Fourier transform for nilpotent Lie groups

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Nilpotent Lie algebras and nilpotent Lie groups

Let $\mathfrak{g}$ be a nilpotent Lie algebra over $\mathbb{R}$, i.e; the sequence of ideals

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Let $G = \exp(\mathfrak{g})$ be the corresponding simply connected connected (nilpotent) Lie group.
Jordan-Hölder basis of $\mathfrak{g}$:

$$\mathcal{Z} = \{Z_1, \cdots, Z_n\}$$

i.e.

$$\mathfrak{g}_j := \text{span}\{Z_j, \cdots, Z_n\} \text{ ideal of } \mathfrak{g}, j = 1, \cdots, n.$$

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The mapping

$$E_{\mathfrak{Y}} : \mathbb{R}^s \times \mathfrak{h} \mapsto G;$$
$$(t_1, \cdots, t_s, U) \to \exp(t_1 Y_1) \cdot \cdots \cdot \exp(t_s Y_s) \cdot h$$

is a diffeomorphism.

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Vergne polarisation

Let $\ell \in \mathfrak{g}^*$. Let $\mathcal{Z} = \{Z_1, \cdots, Z_n\}$ be a Jordan-Hölder basis of $\mathfrak{g}$:
Vergne polarization at ℓ:

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\mathcal{H}_{\ell,\mathfrak{h}} &= L^2(G/H, \chi_\ell) \\
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Proposition

For $F \in L^1(G)$:

$$\sigma_{\ell,\mathfrak{h}}(F)\xi(s) = \int_{G/H} F_{\ell,\mathfrak{h}}(s,t)\xi(t)dt,$$

$$\text{where } F_{\ell,\mathfrak{h}}(s,t) = \int_H F(sht^{-1})\chi_\ell(h)dh.$$

Orbit picture

Theorem

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- *Let* $\ell_i \in \mathfrak{g}^*$ *and let* $\mathfrak{p}_i, i = 1, 2$ *be a polarization at* $\ell_i, i = 1, 2$. *Then*

$$\sigma_{\ell_1,\mathfrak{p}_1} \simeq \sigma_{\ell_2,\mathfrak{p}_2} \Leftrightarrow \mathrm{Ad}^*(G)\ell_2 = \mathrm{Ad}^*(G)\ell_1.$$

Write:

$$[\pi_\ell] := [\sigma_{\pi,\mathfrak{p}}]$$

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Write:

$$[\pi_\ell] := [\sigma_{\pi,\mathfrak{p}}]$$

- *Let* $(\pi, \mathcal{H}_\pi) \in \widehat{G} \Rightarrow \exists \ell \in \mathfrak{g}^*$ *such that*

$$[\pi] = [\pi_\ell]$$

A homeomorphism

Theorem

The mapping $\mathcal{K} : \mathfrak{g}^*/G \to \widehat{G}$ *defined by*

$$\mathcal{K}(\mathrm{Ad}^*(G)\ell) := [\pi_\ell]$$

is a homeomorphism

A partition of the orbit space

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$I(\ell) = \emptyset$ if ℓ is a character. Otherwise, let

$$\begin{aligned} j_1 = j_1(\ell) &= \max\{j \in \{1, \ldots, n\} \mid Z_j \notin \mathfrak{a}(\ell)\} \\ k_1 = k_1(\ell) &= \max\{k \in \{1, \ldots, n\} \mid < l, [Z_{j_1(\ell)}, Z_k] > \neq 0\}. \end{aligned}$$

We let

$$\begin{aligned} \nu_1(\ell) &:= \langle \ell, [Z_{k_1}, Z_{j_1}] \rangle \\ S_1 = S_1(\ell) &:= \frac{1}{\nu_1(\ell)} [Z_{k_1}, Z_{j_1}], \\ Y_1 = Y_1(\ell) &:= Z_{j_1} - \frac{\langle \ell, Y_1 \rangle}{\nu_1(\ell)} S_1 \\ X_1 = X_1(\ell) &:= Z_{k_1} - \frac{\langle \ell, Z_{k_1} \rangle}{\nu_1(\ell)} S_1. \end{aligned}$$

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\end{aligned}$$

Then we have that:

$$\langle \ell, X_1 \rangle = \langle \ell, Y_1 \rangle = 0, \qquad (0.1)$$
$$\langle \ell, [X_1, Y_1] \rangle = 1.$$

We consider

$$\mathfrak{g}^1(\ell) := \{ U \in \mathfrak{g} \mid < l, [U, Y_1(\ell)] >= 0 \} \qquad (0.2)$$

which is an ideal of codimension one in $\mathfrak{g}$.

