

22.5.2013

## Kirillov theory without polarizations

$G$  always denotes an exponential Lie group, i.e.,

$\exp: \mathfrak{g} \rightarrow G$  is a diffeomorphism.  
(This implies that  $G$  is solvable.)

$$\mathfrak{g}^* = \text{Hom}(\mathfrak{g}, \mathbb{R}), \quad \mathfrak{g}^*/G \xrightarrow{\text{bijection}} \widehat{G} \quad \begin{array}{l} \text{unitary} \\ \text{dual} \end{array}$$

In a little more detail:

Each irreducible representation  $\pi$  of  $G$  is monomial,  $\pi = \text{ind}_Q^G \chi$ ,  $\chi \in \text{Hom}(Q, \mathbb{T})$ .

Question Which pairs  $(Q, \chi)$  deliver irreducibles?

Answer Take any  $f \in \mathfrak{g}^*$  s.t.  $f|_{\mathfrak{q}} = d\chi$ .

Then (P) the orbit  $Gf$  contains  $f + \mathfrak{q}^\perp$ , i.e., all extensions of  $d\chi$ , and  $\mathfrak{q}$  is a polarization at  $f$ , which essentially means that  $\mathfrak{q}$  is maximal w.r.t. the property that the restriction of

$f$  delivers a character of  $Q = \exp \mathfrak{q}$ , called  $\chi_f$  below.

Elaboration these ideas yields

a surjective map  $\mathfrak{g}^* \rightarrow \widehat{G}$

(given  $f$  choose a polarization  $\mathfrak{q}$  at  $f$  with (P), and form  $\text{ind}_Q^G \chi_f$ ), which leads to the mentioned bijection between  $\mathfrak{g}^*/G$  and  $\widehat{G}$ .

22.5.2013 - 2

In this approach one has to struggle with polarizations (existence, independence, ...), also the above "answer" requires some work!

Today I wish to present an approach (to the above bijection) without using the word polarization. Also I found (and I find) that it is at least as canonical to construct the map in the opposite direction, i.e.,  $G^\wedge \rightarrow \mathfrak{g}^*/G$ .

The idea is to understand that both spaces,  $G^\wedge$  and  $\mathfrak{g}^*/G$  are built by the same construction laws from corresponding lower dimensional objects.

To illustrate this on the side of  $\mathfrak{g}^*/G$  we observe (easy to prove):

$$(O) \quad G = \exp \mathfrak{g}, \quad f \in \mathfrak{g}^*, \quad \Omega = Gf \\ \mathfrak{o} = \text{abelian ideal in } \mathfrak{g}$$

$$\Omega / \mathfrak{o} \ni \alpha, \alpha', \dots$$

Given  $\alpha$ , consider  $\mathfrak{g}_\alpha$  and

$$\Sigma_\alpha := \{ g / \mathfrak{g}_\alpha \mid g \in \Omega, g / \mathfrak{o} = \alpha \}$$

Then  $\Sigma_\alpha$  is a  $G_\alpha$ -orbit. Form

$$\text{Ext}(\Sigma_\alpha) = \Sigma'_\alpha := \{ h \in \mathfrak{g}^* \mid h / \mathfrak{g}_\alpha \in \Sigma_\alpha \}. \text{ Then} \\ \Omega = \bigcup_\alpha \Sigma'_\alpha.$$

On the side of representation theory our approach requires only an

isomorphism theorem on certain  $C^*$ -algebras, due to Ph. Green, mentioned below in the special case, which is needed. We even don't have to know in advance that  $G$  is of type I.

More specific:

(A)  $\mathcal{P}$  = primitive ideal in  $C^*(G)$   
 $\alpha$  = abelian ideal in  $\mathcal{A}$ ,  $A = \exp \alpha$ .

Then  $h(\mathcal{P}|C^*(A)) = (G\alpha)^-$ ,  $\alpha \in \hat{A}$  suitable.

(Remarks.  $\mathcal{P}|C^*(A)$  is  $G$ -prime, structure of  $G$ -orbits in  $\hat{A} \cong \alpha^*$ ,  $G\alpha$  is open in  $(G\alpha)^-$ , the orbit  $G\alpha$  is determined by the space  $(G\alpha)^-$ .)

(B) Given  $\mathcal{P}$ ,  $A$ ,  $\alpha$  as in (A), form

$$A := C^*(A)/h(\overline{G\alpha}) \triangleright h(\overline{G\alpha} - G\alpha)/h(\overline{G\alpha}) =: \dot{A}$$

$\mathcal{P}$  yields a primitive ideal in

$C^*(G, A, \tau, \tau, \dot{A})$  and in  $C^*(G, A, \dot{\tau}, \dot{\tau}, \dot{A})$ .

The latter algebra is, Ph. Green!, isomorphic to

$$C^*(G_\alpha)_\alpha \otimes \mathcal{K}(L^2(G/G_\alpha)),$$

where  $C^*(G_\alpha)_\alpha$  is the  $C^*$ -completion of

$$\{ \varphi: G_\alpha \rightarrow \mathbb{C} \mid \varphi(xa) = \alpha(a)^{-1} \varphi(x), \sum_{G_\alpha/A} |\varphi| < \infty \}.$$

22.5.2013 - 4

Thus,  $P$  delivers a primitive ideal in  $C^*(G_\alpha)_\alpha$ ; in fact,  $P$  is induced from an ideal in  $C^*(G_\alpha)$ .

