow uT = \sum_i M_{b_i} T M_{a_i}$, where $\sum_i a_i(s) b_i(t) = u(ts^{-1}) \in L^\infty(G) \otimes_{eh} L^\infty(G)$
- $T \in C_r^*(G) \Rightarrow M_b T M_a \in \mathcal{K}(L^2(G))$, $a, b \in C_0(G)$

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- $T \in C_r^*(G) \Rightarrow M_b T M_a \in \mathcal{K}(L^2(G))$, $a, b \in C_0(G)$
- $T \in \mathcal{K}(L^2(G))$, $\mathrm{supp}\, T \subset E^* \Rightarrow E_{a \otimes b}(T) \in C_r^*(G)$, $a, b \in L^2(G)$, where

$$\langle E_{a\otimes b}(T), \varphi \rangle = \langle T, N\varphi(a \otimes b) \rangle$$

$$N\varphi(s,t) = u(ts^{-1})$$

Example

- $(S^n)^*$ is not a set of essential operator synthesis for $n \geq 3$.
- $(S^1)^*$, $(S^2)^*$ are sets of essential synthesis.
- E^* is a set of essential operator synthesis if E is a set of uniqueness.

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Proposition

For $F = \sum a_i \otimes b_i \in L^\infty(X) \otimes_{eh} L^\infty(Y)$ and $T \in \mathcal{B}(L^2(X), L^2(Y))$ let $F \cdot T = \sum_i M_{b_i} T M_{a_i}$.
If $F_1, F_2 \in L^\infty(X) \otimes_{eh} L^\infty(Y)$ and null F_1 = null F_2 is a set of essential synthesis then for $T \in \mathcal{K}$

$$F_1 \cdot T = 0 \text{ iff } F_2 \cdot T = 0.$$

THANK YOU!