A Jordan-Hölder basis of $(\mathfrak{g}^1(\ell), [\cdot, \cdot])$ is given by $\{Z_i^1(\ell) \mid i \neq k_1(\ell)\}$ defined by

$$Z_i^1(\ell) = Z_i - \frac{< l, [Z_i, Y_1(\ell)] >}{\nu_1(\ell)} X_1(\ell), i \neq k_1(\ell). \quad (0.3)$$

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As previously we may now compute the indices $j_2(\ell), k_2(\ell)$ of $l_1 := l|_{\mathfrak{g}^1(\ell)}$ with respect to this new basis and construct the corresponding subalgebra $\mathfrak{g}^2(\ell)$ with its associated basis $\{Z_i^2(\ell) \mid i \neq k_1(\ell), k_2(\ell)\}$.

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This procedure stops after a finite number $r_\ell = r$ of steps. Let

$$I_{\mathcal{Z}}(\ell) = I(\ell) = \big((j_1(\ell), k_1(\ell)), \dots, (j_r(\ell), k_r(\ell))\big)$$

is called the index of ℓ in $\mathfrak{g}$ with respect to the basis $\mathcal{Z} = \{Z_1, \dots .Z_n\}$.

It is known that the last subalgebra $\mathfrak{g}_r(\ell)$ obtained by this construction coincides with the Vergne polarization of ℓ in $\mathfrak{g}$ with respect to the basis $\mathcal{Z}$.

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The length $|I| = 2r$ of the index set $I(\ell)$ gives us the dimension of the coadjoint orbit $\mathrm{Ad}^*(G)\ell$.

Partition of $\mathfrak{g}^*/G$

For an index set $I \in \mathbb{N}^{2j}, j = 0, \cdots, \dim(\mathfrak{g}/2)$:

$$\mathfrak{g}_I^* := \{\ell \in \mathfrak{g}^*, I(\ell) = I, \langle I, X_i(\ell)\rangle = 0, \langle I, Y_i(\ell)\rangle = 0, i = 1, \cdots, r\}.$$

Let

$$\mathcal{I} := \{I \in \bigcup_{j=0}^{\dim(\mathfrak{g}/2)} \mathbb{N}^j, \mathfrak{g}_I^* \neq \emptyset\}.$$

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Then:

$$\mathfrak{g}^*/G \simeq \mathfrak{g}_{\mathcal{I}}^* := \dot{\bigcup}_{I\in\mathcal{I}} \mathfrak{g}_I^*$$

Properties of the $\mathfrak{g}_I^*$:

There exists an index $I^{gen} \in \mathcal{I}$ such that

$$\mathfrak{g}_{gen}^* := \{\ell \in \mathfrak{g}^*, I(\ell) = I^{gen}\}$$

is G-invariant and Zariski open in $\mathfrak{g}^*$.

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There exists an order on $\mathcal{I}$ such that

- I^{gen} is maximal for this order,
- such that

$$\mathfrak{g}_{\leq I}^* := \bigcup_{I' \leq I} \mathfrak{g}_{I'}^*$$

is Zariski closed in $\mathfrak{g}^*$.

Realization on $L^2(\mathbb{R}^r)$

Proposition

- *For every $I \in \mathcal{I}$ the mappings*

$$\mathfrak{g}_I^* \ni \ell \mapsto X_j(\ell), \ell \mapsto Y_j(\ell), \ell \mapsto \mathfrak{p}^{\mathcal{Z}}(\ell)$$

