22.5.2013

Kirillov theory without polarizations

G always denotes an exponential Lie group, i.e.,
$\exp : \mathfrak{g} \to G$ is a diffeomorphism.
(This implies that G is solvable.)

$\mathfrak{g}^* = \mathrm{Hom}(\mathfrak{g}, \mathbb{R})$, $\mathfrak{g}^*/G \xrightarrow[\text{bijection}]{} \hat{G}$ unitary dual

In a little more detail:

Each irreducible representation π of G is monomial, $\pi = \mathrm{ind}_Q^G \chi$, $\chi \in \mathrm{Hom}(Q, \mathbb{T})$.

<u>Question</u> Which pairs (Q, χ) deliver irreducibles?

<u>Answer</u> Take any $f \in \mathfrak{g}^*$ s.t. $f|_{\mathfrak{q}} = d\chi$.
Then (P) the orbit Gf contains $f + \mathfrak{q}^\perp$, i.e., all extensions of $d\chi$,
and $\mathfrak{q}$ is a polarization at f, which essentially means that $\mathfrak{q}$ is maximal w.r.t. the property that the restriction of
f delivers a character of $Q = \exp \mathfrak{q}$, called χ_f below.

Elaboration these ideas yields
a surjective map $\mathfrak{g}^* \longrightarrow \hat{G}$
(given f choose a polarization $\mathfrak{q}$ at f with (P), and form $\mathrm{ind}_Q^G \chi_f$), which leads to the mentioned bijection between $\mathfrak{g}^*/G$ and $\hat{G}$.

In this approach one has to struggle with polarizations (existence, independence, ...), also the above "answer" requires some work!

Today I wish to present an approach (to the above bijection) without using the word polarization. Also I found (and I find) that it is at least as canonical to construct the map in the opposite direction, i.e., $G^\wedge \longrightarrow \mathfrak{g}^*/G$.

The <u>idea</u> is to understand that both spaces, $G^\wedge$ and $\mathfrak{g}^*/G$ are built by the <u>same</u> <u>construction</u> <u>laws</u> from corresponding lower dimensional objects.

To illustrate this on the side of $\mathfrak{g}^*/G$ we observe (easy to prove):

$$(\theta) \qquad G = \exp\mathfrak{g}, \ f \in \mathfrak{g}^*, \ \Omega = Gf$$

$$\mathfrak{a} = \text{abelian ideal in } \mathfrak{g}$$

$$\Omega|_{\mathfrak{a}} \ni \alpha, \alpha', \ldots$$

Given α, consider $\mathfrak{g}_\alpha$ and

$$\Sigma_\alpha := \{ g|_{\mathfrak{g}_\alpha} \mid g \in \Omega, \ g|_{\mathfrak{a}} = \alpha \}$$

Then Σ_α is a G_α-orbit. Form $\mathrm{Ext}(\Sigma_\alpha) = \Sigma'_\alpha := \{ h \in \mathfrak{g}^* \mid h|_{\mathfrak{g}_\alpha} \in \Sigma_\alpha \}$. Then $\Omega = \bigcup_\alpha \Sigma'_\alpha$.

On the side of representation theory our approach requires only an

isomorphism theorem on certain C^*-algebras, due to Ph. Green, mentioned below in the special case, which is needed. We even don't have to know in advance that G is of type I.

More specific:

(A) P = primitive ideal in $C^*(G)$

$\mathfrak{a}$ = abelian ideal in $\mathfrak{g}$, $A = \exp \mathfrak{a}$.

Then $h(P|C^*(A)) = (G\alpha)^-$, $\alpha \in \hat{A}$ suitable.

(Remarks. $P|_{C^*(A)}$ is G-prime, structure of G-orbits in $\hat{A} \cong \mathfrak{a}^*$, $G\alpha$ is open in $(G\alpha)^-$, the orbit $G\alpha$ is determined by the space $(G\alpha)^-$.)

(B) Given P, A, α as in (A), form

$$\mathcal{A} := C^*(A)/k(\overline{G\alpha}) \rhd k(\overline{G\alpha} \setminus G\alpha)/k(\overline{G\alpha}) =: \dot{\mathcal{A}}.$$

P yields a primitive ideal in

$C^*(G, A, T, \tau, \mathcal{A})$ and in $C^*(G, A, \dot{T}, \dot{\tau}, \dot{\mathcal{A}})$.

