Completely Bounded Λ_p-Sets on Compact Groups

Parasar Mohanty

Department of Mathematics and Statistics
Indian Institute of Technology Kanpur

Joint work with Kathryn E. Hare

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Outline of the Talk

Classical Lacunary Sets

Λ_p^{cb} on compact abelian groups

Λ_p^{cb} on compact non-abelian groups

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Concept of Lacunary sets dates back to Weirstrass and Hadamard

In 1861 Riemann guessed that the function $R(x) = \sum_1^\infty \frac{\sin n^2 x}{n^2}$ is nowhere differentiable.

Weirstrass failed to prove it. Gave the famous example $\sum_{n=1}^\infty a_n \cos b^n x$ where $0 < a < 1,\ 1 < b \in \mathbb{N}$ and $ab > 1$.

In 1892, Hadamard proved that the Taylor series $\sum_{n=1}^\infty a_n z^{\lambda_n}$ has $|z| = 1$ as natural boundary whenever $\exists q > 1$ such that $\frac{\lambda_{n+1}}{\lambda_n} > q > 1$.

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Sidon Sets

Sidon proved that the lacunary Fourier series $\sum a_n e^{2\pi i \lambda_n t}$ converges absolutely if $\{\lambda_n\}$ satisfies Hadamard's lacunar conditions.

Let $E \subseteq \mathbb{Z}$. A trigonometric polynomial f is called E-polynomial if $\hat{f}(n) = 0, \ \forall n \notin E$.

Definition

$E \subseteq \mathbb{Z}$ is said to be a Sidon set if $\exists C > 0$ such that $\sum_n |\hat{f}(n)| \leq C \|f\|_\infty$ for all trigonometric E polynomial f.

Kahane called these sets as Sidon sets.

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Denote $\hat{G}$ discrete dual group of G.

Denote $L^p_E(G) = \{f \in L^p(G) : \hat{f}(\gamma) = 0, \ \forall \gamma \notin E\}$.

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Let $2 < p < \infty$. $E \subseteq \hat{G}$ is said to be an Λ_p-set if $\exists\ C > 0$ such that $\|f\|_p \leq C\|f\|_2$ for all $E-$polynomial f. In other words $L^p_E \simeq L^2_E$.

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Conversely if $\forall \phi \in l_E^\infty$ $\exists \mu \in M(G)$ such that $\phi(\gamma) = \hat{\mu}(\gamma)\ \forall \gamma \in \hat{G}$ then E is Sidon.

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Let $1 \leq p < \infty$ a bounded operator $T : L^p(G) \to L^p(G)$ is said to be a multiplier of L^p if $\exists \phi \in l^\infty(\hat{G})$ such that $\widehat{Tf}(\gamma) = \phi(\gamma)\hat{f}(\gamma)$ $\forall f \in L^p \cap L^2$ *and* $\gamma \in \hat{G}$.

Denote $M_p(G)$ as set of all such multipliers. It is a Banach algebra.

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Operator Space Interpolation

G.Pisier has developed complex interpolation for operator spaces.

Let X_0 and X_1 are compatible pair of Banach spaces.

Denote $X_\theta = (X_0, X_1)_\theta$, in Pisier's interpolation theory $\mathbb{M}_n(X_\theta)$ gets the norm of the Banach Space $(\mathbb{M}_n(X_0), \mathbb{M}_n(X_1))_\theta$.

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Let $L^1(G)$ inherits operator space structure form $L^\infty(G)^*$.

$L^1(G)^* \simeq L^\infty(G)$ complete isomorphic with this operator space structure.

Now the canonical operator space structure on $L^p(G)$ is the interpolated operator space structure $(L^1, L^\infty)_{\frac{1}{p}}$.

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Let us denote $S_p, 1 \leq p < \infty$ to be the space of compact operators on l_2 such that $\|T\|_{S_p} = (tr|T|^p)^{1/p}$ where $|T| = tr(T^*T)^{1/2}$.

