Amenability Properties for Various Classes of Polynomial Hypergroups

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Outline

1. Harmonic analysis on polynomial hypergroups: basics
2. General results
3. One-parameter generalizations of ultraspherical polynomials
4. Comparison to the group case

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3. One-parameter generalizations of ultraspherical polynomials
4. Comparison to the group case

Let $a_0 > 0,\ b_0 < 1,\ (a_n)_{n\in\mathbb{N}}, (c_n)_{n\in\mathbb{N}} \subseteq (0,1)$ and $(b_n)_{n\in\mathbb{N}} \subseteq [0,1)$ satisfy $a_0 + b_0 = 1,\ a_n + b_n + c_n = 1\ (n \in \mathbb{N})$;
define polynomials in $\mathbb{R}[x]$ by $P_0(x) := 1,\ P_1(x) := \frac{1}{a_0}(x - b_0)$,

$$P_1(x)P_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x)\ (n \in \mathbb{N})\ .$$

Theorem of Favard (1935): $(P_n(x))_{n\in\mathbb{N}_0}$ is orthogonal w.r.t. a unique probability (Borel) measure μ on $\mathbb{R}$ with $|\text{supp}\ \mu| = \infty$.
Let $(p_n(x))_{n\in\mathbb{N}_0}$ denote the corresponding ortho*norm*al polynomials (with positive leading coefficients).
Linearization formula:

$$P_m(x)P_n(x) = \sum_{k=|m-n|}^{m+n} \underbrace{g(m,n;k)}_{\in\mathbb{R}} P_k(x);\quad \sum_{k=|m-n|}^{m+n} g(m,n;k) = 1,$$

$$g(m,n;|m-n|), g(m,n;m+n) \neq 0\ (m,n \in \mathbb{N}_0)\ .$$

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$$g(m,n;|m-n|), g(m,n;m+n) \neq 0 \ (m,n \in \mathbb{N}_0)\ .$$

Crucial assumption:

Assume the linearization coefficients $g(m,n;k)$ to be non-negative.

Define a *convolution* $\mathbb{N}_0 \times \mathbb{N}_0 \to \mathrm{conv}\,\{\delta_n : n \in \mathbb{N}_0\}$ by

$$(m,n) \mapsto \sum_{k=|m-n|}^{m+n} g(m,n;k)\delta_k \ ;$$

$\rightsquigarrow$ together with the identity map as *involution*, $\mathbb{N}_0$ becomes a commutative discrete hypergroup with unit element 0, a *polynomial hypergroup*.

Haar measure: counting measure weighted by the values of the *Haar function* $h : \mathbb{N}_0 \to [1,\infty)$, $h(n) := p_n^2(1)$.

Let $(\ell^p(h), \|.\|_p)$, $p \in [1,\infty]$, be the ℓ^p-spaces w.r.t. the Haar measure.

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Harmonic analysis on polynomial hypergroups

- *Convolution:* for $f \in \ell^p(h)$ and $g \in \ell^q(h)$, where $p \in [1, \infty)$, $q := \frac{p}{p-1} \in (1, \infty]$, one has $f * g \in \ell^\infty$, where

$$f * g(n) := \sum_{k=0}^{\infty} \sum_{j=|n-k|}^{n+k} g(n, k; j) f(j) g(k) h(k) .$$

  This convolution and complex conjugation make $\ell^1(h)$ a *commutative Banach $*$-algebra with unit δ_0*. $\ell^1(h)$ acts on $\ell^p(h)$ ($p \in [1, \infty]$) by convolution; ℓ^∞ is the dual module of $\ell^1(h)$.
- *Symmetric characters:* ${}_x\alpha : \mathbb{N}_0 \to \mathbb{R}$, ${}_x\alpha(n) := P_n(x)$, where $\widehat{\mathbb{N}_0} := \{x \in \mathbb{R} : \forall n \in \mathbb{N}_0 : |P_n(x)| \leq 1\}$.
- *Fourier transformation and Plancherel isomorphism:* $\widehat{\cdot} : \ell^1(h) \to C(\widehat{\mathbb{N}_0})$, $\widehat{f}(x) = \sum_{k=0}^{\infty} f(k) P_k(x) h(k)$; $\widehat{\cdot}$ extends to an isometric isomorphism $\mathcal{P} : \ell^2(h) \to L^2(\mathbb{R}, \mu)$.

(Lasser 1983)

Let A be a Banach algebra, and let $(X, \cdot)$ be a Banach A-bimodule. A linear mapping $D : A \to X$ is called

- *derivation* if $D(ab) = a \cdot D(b) + D(a) \cdot b \ (a, b \in A)$,
- *inner derivation* if $D(a) = a \cdot x - x \cdot a \ (a \in A)$ for some $x \in X$.

Definition (Johnson 1972; Bade–Curtis–Dales 1987; Johnson 1987)

A is called *amenable* (*weakly amenable*) if for every Banach A-bimodule X every bounded (!) derivation into the dual module X^* (A^*) is an inner derivation.

