WEIGHTED ORLICZ ALGEBRAS

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Young Function and Complementary Young Function

Definition [Rao and Ren, 2002] (Young Function)

A function $\Phi : [0, +\infty) \to [0, +\infty]$ is called a Young function if

(i) Φ is convex,

(ii) $\lim_{x \to 0^+} \Phi(x) = \Phi(0) = 0$,

(iii) $\lim_{x \to +\infty} \Phi(x) = +\infty$.

Definition(Complementary Young Function)

A Young function Ψ complementary to Φ is defined by

$$\Psi(y) = \sup\{xy - \Phi(x) : x \geq 0\}$$

for $y \geq 0$. Then (Φ, Ψ) is called a complementary pair of Young functions.

Young Function and Complementary Young Function

Example

1) Let $1 < p < +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then $\Phi(x) = \frac{x^p}{p}$, $x \geq 0$, and $\Psi(x) = \frac{x^q}{q}$, $x \geq 0$, are a complementary pair of Young functions.

Example

2) Especially if $p = 1$, then the complementary Young function of $\Phi(x) = x$ is

$$\Psi(x) = \begin{cases} 0, & 0 \leq x \leq 1 \\ +\infty, & x > 1 \end{cases}$$

Example

3) If $\Phi(x) = e^x - 1$, $x \geq 0$, then

$$\Psi(x) = \begin{cases} 0, & 0 \leq x \leq 1 \\ x \ln x - x + 1, & x > 1 \end{cases}$$

Weighted Orlicz Spaces

Definition (Weighted Orlicz Space)

Let G be a locally compact group with left Haar measure μ and w be a weight on G (i.e. w is a positive, Borel measurable function such that $w(xy) \leq w(x)w(y)$ for all $x, y \in G$). Given a Young function Φ, the weighted Orlicz space $L^{\Phi}_{w}(G)$ is defined by

$$L^{\Phi}_{w}(G) := \{f : G \to K | \exists \alpha > 0, \int_G \Phi(\alpha |f\, w|) d\mu < +\infty\}$$

Then $L^{\Phi}_{w}(G)$ becomes a Banach space under the norm $|| \cdot ||_{\Phi,w}$ (called the weighted Orlicz norm) defined for $f \in L^{\Phi}_{w}(G)$ by

$$||f||_{\Phi,w} := \sup \left\{ \int_G |f\, w\, v| d\mu : v \in L^{\Psi}(G), \int_G \Psi(|v|) d\mu \leq 1 \right\}$$

where Ψ is the complementary function to Φ.

Weighted Orlicz Spaces

For $f \in L_w^\Phi(G)$, one can also define the norm

$$||f||^\circ_{\Phi,w} = \inf\left\{k > 0 : \int_G \Phi\left(\frac{f\,w}{k}\right)d\mu \leq 1\right\},$$

which is called the weighted Luxemburg norm and is equivalent to the weighted Orlicz norm.

Recall...

Notice that if $\Phi(x) = \frac{x^p}{p}$, $1 \leq p < +\infty$, $L_w^\Phi(G)$ becomes the classical weighted Lebesgue space $L^p(G)$.

Comparison of Weighted Orlicz Spaces $L^{\Phi}_w(G)$

We compare the weighted Orlicz spaces with respect to Young function Φ and weight w. We need some definitions to do this.

Definition

Let w_1 and w_2 be two weights on G. Then

$$w_1 \preccurlyeq w_2 \Leftrightarrow \exists c > 0, \forall x \in G, w_1(x) \leq c w_2(x)$$

If $w_1 \preccurlyeq w_2$ and $w_2 \preccurlyeq w_1$, then we write $w_1 \approx w_2$.

Definition

Let Φ_1 and Φ_2 be two Young functions. Then

$$\Phi_1 \prec \Phi_2 \Leftrightarrow \exists d > 0, \forall x \geq 0, \Phi_1(x) \leq \Phi_2(dx).$$

Comparison Between $L^{\Phi_1}_{w_1}(G)$ and $L^{\Phi_2}_{w_2}(G)$

Theorem

Let w_1, w_2 be two weights on G and let Φ_1, Φ_2 be two Young functions. Then

$$w_1 \preccurlyeq w_2 \text{ ve } \Phi_1 \prec \Phi_2 \Rightarrow L^{\Phi_2}_{w_2}(G) \subseteq L^{\Phi_1}_{w_1}(G).$$

Notice that the converse is not true.