Denote $L^p(G, S_p)$ be the space of S_p valued measurable functions f such that

$$\|f\|_{L^p(G,S_p)} = \left(\int_G \|f(x)\|_{S_p}^p dx \right)^{1/p} < \infty.$$

Proposition (Pisier)

Let $1 \leq p < \infty$. A linear map $T : L^p(G) \to L^p(G)$ is completely bounded if and only if the mapping $T \otimes I_{S_p}$ is bounded on $L^p(G, S_p)$. Moreover,

$$\|T\|_{cb} = \|T \otimes I_{S_p}\|_{L^p(G,S_p) \to L^p(G,S_p)}.$$

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$M_p^{cb}(G) = M_p(G)$ if $p = 1, 2$. What about other p's.

Theorem

Let G be a locally compact abelian group. Then $M_p^{cb}(G) \subsetneq M_p(G)$ for $1 < p \neq 2 < \infty$.

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Comletely bounded Λ_p sets

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Definition

Let $2 < p < \infty$. A subset $E \subseteq \hat{G}$ is called Λ_p^{cb} set if there exists a constant C, depending only on p and E such that

$$\|f\|_{L^p(G,S_p)} \leq C \max\{\|(\sum_{\gamma\in E} \hat{f}(\gamma)^* \hat{f}(\gamma))^{1/2}\|_{S_p}, \|(\sum_{\gamma\in E} \hat{f}(\gamma)\hat{f}(\gamma)^*)^{1/2}\|_{S_p}\}$$

for all S_p valued E- polynomials f defined on G. We denote $\lambda_p^{cb}(E)$ the least constant C for which above inequality holds.

She showed that for $2 < p < \infty$, Λ_p^{cb}sets are interpolation sets for M_p^{cb}.

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Remark

1. As an application of Jensen's inequality RHS is dominated by $\|f\|_{L^p(G,S_p)}$.
2. Unlike classical setting for Λ_p^{cb}set E we cannot have $L_E^p(G,S_p) \approx L_E^2(G,S_2)$. However, if E is Λ_p^{cb} then $L_E^2(G,S_2) \subset L_E^p(G,S_p)$.
3. By considring $f = g \otimes x$ where g is an E-polynomial on G and $x \in S_p$ with $\|x\|_{S_p} = 1$ it is straight forward to see that $\Lambda_p^{cb} \subseteq \Lambda_p$.

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Given that all Sidon sets are Λ_p^{cb} for all $p > 2$,it is natural to ask, “Is there a non-Sidon, Λ_p^{cb} set?"

Harcharras and Banks answered this question for the circle group $\mathbb{T}$ in by showing that for any finite set Q of prime numbers, the set of natural numbers whose prime divisors all lie in Q is Λ_p^{cb} for all p, but not Sidon if $|Q| > 2$.

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Theorem (Hare and M)

Let G be an arbitrary compact abelian group. Then there exists $E \subseteq \hat{G}$ which is Λ_p^{cb} for all $p \in (2, \infty)$ but not Sidon.

Sketch of the proof

Rudin introduced following combinatorial property to construct Λ_p sets.

For $p \geq 2$ an integer, let

$$A_p(E) = \sup_{\gamma \in \Gamma} \left|\{(\gamma_1, \ldots, \gamma_p) \in E^p : \gamma_1\gamma_2 \ldots \gamma_p = \gamma\}\right| \text{ and}$$

$$B_p(E) = \sup_{\gamma \in \Gamma} \left|\{(\gamma_1, \ldots, \gamma_p) \in E^p : \gamma_1^{-1}\gamma_2 \ldots \gamma_p^{(-1)^p} = \gamma\}\right|.$$

Characters γ_j may be repeated and this can cause complications with the counting

Harcharras was able to extend this result to Λ_p^{cb} sets, under the weaker assumption that the characters γ_j were distinct.

