Operator synthesis: unions and products

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(joint work with G. K. Eleftherakis)

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- Tensor products and property S_σ
- Preservation properties

Spectral synthesis

Let G be a locally compact group.

$B(G)$ is the collection of all functions $u : G \to \mathbb{C}$ of the form $u(t) = (\pi(t)\xi, \eta)$, where π is a continuous unitary representation of G on H and $\xi, \eta \in H$.

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$B(G)$ is a Banach algebra under pointwise operations and $A(G)$ is an ideal of $B(G)$.

$\|u\|_{B(G)} = \inf\{\|\xi\|\|\eta\| : u(\cdot) = (\pi(\cdot)\xi, \eta)\}$.

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$A(G)$ is a regular commutative semi-simple Banach algebra.

If $J \subseteq A(G)$ is an ideal, we let

$$\mathrm{null}\, J = \{t \in G : u(t) = 0, \text{ for all } u \in I\}.$$

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E satisfies spectral synthesis if $J(E) = I(E)$.

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(v) An operator $T \in \mathcal{B}(L^2(X), L^2(Y))$ is *supported on* κ if

$$(\alpha \times \beta) \cap \kappa \simeq \emptyset \;\Rightarrow\; P(\beta) T P(\alpha) = 0,$$

where $P(\alpha)$ is the projection from $L^2(X)$ onto $L^2(\alpha)$.

Operator synthesis

Let $\Gamma(X, Y) = L^2(X)\hat{\otimes}L^2(Y)$.

An element $h \in \Gamma(X, Y)$ can be identified with a function, defined up to a marginally null set,

$$h(x, y) = \sum_{i=1}^{\infty} f_i(x)g_i(y), \quad x \in X, y \in Y.$$

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Define the null set of V as the biggest (with respect to marginal inclusion) ω-closed set $E \subseteq X \times Y$ such that $h|_E = 0$ for every $h \in V$.

Operator synthesis

Given an ω-closed set $\kappa \subseteq X \times Y$, let

$$\Phi(\kappa) = \{h \in \Gamma(X, Y) : h|_{\kappa} = 0\},$$

$$\Psi(\kappa) = \overline{\{h \in \Gamma(X, Y) : h = 0 \text{ on an } \omega\text{-open neighbhd of } \kappa\}}.$$

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κ satisfies *operator synthesis* if $\Phi(\kappa) = \Psi(\kappa)$.

The dual perspective

Let $H_1 = L^2(X)$, $H_2 = L^2(Y)$.

$\Gamma(X, Y)^* = \mathcal{B}(H_1, H_2)$.

$\mathcal{D}_X \equiv L^\infty(X)$, $\mathcal{D}_Y \equiv L^\infty(Y)$.

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Connections with reflexivity

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The space $\mathcal{U}$ is called *reflexive* if $\mathcal{U} = \mathrm{Ref}\,\mathcal{U}$.

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Arvseon's Transitivity Theorem, 1974

If $\mathcal{U}$ is a transitive masa-bimodule then $\mathcal{U}$ is weak* dense in $\mathcal{B}(H_1, H_2)$.

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Erdos, Katavolos, Shulman, 1998: A weak* closed masa-bimodule $\mathcal{U} \subseteq \mathcal{B}(H_1, H_2)$ is reflexive if and only if $\mathcal{U} = \mathfrak{M}_{\max}(\kappa)$ for some ω-closed set $\kappa \subseteq X \times Y$.

Connections between specral and operator synthesis

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Theorem (Ludwig-Turowska, 2006)

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Spronk-Turowska, 2002: the case G is compact.

Note that in this case local spectral synthesis is equivalent to spectral synthesis.

Froelich, 1988: the case G is abelian

Examples of (operator) synthetic sets

- *Ternary sets*

  $f : X \to \mathbb{R}$, $g : Y \to \mathbb{R}$

  $\{(x, y) \in X \times Y : f(x) = g(y)\}$ satisfies operator synthesis (Shulman, Katavolos-T)

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- *Sets of finite width*

  $f_i : X \to \mathbb{R}$, $g_i : Y \to \mathbb{R}$

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  In Harmonic Analysis, the analogous sets are $\{t \in G : \omega_i(s) \leq r_i, i = 1, \dots, n\}$, where $\omega_i : G \to \mathbb{R}^+$ continuous homomorphisms.