Comparison Between $L^{\Phi_1}_{w_1}(G)$ and $L^{\Phi_2}_{w_2}(G)$

Let Φ be a Young function. Putting $\Phi_1 = \Phi_2 = \Phi$, we compare the weighted spaces $L^{\Phi}_{w_1}(G)$ and $L^{\Phi}_{w_2}(G)$. To investigate this we need some definitions.

Definition (Δ_2 Condition)

Let Φ be a Young function

$$\Phi \in \Delta_2 \Leftrightarrow \exists K > 0, \forall x \geq 0, \quad \Phi(2x) \leq K\Phi(x)$$

Mostly we consider the Δ_2 condition for the Young function Φ.

Examples

- If $1 \leq p < \infty$, then for the Young function $\Phi(x) = \frac{x^p}{p}$, $x \geq 0$, $\Phi \in \Delta_2$.
- If $\Phi(x) = e^x - 1, x \geq 0$, then $\Phi \notin \Delta_2$.
- If $\Phi(x) = x + x^p, x \geq 0$, $1 < p < \infty$, then $\Phi \in \Delta_2$.
- If $\Phi(x) = (e + x)\ln(e + x) - e, x \geq 0$, then $\Phi \in \Delta_2$.

Comparison Between $L^{\Phi}_{w_1}(G)$ and $L^{\Phi}_{w_2}(G)$

Theorem

Let w_1, w_2 be two weights on G and let Φ be a continuous Young function such that $\Phi \in \Delta_2$. Then

$$w_1 \preccurlyeq w_2 \Leftrightarrow L^{\Phi}_{w_2}(G) \subseteq L^{\Phi}_{w_1}(G).$$

Note

If $w_1 \preccurlyeq w_2$, then it is clear that $L^{\Phi}_{w_2}(G) \subseteq L^{\Phi}_{w_1}(G)$ for any Young function Φ. The converse is not true in general. But if Φ is a continuous Young function such that $\Phi \in \Delta_2$, then the converse becomes true.

Corollary

Under the same conditions as in previous theorem,

$$w_1 \approx w_2 \Leftrightarrow L^{\Phi}_{w_1}(G) = L^{\Phi}_{w_2}(G).$$

Dual Space of $L^{\Phi}_{w}(G)$

Theorem(Dual Space)

Let G be a locally compact group and w be a weight on G. If (Φ, Ψ) is a complementary pair of Young functions such that $\Phi \in \Delta_2$, then the dual space of $(L^{\Phi}_{w}(G), ||\cdot||_{\Phi,w})$ is $L^{\Psi}_{w^{-1}}(G)$ formed by all measurable functions g on G such that $\frac{g}{w} \in L^{\Psi}(G)$ and endowed with the norm $||\cdot||^{\circ}_{\Psi,w^{-1}}$ defined for $g \in L^{\Psi}_{w^{-1}}(G)$ by

$$||g||^{\circ}_{\Psi,w^{-1}} := ||\frac{g}{w}||^{\circ}_{\Psi}.$$

Basic Properties of $L^{\Phi}_{w}(G)$

Proposition

Let Φ be a continuous Young function such that $\Phi \in \Delta_2$ and $f \in L^{\Phi}_{w}(G)$. Then

i) $\overline{C_c(G)}^{||\cdot||_{\Phi,w}} = L^{\Phi}_{w}(G)$.

ii) For every $x \in G$, $L_x f \in L^{\Phi}_{w}(G)$ and $||L_x f||_{\Phi,w} \leq w(x)||f||_{\Phi,w}$.

iii) The map

$$\begin{array}{ccc} G & \rightarrow & L^{\Phi}_{w}(G) \\ x & \mapsto & L_x f \end{array}$$

is left continuous.

Banach Algebra with Respect to Pointwise Multiplication

H.Hudzik (1985) gives necessary and sufficient conditions for an Orlicz space to be a Banach algebra with respect to pointwise multiplication on the measure space (X, Σ, μ). We adapt the results of H. Hudzik to the locally compact group G.

Proposition

Let G be a locally compact group and w be a weight on G. If Φ is a strictly increasing continuous Young function, then the following statements are equivalent for $\lim_{x\to\infty} \frac{\Phi(x)}{x} = +\infty$

i) $L^{\Phi}_w(G) \subseteq L^{\infty}_w(G)$.

ii) G is discrete.

iii) $L^1_w(G) \subseteq L^{\Phi}_w(G)$.