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  are smooth.
- *The family of vectors* $\mathfrak{X}(\ell) = \{X_j(\ell), j = 1, \cdots, r\}$ *form a Malcev-basis of* $\mathfrak{g}$ *modulo* $\mathfrak{p}^{\mathcal{Z}}(\ell)$*,*
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  the vectors $\{Y_j(\ell), j = 1, \cdots, r\}$ *form a Malcev basis of* $\mathfrak{p}^{\mathcal{Z}}(\ell)$ *modulo* $\mathfrak{g}(\ell)$*.*
- *We identify the Hilbert space* $L^2(G/P^{\mathcal{Z}}(\ell), \chi_\ell)$ *with* $L^2(\mathbb{R}^{r_\ell})$ *using the unitary operator:*

$$U_\ell(\eta) = \eta \circ E_\ell^{\mathcal{Z}} \in L^2(\mathbb{R}^{r_\ell}), \eta \in L^2(G/P^{\mathcal{Z}}(\ell), \chi_\ell).$$

An example

Let $\mathfrak{g} = \text{ span } \{A, B, C, D, U, V\}$.

$$[A, B] = U, [C, D] = V, [A, C] = V, [B, D] = sU$$

$(s \in R^*)$.

Let $\ell \in \mathfrak{g}^*$

$$\mu = \langle \ell, U \rangle, \langle \ell, V \rangle = \nu.$$

- $\nu \neq 0 \Rightarrow$

$$\mathfrak{g}^1(\ell) = \operatorname{span}\{A, B - \frac{s\mu}{\nu} C, D, U, V\},$$
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$$[Z_1^1, Z_2^1]_{s,\mu,\nu} = Z_5^1 - \frac{s\mu}{\nu} Z_6^1,$$
$$[Z_2^1, Z_4^1]_{s,\mu,\nu} = sZ_5^1 - \frac{s\mu}{\nu} Z_6^1.$$
$$j_2(\ell) = 2, k_2(\ell) = 1, \text{if } s \neq 1.$$

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Definition

A variable locally compact group is a pair

$$(\mathcal{B}, G)$$

where $\mathcal{B}$ and G are locally compact topological spaces, such that for every $\beta \in \mathcal{B}$ there exists a group multiplication $\cdot_\beta$ on G, which turns $(G, \cdot_\beta)$ into a topological group, such that

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$$\mathcal{B} \times (G \times G) \mapsto G, (\beta, (s, t)) \to s \cdot_\beta t$$

is continuous.

Definition

A variable nilpotent Lie algebra is a triple

$$(\mathfrak{g}, \mathcal{Z}, \mathcal{B})$$

of a real finite dimensional vector space $\mathfrak{g}$, of a basis $\mathcal{Z} = \{Z_1, \cdots, Z_n\}$ of $\mathfrak{g}$ and a smooth manifold $\mathcal{B}$, such that

- for every $\beta \in \mathcal{B}$ there is a Lie algebra product $[,]_\beta$ on $\mathfrak{g}$,
- $[Z_i, Z_j]_\beta = \sum_{k=j+1}^{n} c_k^{i,j}(\beta) Z_k$, $1 \leq i < j \leq n$
- and such that the functions $\beta \to c_k^{i,j}(\beta)$ are all smooth.

Fourier transform

Definition

$$l^{\infty}(\widehat{G}) := \{(\varphi(\ell) \in \mathcal{K}(\mathcal{H}_{\ell})_{\ell \in \mathfrak{g}^*_{\mathcal{I}}}, \|\varphi\|_{\infty} := \sup_{\ell \in \mathfrak{g}^*_{\mathcal{I}}} \|\varphi(\ell)\|_{\mathrm{op}} < \infty\}.$$

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For $F \in L^1(G)$, let

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For $F \in L^1(G)$, let

$$\mathcal{F}(F)(\ell) = \widehat{F}(\ell) := \pi_\ell(F),\ \ell \in \mathfrak{g}^*_{\mathcal{I}}.$$

For $u \in \mathcal{U}(\mathfrak{g})$ let

$$\widehat{u}(\ell) = d\pi_\ell(u) \in \mathcal{PD}(\mathbb{R}^{r_l}), \ell \in \mathfrak{g}^*_{\mathcal{I}}$$