Schur multipliers

Let (X, μ) and (Y, ν) be standard measure spaces.

For a function $\varphi \in L^\infty(X \times Y)$, let $S_\varphi : L^2(X \times Y) \to L^2(X \times Y)$ be the corresponding multiplication operator

$$S_\varphi \xi = \varphi \xi.$$

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The space $L^2(X \times Y)$ can be identified with the Hilbert-Schmidt class in $\mathcal{B}(L^2(X), L^2(Y))$ by

$$\xi \longrightarrow T_\xi, \qquad T_\xi f(y) = \int_X \xi(x, y) f(x) d\mu(x).$$

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A function $\varphi \in L^\infty(X \times Y)$ is called a *Schur multiplier* if there exists $C > 0$ such that

$$\|S_\varphi \xi\|_{\mathrm{op}} \leq C \|\xi\|_{\mathrm{op}}, \quad \xi \in L^2(X \times Y).$$

Schur multipliers as completely bounded modular maps

Let $\mathfrak{S}(X, Y)$ be the class of all Schur multipliers and write $\mathcal{D}_X$ (resp. $\mathcal{D}_Y$) for the multiplication masa of $L^\infty(X)$ (resp. $L^\infty(Y)$).

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If $\varphi \in \mathfrak{S}$ then S_φ extends by continuity to a bounded operator on the space of compact operators:

$$S_\varphi : \mathcal{K}(L^2(X), L^2(Y)) \to \mathcal{K}(L^2(X), L^2(Y)),$$

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S_φ is moreover modular in the sense that

$$S_\varphi(BTA) = BS_\varphi(T)A, \qquad A \in \mathcal{D}_X, B \in \mathcal{D}_Y, T \in \mathcal{B}(L^2(X), L^2(Y)).$$

By a result of Smith, S_φ is completely bounded.

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Weak* closed masa-bimodules are precisely the weak* closed invariant subspaces of Schur multipliers.

A characterisation of Schur multipliers

Peller's Theorem (1985)

The following are equivalent:

(i) φ is a Schur multiplier;

(ii) there exist families $\{a_k\}_{k=1}^{\infty} \subseteq L^{\infty}(X)$ and $\{b_k\}_{k=1}^{\infty} \subseteq L^{\infty}(Y)$ and a constant $C > 0$ such that $\mathrm{esssup}_{x\in X} \sum_{k=1}^{\infty} |a_k(x)|^2 \leq C$, $\mathrm{esssup}_{y\in Y} \sum_{k=1}^{\infty} |b_k(y)|^2 \leq C$ and

$$\varphi(x,y) = \sum_{k=1}^{\infty} a_k(x) b_k(y), \quad \text{a.e. on } X \times Y;$$

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$S_{\varphi} S_{\psi} = S_{\varphi\psi}$.

Schur idempotents, $\mathfrak{I}$: idempotent Schur multipliers ($\phi^2 = \phi$), Katavolos-Pauslen (2006).

Multipliers of Fourier algebras

G locally compact group.

$MA(G) = \{g \in C(G) : gf \in A(G), \text{ for all } f \in A(G)\}.$

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Proof also given by Jolissaint (1992), extended by Spronk (2004).

The union problem

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Given two ω-closed operator synthetic sets $\kappa, \lambda \subseteq X \times Y$, is $\kappa \cup \lambda$ operator synthetic?

Ludwig-Turowska, Shulman-Turowska, T: The union of an operator synthetic set and a ternary set is operator synthetic.

Ternary are the sets $\{(x, y) : f(x) = g(y)\}$. They are the supports of "ternary" masa-bimodules, that is, masa-bimodules $\mathcal{U}$ such that $\mathcal{U}\mathcal{U}^*\mathcal{U} \subseteq \mathcal{U}$.

Extension of the union problem

Question

Suppose that $\mathcal{U}$ and $\mathcal{V}$ are reflexive masa-bimodules. Is $\overline{\mathcal{U}+\mathcal{V}}^{w^*}$ reflexive?

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Suppose that $\mathcal{U} = \mathfrak{M}_{\max}(\kappa)$ and $\mathcal{V} = \mathfrak{M}_{\max}(\lambda)$ with κ and λ operator synthetic. If $\overline{\mathcal{U}+\mathcal{V}}^{w^*}$ is reflexive then $\kappa \cup \lambda$ is operator synthetic.