### Consequences

(B1)  $G$  is of type I,  $G^\wedge \leftrightarrow \text{Prim } C^*(G)$

(B2) Given  $P (\leftrightarrow \pi \in G^\wedge)$ ,  $A = \exp \mathfrak{a}$  as above and chosen  $\alpha \in \hat{A}$ , there is a unique  $\sigma = \sigma_\alpha \in G_\alpha^\wedge$  s.t.  
$$\pi = \text{ind}_{G_\alpha}^G \sigma.$$

In turn

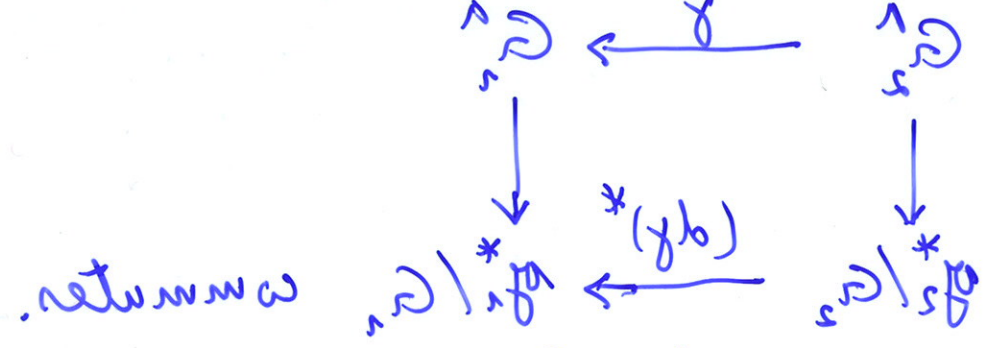
(B3) For any abelian  $A = \exp \mathfrak{a} \triangleleft G$ , any  $\alpha \in \hat{A}$ , any  $\sigma \in G_\alpha^\wedge$  with  $\sigma|_A = \alpha$  the induced representation

$$\text{ind}_{G_\alpha}^G \sigma \text{ is irreducible.}$$

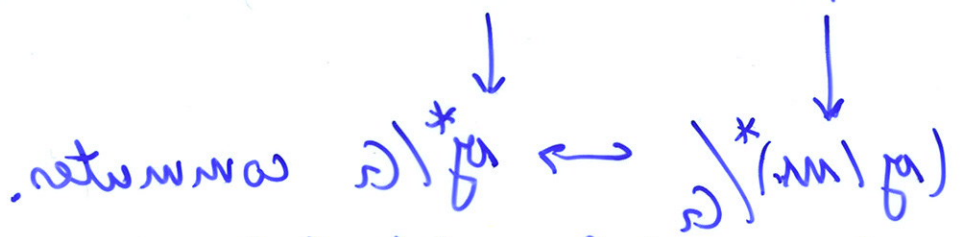
### Theorem (case of exponential groups)

There exists a family  $\kappa_G : G^\wedge \rightarrow \mathfrak{g}^*/G$  of bijections,  $G$  an exponential Lie group, with the following properties.

(i) If  $\sigma \in \pi^{-1}(x)$ , i.e.,  $\pi(\exp \sigma) = x$ , then  $\pi(\sigma) = x$ .  
 (ii) If  $\gamma: G_1 \rightarrow G_2$  is an isomorphism then the



(iii) If  $m$  is an ideal in  $G$ ,  $M = \exp m$ , then the diagram

$$\begin{array}{ccc} \hat{G} & \xrightarrow{\quad} & \hat{G/M} \\ \downarrow & & \downarrow \\ G/G^* & \xrightarrow{\quad} & (M/M)^* / G^* \end{array}$$


(iv) If  $\sigma$  is an arbitrary ideal in  $G$ ,  $A = \exp \sigma$ ,  $\pi \in G$ , then

$$\pi_A ( \mathcal{L}(\ker \pi / A) ) = ( \mathcal{L}(\pi^{-1} \sigma) / \sigma )$$

(compare (A))

(v) Let  $\sigma, \tau, A, \pi$  be as in (iv). For any  $\alpha \in \hat{A}$ , or unique  $\sigma \in G_\alpha^*$  s.t.  $\pi = \exp \sigma$ . One can write  $\pi(\sigma) = \bigcup_x \mathcal{B}_x \cup \mathcal{B}_x(\sigma)$

(compare (D)); in particular the right hand side is  $G$ -orbit. Moreover, the family  $\pi_\alpha^{-1}$  is uniquely determined by (i) - (v).

22.5.2013 - 6

## Remarks to the proof

The uniqueness is obvious - (v) contains a recipe. The crucial point is the independence on  $\alpha$ , in (v).

### Theorem (case of nilpotent groups)

There exists a family of bijections  $\hat{N} \rightarrow \mathfrak{m}^*/N$ ,  $N$  a simply connected nilpotent Lie group, with the following properties.

(i), (ii), (iii) as above

(iv)' If  $M$  is a connected normal subgroup of  $N$ , if  $\pi \in N^\wedge$  then

$$\kappa_M(\ker \pi|_M) = \kappa_G(\pi)|_M.$$

Moreover, the family  $(\kappa_N)$  is uniquely determined by (i), (ii), (iii), (iv)'.  
2