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Definition

Let $p \geq 2$ be an integer. The set E has the $Z(p)$ property if

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Theorem (Harcharras)

Let $p \geq 2$ be an integer. Then every subset E of $\hat{G}$ with the $Z(p)$ property is a Λ_{2p}^{cb} set. Moreover, there exists a constant C_p, depending only on p, such that $\lambda_{2p}^{cb}(E) \leq C_p Z_p(E)^{1/2p}$ for each $E \subseteq \hat{G}$.

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Our strategy will be to find size limitations on the arithmetic structures that Sidon sets can contain, and then to construct Λ_p^{cb} sets, using Harcharras' sufficient condition, which violate this size limitation.

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Lemma

Let $E = \{\chi_j\}_{j=1}^{\infty} \subseteq \hat{G}$, where χ_j^2 are all distinct and non-trivial. We can choose an infinite subset $E' \subseteq E$ such that $Z_2(E') = 1$ and for sufficiently large n,

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One of the following three possibilities will occur in Γ.

1. $\hat{G}$ contains an element χ of infinite order.
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Case 1: Let χ be an element of Γ of infinite order and consider $\chi_j = \chi^j$.

Applying Lemma we obtain $E' \subseteq \{\chi^j\}$ with $Z_2(E') = 1$ and $\left|E' \cap \{\chi_j\}_{2^{n+1}}^{2^{n+2}}\right| \geq n^2$.

By Harcharras's Theorem E' is a Λ_4^{cb} set, but it is not Sidon.

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Case 2: For each j, choose $\chi_j \in \Gamma$ with order $N_j > \max(2N_{j-1}, 3 \cdot 2^j)$ and consider

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Case 3: Suppose Γ is of bounded order K.

Being an infinite abelian group, $\Gamma = \oplus_{j \in J} \mathbb{Z}_{n_j}$ where $n_j \leq K$ and $|J| = \infty$.

$\hat{G}$ contains an infinite subgroup $\oplus_{j=1}^{\infty} \mathbb{Z}_p^{(j)}$ for some prime p dividing n.

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Let $E \subseteq \hat{G}$. A trigonometric polynomial f is called an E-polynomial if $\hat{f}(\sigma) = 0, \ \forall \sigma \notin E$.

Definition

1. A set $E \subseteq \hat{G}$ is called a *Sidon set* if there is a constant C such that $\sum_{\sigma \in E} d_\sigma tr|\hat{f}(\gamma)| \leq C\|f\|_\infty$ for all E-polynomial f.
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Every Sidon set is a Λ_p set

Unlike the abelian situation there are compact non-abelian group G which has no infinte Sidon sets in $\hat{G}$.

Definition

An operator $T : L^p(G) \to L^p(G)$ defined by

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In the case of non-abelian G, Figá-Talamanca and Rider proved the following result.

Theorem (Figá-Talamanca , Rider)

Let $E \subseteq \hat{G}$ and $2 < p < \infty$.

(i) *E is Λ_p if and only if for every $T \in M_2$ there exists $S \in M_p$ such that $Tf = Sf$ for all $f \in L_E^p(G)$.*

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Fourier transform of Vector valued functions

Let G be a compact group.

Let $f \in L^1(G, S_p)$. For $\sigma \in \hat{G}$ the vector valued Fourier coefficient of f at σ with degree d_σ is defined as

$$\hat{f}(\sigma) = \int_G f(x)\sigma(x^{-1})dx.$$

The integral is interpreted as an element of $\mathcal{B}(\mathbb{C}^{d_\sigma}, S_p^{d_\sigma})$ in weak sense.

By fixing an orthonormal basis $e_1, \ldots, e_{d_\sigma}$ for $\mathcal{H}_\sigma$, $\hat{f}(\pi)$ can be viewed as a $d_\sigma \times d_\sigma$ matrix of entries from S_p.

In particular the $(i,j)^{th}$ entry of $\hat{f}(\sigma)$ with respect to this basis is given by

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By fixing an orthonormal basis $e_1, \ldots, e_{d_\sigma}$ for $\mathcal{H}_\sigma$, $\hat{f}(\pi)$ can be viewed as a $d_\sigma \times d_\sigma$ matrix of entries from S_p.