Given some $x \in \widehat{\mathbb{N}_0}$, a linear functional $D_x : \ell^1(h) \to \mathbb{C}$ is called a *point derivation* on $\ell^1(h)$ at x if

$D_x(f * g) = \widehat{f}(x) D_x(g) + \widehat{g}(x) D_x(f) \ (f, g \in \ell^1(h))$.

We call $\ell^1(h)$ *point-amenable* if there is no $x \in \widehat{\mathbb{N}_0}$ for which there exists a non-zero bounded point derivation.

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Outline

General results: weak amenability

Theorem (Lasser 2007; weak amenability: characterization)

$\ell^1(h)$ is weakly amenable if and only if $\{\|\mathcal{P}^{-1}(P_n') * \varphi\|_\infty : n \in \mathbb{N}_0\}$ is unbounded for all $\varphi \in \ell^\infty \backslash \{0\}$.

Theorem (K. 2013; weak amenability: necessary criteria)

Let $\ell^1(h)$ be weakly amenable. Then $\{\mathcal{P}^{-1}(P_n')(0) : n \in \mathbb{N}_0\}$ is unbounded; moreover, μ has a singular part or μ' is not absolutely continuous on $[\min \operatorname{supp} \mu, \max \operatorname{supp} \mu]$.

Theorem (K. 2013; weak amenability: sufficient criterion)

If $\{\|\mathcal{P}^{-1}(P_n') * \varphi\|_\infty : n \in \mathbb{N}_0\}$ is unbounded for all $\varphi \in \ell^\infty \backslash \mathcal{O}(n^{-1})$ and

- *μ is absolutely continuous, $\operatorname{supp} \mu = [-1, 1]$, $\mu' > 0$ a.e. in $[-1, 1]$,*
- *$h \in \mathcal{O}(n^\alpha)$ for some $\alpha \in [0, 1)$, $\sup_{n \in \mathbb{N}_0} \int_\mathbb{R} p_n^4(x)\, d\mu(x) < \infty$,*

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Ingredients for the proofs:

- harmonic analysis on polynomial hypergroups, Plancherel isomorphism,
- asymptotics of orthogonal polynomials (Nevai et al. 1979 –),
- basic Hilbert space theory,
- elementary approximation theory,
- Christoffel–Darboux formula,
- fundamental lemma of the calculus of variations,
- integration by parts.

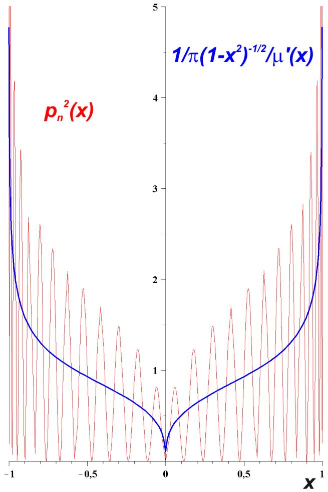

Figure : "Oscillations" around a weak limit

General results: point-amenability and amenability

Theorem (Lasser 2009, K. 2013; point-amenability: characterizations)

TFAE:

1. $\ell^1(h)$ *is point-amenable,*
2. $\{P_n'(x) : n \in \mathbb{N}_0\}$ *is unbounded for all* $x \in \widehat{\mathbb{N}_0}$*,*
3. $\{\|\mathcal{P}^{-1}(P_n') * {}_x\alpha\|_\infty : n \in \mathbb{N}_0\}$ *is unbounded for all* $x \in \widehat{\mathbb{N}_0}$*.*

Proposition (K. 2013; point-amenability: necessary criteria)

Let $b_n \equiv 0$*. If* $\ell^1(h)$ *is point-amenable, then*

- $c_n a_{n-1} > \frac{1}{4}$ *for some* $n \in \mathbb{N}$*,*
- $\limsup_{n\to\infty} c_n \geq \frac{1}{2}$*.*

Theorem (Lasser 2007; amenability: necessary criterion)

If $\ell^1(h)$ *is amenable, then* $h(n) \not\to \infty$ $(n \to \infty)$*.*

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One-parameter generalizations of ultraspherical polynomials

Let $\alpha \geq -\frac{1}{2}$; let $(P_n(x))_{n\in\mathbb{N}_0}$ be orthogonal w.r.t.

$$\mathrm{d}\mu(x) = (\ldots)(1-x^2)^\alpha \chi_{(-1,1)}(x)\,\mathrm{d}x$$

("ultraspherical").

Dougall 1919, Hsü 1938: polynomial hypergroup induced.

$\ell^1(h)$ weakly amenable $\Leftrightarrow \alpha < 0$ (K. AHA2011).

Most important one-parameter generalizations:

- class of *Jacobi polynomials*,
- class of *symmetric Pollaczek polynomials*,
- class of *associated ultraspherical polynomials*,
- class of *continuous q-ultraspherical (Rogers) polynomials* (contains the ultraspherical polynomials as limiting cases).

Theorem (Lasser 2009; Vogel 1987)

If $(P_n(x))_{n\in\mathbb{N}_0}$ belongs to the class of continuous q-ultraspherical polynomials, then $\ell^1(h)$ is not even point-amenable.