Properties of $\widehat{u}$

- For every $u \in \mathcal{U}(\mathfrak{g})$, for $\ell \in \mathfrak{g}_I$,

$$d\sigma_{\ell,\mathfrak{p}^{\mathcal{Z}}(\ell)}(u) = \widehat{u}(\ell) = \sum_{\alpha \in \mathbb{R}^{r_I}} p_\alpha^u(\ell)\partial^\alpha$$

with polynomial coefficients $p_\alpha^u(\ell)$ which depend smoothly on $\ell \in \mathfrak{g}_I^*$.
Let

$$d\mu(u) := (d\sigma_{\ell,\mathfrak{p}^{\mathcal{Z}}(\ell)}(u))_{\ell \in I^{gen}}$$

- For every $D = \sum_{\alpha \in \mathbb{N}^{r_I}} p_\alpha \partial^\alpha$ there exists a smooth mapping $\rho_{D,I} : \mathfrak{g}_I^* \to \mathcal{U}(\mathfrak{g})$, such that

$$d\sigma_{\ell,\mathfrak{p}^{\mathcal{Z}}(\ell)}(\rho_{D,I}(\ell)) = D, \ell \in \mathfrak{g}_I^*.$$

Properties of $\widehat{F}, F \in \mathcal{S}(G)$

- With respect to the basis $\mathfrak{X}(\ell) = \{X_1(\ell), \cdots, X_r(\ell)\}$ the kernel functions of the operators $\sigma_{\ell, \mathfrak{p}^{\mathcal{Z}}(\ell)}(F)$:

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- Let $Q \in \mathbb{C}[\mathfrak{g}]$. For every $I = I^{gen}$, there exists a partial differential operator $D_Q(I)$ on $\mathfrak{g}_I^* \times \mathbb{R}^{r_I}$ with polynomial coefficients in the variable $(x, x') \in \mathbb{R}^{r_I} \times \mathbb{R}^{r_I}$ and smooth coefficients in $\ell \in \mathfrak{g}_I^*$, such that for every $F \in S(G)$:

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Properties of $\widehat{F}, F \in L^1(G)$:

1. the operator field $\widehat{F}$ is contained in $l^\infty(\widehat{G})$.
2. on the subsets $\mathfrak{g}_I^*, I \in \mathcal{I}$, the mappings

   $\ell \mapsto \widehat{F}(\ell) \in \mathcal{K}(L^2(\mathbb{R}^{r_I}))$ are operator -norm continuous.

3. For every sequence $(\mathrm{Ad}^*(G)\ell_k)_{k\in\mathbb{N}}$ which goes to infinity in $\mathfrak{g}^*/G$, we have that

$$\lim_{k\to\infty} \|\widehat{F}(\ell_k)\|_{\mathrm{op}} = 0.$$

Questions:

- Characterize the image of $C^*(G)$ in $l^\infty(\widehat{G})$ under the Fourier transform,i.e.
  understand how $\pi_\ell(F)$ varies if $\ell \in \mathfrak{g}_I^*$ approaches the boundary of $\mathfrak{g}_I^*$.

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  understand how $\pi_\ell(F)$ varies if $\ell \in \mathfrak{g}_I^*$ approaches the boundary of $\mathfrak{g}_I^*$.
- Characterize the image of $\mathcal{S}(G)$ in $l^\infty(\widehat{G})$ under the Fourier transform.

Properly converging sequences in $\widehat{G}$

Let $I \in \mathcal{I}$ and let $\overline{\mathcal{O}} = (\pi_{\mathcal{O}_k})$ be a properly converging sequence in $\widehat{G}_I$ with limit set $L(\overline{\mathcal{O}})$ contained in $\widehat{G}_{<I}$, then the elements $\rho \in L(\overline{\mathcal{O}})$ are "entangled " by $\overline{\mathcal{O}}$:

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Question: What is the relation between the sequence of operators

$$(\pi_{\mathcal{O}_k}(F) \in \mathcal{B}(L^2(\mathbb{R}^{r_I})))_k$$

and the operator field

$$(\rho(F))_{\rho \in L(\overline{\mathcal{O}})}?$$

Definition

Let

$$L^2(\widehat{G}) = \{(\varphi(\ell))_{\ell\in\mathfrak{g}^*_{Igen}}, \ell \to \varphi(\ell) \text{ measurable}, \int_{\widehat{G}} \|\varphi(\ell)\|^2_{H-S} d\mu(\ell) < \infty\}$$

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Theorem

The Fourier transform maps $\mathcal{S}(G)$ onto $\mathcal{S}(\widehat{G})$.