In particular the $(i,j)^{th}$ entry of $\hat{f}(\sigma)$ with respect to this basis is given by

$$\int_G f(x) < \sigma(x^{-1}e_j, e_i > dx.$$

Fourier transform of Vector valued functions

Let G be a compact group.

Let $f \in L^1(G, S_p)$. For $\sigma \in \hat{G}$ the vector valued Fourier coefficient of f at σ with degree d_σ is defined as

$$\hat{f}(\sigma) = \int_G f(x)\sigma(x^{-1})dx.$$

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Λ_p^{cb}-sets

For matrix $A = (A_{ij})_{i,j=1}^n,\ A_{ij} \in S_p$ denote $TrA = \sum_{i=1}^n A_{ii}$.

trV we mean the usual trace of $n \times n$ complex matrix V.

Definition

Let $E \subseteq \hat{G}$. For $2 < p < \infty$, we say that E is Λ_p^{cb} if there exists a constant C such that

$$\|f\|_{L^p(G,S_p)} \leq C \left[\|(\sum_{\sigma\in E} d_\sigma Tr(A_\sigma(A_\sigma)^*)^{1/2}\|_{S_p} + \|(\sum_{\sigma\in E} d_\sigma Tr((A_\sigma)^* A_\sigma))\|_{S_p} \right]$$

where $f = \sum_{\sigma\in E} d_\sigma Tr(A_\sigma \sigma)$ a S_p valued trigonometric E-polynomial.

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Proposition

Let $V \in U(n)$ *and* $A \in \mathcal{B}(\mathbb{C}^n, S_p^n)$, $n > 4$. *Then*

$$\int_{U(n)} \|TrAV\|_{S_4}^4 dV \leq \frac{8c}{(degV)^2}\ tr\left[(\sum_{m,n} A_{mn}^* A_{mn})^2 + (\sum_{m,n} A_{mn} A_{mn}^*)^2 \right]$$

In the abelian case Λ_p-sets are interpolation sets of M_p as a consequence of Khintchine inequality. For non-abelian groups G, it can be proved that if $\sum_{\sigma \in \hat{G}} d_\sigma tr(A_\sigma A_\sigma^*) < \infty$ then given $p < \infty$ there exists unitary transformations $\{U_\sigma\}$ such that $\sum_\sigma d_\sigma tr(U_\sigma A_\sigma \sigma(x))$ is the Fourier series of an $L^p(G)$ function.

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Theorem

Let $V = \{V_\sigma\} \in \prod U(d_\sigma)$ *and* $A^\sigma \in \mathcal{B}(\mathbb{C}^{d_\sigma}, S_p^{d_\sigma})$ *for* $\sigma \in \hat{G}$ *then*

$$\int_{\mathcal{G}} \| \sum_\sigma d_\sigma Tr A^\sigma V_\sigma \|_{S_4}^4 dV \leq Ctr[(\sum_\sigma d_\sigma \sum_{j,l} A_{jl}^\sigma (A^\sigma)_{jl}^*)^2 + (\sum_\sigma d_\sigma \sum_{j,l} (A^\sigma)_{jl}^* A_{jl}^\sigma)^2]$$

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Theorem (Hare & M)

Let $T \in M_2(G)$ and $E \subseteq \hat{G}$. If E is Λ_4^{cb} and $\widehat{Tf}(\sigma) = \phi(\sigma)\hat{f}(\sigma)$ with $\phi(\sigma) = 0$ for all $\sigma \notin E$ then $T \in M_4^{cb}$. Conversely if $T_\phi \in M_2$ implies $T_\phi \in M_4^{cb}$ for all $\phi \in l_E^\infty(\hat{G}) = \{\phi(\sigma) \in \mathcal{B}(\mathcal{H}_\sigma) : \phi(\sigma) = 0\ \forall \sigma \notin E\}$ then E is Λ_4^{cb}.

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