We need the limit condition for $iii) \Rightarrow ii)$.

Banach Algebra with Respect to Pointwise Multiplication

Corollary

If $G = \mathbb{Z}$, then the weighted Orlicz sequence spaces are denoted by $L_w^\Phi(\mathbb{Z}) = l_w^\Phi$ and

$$l_w^1 \subseteq l_w^\Phi \subseteq l_w^\infty .$$

Theorem (Banach Algebra with Respect to Pointwise Multiplication)

Let G be a locally compact group and w a weight on G such that $w(x) \geq 1$ for all $x \in G$. If Φ is a strictly increasing, continuous Young function, then

$L_w^\Phi(G)$ is Banach algebra w.r.t. pointwise multiplication $\Leftrightarrow L_w^\Phi(G) \subseteq L_w^\infty(G)$.

Observation

Under the same conditions as in the previous theorem,

$L_w^\Phi(G)$ is Banach algebra w.r.t. pointwise multiplication $\Leftrightarrow G$ is discrete.

Banach Algebra with Respect to Convolution

Theorem [H. Hudzik, 1985]

$$L^{\Phi}(G) \text{ is Banach algebra w.r.t. convolution } \Leftrightarrow L^{\Phi}(G) \subseteq L^{1}(G)$$

Theorem (Banach Algebra with Respect to Convolution)

Let w be a weight on G and let Φ be a Young function. If $L_w^{\Phi}(G) \subseteq L_w^1(G)$, then the weighted Orlicz space $(L_w^{\Phi}(G), ||\cdot||_{\Phi,w})$ is a Banach algebra w.r.t. convolution.

Note that the converse is not true in general. For $\Phi(x) = \frac{x^p}{p}$, $p > 1$, $L_w^p(G)$ is a Banach algebra, but it is not in $L_w^1(G)$. (Kuznetsova, 2006)

Banach Algebra with Respect to Convolution

Observation

If Φ is a continuous Young function such that $\Phi'_+(0) > 0$, then we have the inclusion $L^\Phi_w(G) \subseteq L^1_w(G)$. So the weighted Orlicz space $(L^\Phi_w(G), ||\cdot||_{\Phi,w})$ is a Banach algebra w.r.t. convolution.

Theorem

Let Φ be a continuous Young function such that $\Phi'_+(0) > 0$ and $\Phi \in \Delta_2$. Then the weighted Orlicz algebra $L^\Phi_w(G)$ has a left approximate identity bounded in $L^1_w(G)$.

Theorem

Let Φ be a continuous Young function such that $\Phi'_+(0) > 0$ and $\Phi \in \Delta_2$. If G is non-discrete, then the weighted Orlicz algebra $L^\Phi_w(G)$ has no bounded approximate identity.

Banach Algebra with Respect to Convolution

Proposition

Let Φ be a continuous Young function such that $\Phi'_+(0) > 0$. Then the weighted Orlicz algebra $L^\Phi_w(G)$ is a left ideal in $L^1_w(G)$.

Observation

Without any assumption on Young function Φ, we can have the weighted Orlicz space $L^\Phi_w(G)$ as a left Banach $L^1_w(G)$-module w.r.t. convolution.

Weighted Orlicz Algebra $L_w^\Phi(G)$

The next step is to describe the maximal ideal space of the algebra $L_w^\Phi(G)$ on an abelian group G. From now on, we assume that $w(x) \geq 1$ for all $x \in G$ and Φ is a continuous Young function satisfying $\Phi \in \Delta_2$.

Note

$$L_w^\Phi(G) \text{ is a commutative Banach algebra } \Leftrightarrow G \text{ is abelian.}$$

Theorem

If the space $L_w^\Phi(G)$ is a convolution algebra, then its maximal ideal space can be identified with the subset of $L_{w^{-1}}^\Psi(G)$ consisting of continuous homomorphisms $\chi : G \to \mathbb{C} \backslash \{0\}$. Each character of this algebra can be expressed via the corresponding function χ by the formula

$$X(f) = \int_G f \chi d\mu, f \in L_w^\Phi(G).$$

Lemma

The weighted Orlicz algebra $L_w^{\Phi}(G)$ is not radical.

Theorem

If the space $L_w^{\Phi}(G)$ is an algebra, then

(i) it is semisimple,

(ii) its maximal ideal space contains a homeomorphic image of the group $\widehat{G}$,

(iii) it is unital if and only if G is discrete.