The latter algebra is, Ph. Green!, isomorphic to

$$C^*(G_\alpha|\alpha \otimes \mathcal{K}(\mathcal{L}^2(G/G_\alpha)),$$

where $C^*(G_\alpha)_\alpha$ is the C^*-completion of

$$\{\varphi : G_\alpha \to \mathbb{C} \mid \varphi(xa) = \alpha(a)^{-1}\varphi(x),\ \int_{G_\alpha/A} |\varphi| < \infty\}.$$

Thus, P delivers a primitive ideal in $C^*(G_\alpha)_\alpha$; in fact, P is induced from an ideal in $C^*(G_\alpha)$.

Consequences

(B1) G is of type I, $G^\wedge \longleftrightarrow \mathrm{Priv}\, C^*(G)$

(B2) Given P $(\leftrightarrow \pi \in G^\wedge)$, $A = \exp \mathfrak{r}$ as above and <u>chosen</u> $\alpha \in \hat{A}$, there is a <u>unique</u> $\sigma = \sigma_\alpha \in G_\alpha^\wedge$ s.t.

$$\pi = \mathrm{ind}_{G_\alpha}^{G} \sigma .$$

In turn

(B3) For any abelian $A = \exp \mathfrak{r} \lhd G$, any $\alpha \in \hat{A}$, any $\sigma \in G_\alpha^\wedge$ with $\sigma|_A = \alpha$ the induced representation

$$\mathrm{ind}_{G_\alpha}^{G} \sigma \text{ is irreducible.}$$

<u>Theorem</u> (case of exponential groups)
There exists a family $\varkappa_G : G^\wedge \longrightarrow \mathfrak{g}^*/G$ of bijections, G an exponential Lie group, with the following properties.

22.5.2013 - 5

(i) If G is abelian then $\kappa_G(\pi) = d\pi \in \mathfrak{g}^*$, i.e., $\pi(\exp X) = e^{i\,d\pi(X)}$, $X \in \mathfrak{g}$.

(ii) If $\gamma : G_1 \to G_2$ is an isomorphism then the diagram

$$\begin{tikzcd} \hat{G}_2 \arrow[r, "\hat{\gamma}"] \arrow[d] & \hat{G}_1 \arrow[d] \\ \mathfrak{g}_2^*/G_2 \arrow[r, "(d\gamma)^*"] & \mathfrak{g}_1^*/G_1 \end{tikzcd}$$

commutes.

(iii) If $\mathfrak{m}$ is an ideal in $\mathfrak{g}$, $M = \exp \mathfrak{m}$ then the diagram

$$\begin{tikzcd} (G/M)^\wedge \arrow[r, hook] \arrow[d] & \hat{G} \arrow[d] \\ (\mathfrak{g}/\mathfrak{m})^*/G \arrow[r, hook] & \mathfrak{g}^*/G \end{tikzcd}$$

commutes.

(iv) If $\mathfrak{a}$ is an abelian ideal in $\mathfrak{g}$, $A = \exp \mathfrak{a}$, $\pi \in \hat{G}$ then

$$\kappa_A\left(h\left(\ker \pi|_A\right)\right) = \left(\kappa_G(\pi)|\mathfrak{a}\right)^-.$$

(compare (A))

(v) Let $\mathfrak{g}, \mathfrak{a}, A, \pi$ be as in (iv). For any $\alpha \in \hat{A}$ s.t. $h(\ker \pi|_A) = (G\alpha)^-$ there exists, compare (B2), a unique $\sigma = \sigma_\alpha \in \hat{G}_\alpha$ s.t. $\pi = \mathrm{ind}_{G_\alpha}^{G} \sigma$. One has

$$\kappa_G(\pi) = \bigcup_\alpha \mathrm{Ext}\left[\kappa_{G_\alpha}(\sigma_\alpha)\right]$$

(compare (C); in particular the right hand side is a G-orbit)

Moreover, the family κ_G is uniquely determined by (i) - (v).

Remarks to the proof

The uniqueness is obvious – (v) contains a recipe. The crucial point is the independence on κ, in (v).

Theorem (case of nilpotent groups)

There exists a family of bijections $\hat{N} \to \mathfrak{n}^*/N$, N a simply connected nilpotent Lie group, with the following properties.

(i), (ii), (iii) as above

(iv)' If M is a connected normal subgroup of N, if $\pi \in N^\wedge$ then

$$\kappa_M\left(h\left(\ker \pi|_M\right)\right) = \kappa_G(\pi)|_{\mathfrak{m}}.$$

Moreover, the family (κ_N) is uniquely determined by (i), (ii), (iii), (iv)'.