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Jacobi polynomials

Let $\alpha \geq -\frac{1}{2}$, $-1 < \beta \leq \alpha$ satisfy $a(a+5)(a+3)^2 \geq (a^2 - 7a - 24)b^2$, where $a := \alpha + \beta + 1$, $b := \alpha - \beta$; let $(P_n(x))_{n\in\mathbb{N}_0}$ be orthogonal w.r.t.

$$\mathrm{d}\mu(x) = (\ldots)(1-x)^\alpha(1+x)^\beta\chi_{(-1,1)}(x)\,\mathrm{d}x\ .$$

- Gasper 1970: polynomial hypergroup induced.
- Rahman 1981: explicit computation of the $g(m,n;k)$ as a ${}_9F_8$ hypergeometric series.
- K. 2013: explicit expansions of $\mathcal{P}^{-1}(P_n')$.

Theorem (K. 2013)

- $\ell^1(h)$ *amenable* $\Leftrightarrow \alpha = -\frac{1}{2}$,
- $\ell^1(h)$ *weakly amenable* $\Leftrightarrow \alpha < 0$,
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Main problem: weak amenability for $\alpha < 0$.

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Symmetric Pollaczek polynomials

Let $-\frac{1}{2} < \alpha < 0$, $0 \leq \lambda < \alpha + \frac{1}{2}$, or let $\alpha, \lambda \geq 0$; let $(P_n(x))_{n\in\mathbb{N}_0}$ be orthogonal w.r.t.

$$\mathrm{d}\mu(x) = (\ldots)(1-x^2)^{\alpha} \frac{\left|\Gamma\left(\alpha + \frac{1}{2} + i\frac{\lambda x}{\sqrt{1-x^2}}\right)\right|^2}{e^{\frac{\lambda x(\pi - 2\arccos x)}{\sqrt{1-x^2}}}} \chi_{(-1,1)}(x)\,\mathrm{d}x\,.$$

Lasser 1994: polynomial hypergroup induced.

Theorem (K. 2013)

- $\ell^1(h)$ *not amenable,*
- $\ell^1(h)$ *weakly amenable* $\Leftrightarrow \alpha < 0, \lambda = 0$,
- $\ell^1(h)$ *point-amenable* $\Leftrightarrow \alpha < \frac{1}{2}, \lambda = 0$.

Main problem: failure of point-amenability if $0 < \lambda < -\alpha + \frac{\sqrt{8\alpha+5}}{2} - 1$.

Associated ultraspherical polynomials

Let $\alpha > -\frac{1}{2}$, $\nu \geq 0$; let $(P_n(x))_{n\in\mathbb{N}_0}$ be orthogonal w.r.t.

$$\mathrm{d}\mu(x) = (\ldots)(1-x^2)^{\alpha} \frac{1}{\left|{}_2F_1\left(\begin{matrix}\frac{1}{2}-\alpha,\nu\\ \alpha+\nu+\frac{1}{2}\end{matrix}\middle| e^{2i\arccos x}\right)\right|^2} \chi_{(-1,1)}(x)\,\mathrm{d}x\,.$$

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- $\ell^1(h)$ *weakly amenable* $\Leftrightarrow \alpha < 0, \nu = 0$,
- $\ell^1(h)$ *point-amenable* $\Leftrightarrow \alpha < \frac{1}{2}$.

Main problem: failure of weak amenability if $\alpha < 0$, $\nu > 0$.

Outline

Comparison to the group case

Amenability notions for the L^1-algebras:

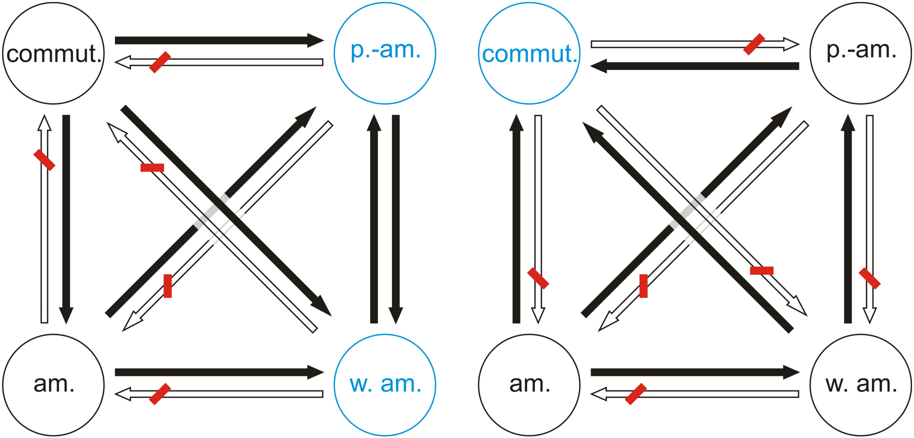

Figure : Locally compact groups (left) vs. polynomial hypergroups (right); "blue" means "always satisfied"

Selected references

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