Inverse Fourier transform

Theorem

There exists a G-invariant polynomial function P_{gen} on $\mathfrak{g}^*$ such that for every $F \in \mathcal{S}(G)$:

$$\begin{aligned} F(g) &= \int_{\mathfrak{g}^*_{l\,gen}} \mathrm{tr}\,(\pi_\ell(g^{-1}) \circ \widehat{F}(\ell)) |P_{gen}(\ell)| d\ell, \\ &= \int_{\widehat{G}} \mathrm{tr}\,(\pi(g^{-1}) \circ \pi(F)) d\mu(\pi), g \in G. \end{aligned}$$

Smooth compactly supported operator fields

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& \text{support } (\varphi) \text{ compact in } \mathfrak{g}^*_{I^{gen}}, \\
& \text{the function } (\ell, x, x') \to \varphi(\ell)(x, x') \\
& \text{is smooth in } \ell \\
& \text{and Schwartz in } (x, x') \in \mathbb{R}^{r_{gen}} \times \mathbb{R}^{r_{gen}}.\}
\end{aligned}
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Theorem

For every $\varphi \in C_c^\infty(\widehat{G})$ there exists a unique $F \in \mathcal{S}(G)$, such that

$$\widehat{F} = \varphi.$$

Un-sufficient data

What can we do, if we have only a smooth field $(\varphi(\ell) \in \mathcal{K}(L^2(\mathbb{R}^r)))_{\ell \in M}$ defined on a smooth submanifold of $\widehat{G}$?

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Theorem

(R. Howe) For every $\varphi \in \mathcal{S}(\mathbb{R}^r \times \mathbb{R}^r)$ there exists $F \in S(G)$ such that

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This means that

$$\sigma_{\ell,\mathfrak{p}}(\mathcal{S}(G)) = \mathcal{B}(\mathcal{H}_{\ell,\mathfrak{p}})^\infty.$$

Fourier inversion for sub-manifolds

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Theorem

(Currey-L-Molitor-Braun) Let $\mathfrak{g}_I^*$ be a fixed layer of $\mathfrak{g}^*$. Let M be a smooth sub-manifold of $\mathfrak{g}_I^*$. There exists an open subset M^0 of M such that for any smooth kernel function Φ with compact support $C \subset M^0$, there is a function F in the Schwartz space $\mathcal{S}(G)$ such that $\pi_\ell(F)$ has $\Phi(\ell)$ as an operator kernel for all $\ell \in M^0$. Moreover, the Schwartz function F may be chosen such that $\pi_\ell(F) = 0$ for all $\ell \in M \setminus M^0$ and for any ℓ in $\mathfrak{g}_{<I}^*$ and such that the map $\Phi \mapsto F$ is continuous with respect to the corresponding function space topologies.

An application

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For instance : (ρ, E) an irreducible bounded representation ρ of $\mathbb{G}$ on a Banach space E and

$$J = \ker(\rho_{|G})_{L^1(G)}.$$

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$\widehat{G}$ is Baire space, $L^1(G)$ has the Wiener property and J is A-prime $\Rightarrow$ the hull $h(J)$ of J in $\widehat{G}$ is the closure of an A-orbit in $\widehat{G}$:

$$h(J) = \overline{A \cdot \pi_\ell} \text{ for some } \ell \in \mathfrak{g}^*.$$

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$$ker(h(J))_{\mathcal{S}}/j(h(J))_{\mathcal{S}} \text{ is nilpotent} \Rightarrow J_{\mathcal{S}} = ker(h(J))_{\mathcal{S}}.$$

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Is $\varphi = 0$ on J?

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and φ defines a tempered distribution d_φ on $\mathcal{S}(A \cdot \pi_\ell)$
From this one can show that

$$|\langle \varphi, F \rangle| \leq \sup_{\pi \in A \cdot \pi_\ell} \|\pi(F)\|_{\mathrm{op}}, F \in L^1(G).$$

Theorem

Suppose that $J \subset L^1(G)$ *is A-prime and* $h(J) = A \cdot \pi$ *is a closed A-orbit in* $\widehat{G}$*, then* $J = ker(A \cdot \pi)$.