Schur idempoents

If $\phi = S_{\chi_\kappa}$ is a Schur idempotent then S_{χ_κ} has range $\mathfrak{M}_{\max}(\kappa)$ and κ is ω-clopen and hence operator synthetic.

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Theorem (Eleftherakis-T, 2011)

The weak* closed linear span of a reflexive masa-bimodule and an approximately injective masa-bimodule is automatically reflexive.

The simplest case

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Thus, $T = \phi^{\perp}(T) + \phi(T) \in \mathcal{U} + \mathcal{V}$.

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Then $T = (T - S) + S \in \overline{\mathcal{U} + \mathcal{V}}^{w^*}$.

Approximately $\mathfrak{I}$-decomposable masa-bimodules

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A masa-bimodule $\mathcal{U}$ is called approximately $\mathfrak{I}$-decomposable if there exists a constant $C > 0$ and, for each $n \in \mathbb{N}$, Schur idempotents ϕ_n and ψ_n such that

- $\mathcal{U} \subseteq \mathrm{Ran}\, \phi_n + \mathrm{Ran}\, \psi_n$, for all n;
- $\mathrm{Ran}\, \phi_n \subseteq \mathcal{U}$, for all n;
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Key technical tool: the intersection formula

$$\cap_{i=1}^{n} \overline{\mathcal{U} + \mathcal{V}_i}^{w^*} = \overline{\mathcal{U} + \cap_{i=1}^{n} \mathcal{V}_i}^{w^*}.$$

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Theorem, Eleftherakis-T, 2011

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Let κ be the support of an operator synthetic approximately $\mathfrak{I}$-decomposable masa-bimodule. If λ is an operator synthetic set then $\kappa \cup \lambda$ is operator synthetic.

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In particular, this conclusion holds if κ is a set of finite width:

$$\kappa = \{(x, y) : f_i(x) \leq g_i(y), \;\; i = 1, \ldots, n\}.$$

A consequence for spectral synthesis

Let us call s set $E \subseteq G$ a *level set* if

$$E = \{t \in G : \omega(t) \leq r_i\},$$

where $\omega : G \to \mathbb{R}$ is a continuous homomorpshim.

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Corollary

Suppose that $F \subseteq G$ satisfies spectral synthesis, while E_i is a level set, $i = 1, \ldots, n$. Then $F \cup (\cap_{i=1}^{n} E_i)$ satisfies spectral synthesis.

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Question

If κ_1 and κ_2 are (operator) synthetic sets, is $\kappa_1 \times \kappa_2$ (operator) synthetic?

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$\kappa_1 \subseteq X_1 \times Y_1$, $\kappa_2 \subseteq X_2 \times Y_2$.

$\mathfrak{M}_{\max}(\kappa_1)\bar{\otimes}\mathfrak{M}_{\max}(\kappa_2)$ is an $L^\infty(Y_1 \times Y_2)$,$L^\infty(X_1 \times X_2)$-module.

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Thus, we should be looking at the set $\rho(\kappa_1 \times \kappa_2)$, where

$$\rho : (x_1, y_1, x_2, y_2) \longrightarrow (x_1, y_1, x_2, y_2).$$

Kraus' property S_σ

Let $\omega \in \mathcal{B}(H_1, H_2)_*$. The *right Tomiyama's slice map*

$$R_\omega : \mathcal{B}(H_1 \otimes K_1, H_2 \otimes K_2) \to \mathcal{B}(K_1, K_2)$$

is given by

$$R_\omega(A \otimes B) = \omega(A)B, \quad A \in \mathcal{B}(H_1, H_2), B \in \mathcal{B}(K_1, K_2).$$

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The *Fubini product* of $\mathcal{V}$ and $\mathcal{U}$ is

$$\mathcal{F}(\mathcal{V},\mathcal{U}) = \{T \in \mathcal{V}\bar{\otimes}\mathcal{B}(K_1 \otimes K_2) : R_\omega(T) \in \mathcal{U},\ \forall \omega \in \mathcal{B}(K_1, K_2)_*\}.$$

A weak* closed subspace $\mathcal{V} \subseteq \mathcal{B}(H_1, H_2)$ possesses *property S_σ* if

$$\mathcal{V}\bar{\otimes}\mathcal{U} = \mathcal{F}(\mathcal{V},\mathcal{U}), \quad \forall\, \mathcal{U} \subseteq \mathcal{B}(K_1, K_2).$$

Preservation of S_σ

Kraus, 1983: $\mathcal{B}(H_1, H_2)$ possess property S_σ. Thus

$$\mathcal{F}(\mathcal{V},\mathcal{U}) = (\mathcal{V}\bar{\otimes}\mathcal{B}(K_1, K_2)) \cap (\mathcal{B}(H_1, H_2)\bar{\otimes}\mathcal{U}).$$

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Hopenwasser-Kraus, 1983 (can be generalised to):

Theorem

Every masa-bimodule of finite width possesses S_σ.

$\mathcal{V}$ is of finite width if $\mathcal{V} = \mathfrak{M}_{\max}(\kappa)$, foe some subset κ of finite width.

Theorem (Eleftherakis-T, 2013)

If $\mathcal{B}$ is a masa-bimodule of finite width then

$$\mathcal{F}(\overline{\mathcal{V}+\mathcal{B}}^{w^*},\mathcal{U}) = \overline{\mathcal{F}(\mathcal{V},\mathcal{U}) + \mathcal{B}\otimes\mathcal{U}}^{w^*}.$$

In particular, if $\mathcal{V}$ has S_σ then so does $\overline{\mathcal{V}+\mathcal{B}}^{w^*}$.

Connections between synthesis and property S_σ

Proposition

Let κ and λ be operator synhetic sets. The following are equivalent:

(i) $\rho(\kappa \times \lambda)$ is operator synthetic;

(ii) $\mathcal{F}(\mathfrak{M}_{\mathrm{max}}(\kappa), \mathfrak{M}_{\mathrm{max}}(\lambda)) = \mathfrak{M}_{\mathrm{max}}(\kappa) \bar{\otimes} \mathfrak{M}_{\mathrm{max}}(\lambda)$.

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Corollary

Suppose that $\mathfrak{M}_{\max}(\kappa)$ possesses property S_σ. Then $\rho(\kappa \times \lambda)$ is operator synthetic for every operator synthetic set λ.

More intersection formulas

Can we formulate sufficient conditions for the operator synthesis of sets of the form

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Theorem (Eleftherakis-T, 2013)

If $\{\mathcal{B}_{j_p}^{p}\}_{j_p=1}^{m_p}$ are families of masa-bimodules of finite width and $\mathcal{U}_p$ are weak* closed spaces of operators, $p = 1, \dots, r$, then

$$\bigcap_{j_1,\dots,j_r} \overline{\mathcal{B}_{j_1}^{1} \otimes \mathcal{U}_1 + \dots + \mathcal{B}_{j_r}^{r} \otimes \mathcal{U}_r} = \overline{(\cap_{j_1}\mathcal{B}_{j_1}^{1}) \otimes \mathcal{U}_1 + \dots + (\cap_{j_r}\mathcal{B}_{j_r}^{1}) \otimes \mathcal{U}_r}.$$

Consequences for operator and spectral synthesis

Theorem (Eleftherakis-T, 2013)

Let $\kappa_i \subseteq X_1 \times X_2$ be a set of finite width, and let $\lambda_i \subseteq Y_1 \times Y_2$ be an ω-closed set, $i = 1, \dots, r$. Suppose that $\cup_{k=1}^{p} \lambda_{m_k}$ is operator synthetic whenever $1 \leq m_1 < m_2 < \cdots < m_p \leq r$. Then the set $\rho(\cup_{i=1}^{r} \kappa_i \times \lambda_i)$ is operator synthetic.

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Corollary

Let G and H be second countable locally compact groups. Suppose that $E_1, \dots, E_r$ are finite intersections of level sets in G and $F_1, \dots, F_r$ are closed subsets of H such that $\cup_{k=1}^{p} F_{m_k}$ is a set of local spectral synthesis whenever $1 \leq m_1 < m_2 < \cdots < m_p \leq r$. Then the set $\cup_{i=1}^{r} E_i \times F_i$ is a set of local spectral synthesis of $G \times H$.

MUCHAS GRACIAS