Interpolation sets and quotients of function spaces on a locally compact group II

Jorge Galindo

Instituto de Matemáticas y Aplicaciones de Castellón, Universitat Jaume I

International Conference on Abstract Harmonic Analysis, Granada May 20-24, 2013

Based on Joint work with Mahmoud Filali (University of Oulu, Finland).

Theorem 1

Let $\mathcal{A} \subset \mathcal{B} \subset \mathcal{LUC}(G)$, two admissible subalgebras of $\mathcal{LUC}(G)$. Let $U \in \mathcal{N}(e)$ be compact such that T is right U-uniformly discrete. If G contains a family of sets $\{\mathbf{T}_{\eta}: \eta < \kappa\}$ with:

2 T_{η} fails to be an \mathcal{A} -interpolation set for any $\eta < \kappa$.

 $\bigcup_{\eta < \kappa} T_{\eta} \text{ is an aproximable } \mathbb{B}\text{-interpolation set.}$

Then, there is a linear isometry $\Psi: \ell_{\infty}(\kappa) \to \mathcal{B}/\mathcal{A}$.

Back to ENAR

Theorem 1

Let $\mathcal{A} \subset \mathcal{B} \subset \mathcal{LUC}(G)$, two admissible subalgebras of $\mathcal{LUC}(G)$. Let $U \in \mathcal{N}(e)$ be compact such that T is right U-uniformly discrete. If G contains a family of sets $\{\mathbf{T}_{\eta}: \eta < \kappa\}$ with:

$$\ \, \mathbf{0} \ \ \, T_{\eta} \cap T_{\eta'} = \emptyset \ \, \text{for every} \ \, \eta < \eta' < \kappa.$$

2) T_{η} fails to be an A-interpolation set for any $\eta < \kappa$.

$$\bigcup_{\eta < \kappa} T_{\eta} \text{ is an aproximable } \mathbb{B}\text{-interpolation set.}$$

Then, there is a linear isometry $\Psi: \ell_{\infty}(\kappa) \to \mathcal{B}/\mathcal{A}$.

Back to ENAR

Theorem 1

Let $\mathcal{A} \subset \mathcal{B} \subset \mathcal{LUC}(G)$, two admissible subalgebras of $\mathcal{LUC}(G)$. Let $U \in \mathcal{N}(e)$ be compact such that T is right U-uniformly discrete. If G contains a family of sets $\{\mathbf{T}_{\eta}: \eta < \kappa\}$ with:

$$\ \, \mathbf{0} \ \ \, T_{\eta} \cap T_{\eta'} = \emptyset \ \, \text{for every} \ \, \eta < \eta' < \kappa.$$

2 T_{η} fails to be an \mathcal{A} -interpolation set for any $\eta < \kappa$.

(a) $\bigcup_{\eta < \kappa} T_{\eta}$ is an **aproximable** \mathcal{B} -interpolation set.

Then, there is a linear isometry $\Psi: \ell_{\infty}(\kappa) \to B/A$.

Back to ENAR

Theorem 1

Let $\mathcal{A} \subset \mathcal{B} \subset \mathcal{LUC}(G)$, two admissible subalgebras of $\mathcal{LUC}(G)$. Let $U \in \mathcal{N}(e)$ be compact such that T is right U-uniformly discrete. If G contains a family of sets $\{\mathbf{T}_{\eta}: \eta < \kappa\}$ with:

$$\ \, \mathbf{0} \ \ \, \mathbf{T}_\eta \cap \mathbf{T}_{\eta'} = \emptyset \ \, \text{for every} \ \, \eta < \eta' < \kappa.$$

2 T_{η} fails to be an \mathcal{A} -interpolation set for any $\eta < \kappa$.

$$\bigcup_{\eta<\kappa} T_{\eta} \text{ is an aproximable } \mathbb{B}\text{-interpolation set.}$$

Then, there is a linear isometry $\Psi: \ell_{\infty}(\kappa) o \mathcal{B}/\mathcal{A}$. Back to ENAI

Theorem 1

Let $\mathcal{A} \subset \mathcal{B} \subset \mathcal{LUC}(G)$, two admissible subalgebras of $\mathcal{LUC}(G)$. Let $U \in \mathcal{N}(e)$ be compact such that T is right U-uniformly discrete. If G contains a family of sets $\{\mathbf{T}_{\eta}: \eta < \kappa\}$ with:

$$\ \, \mathbf{0} \ \ \, \mathbf{T}_\eta \cap \mathbf{T}_{\eta'} = \emptyset \ \, \text{for every} \ \, \eta < \eta' < \kappa.$$

2 T_{η} fails to be an \mathcal{A} -interpolation set for any $\eta < \kappa$.

$$\bigcup_{\eta<\kappa} T_{\eta} \text{ is an aproximable } \mathbb{B}\text{-interpolation set.}$$

Then, there is a linear isometry $\Psi: \ell_{\infty}(\kappa) o \mathcal{B}/\mathcal{A}$. Back to ENAI

Theorem 1

Let $\mathcal{A} \subset \mathcal{B} \subset \mathcal{LUC}(G)$, two admissible subalgebras of $\mathcal{LUC}(G)$. Let $U \in \mathcal{N}(e)$ be compact such that T is right U-uniformly discrete. If G contains a family of sets $\{\mathbf{T}_{\eta}: \eta < \kappa\}$ with:

$$\ \, \mathbf{0} \ \ \, \mathbf{T}_\eta \cap \mathbf{T}_{\eta'} = \emptyset \ \, \text{for every} \ \, \eta < \eta' < \kappa.$$

2 T_{η} fails to be an \mathcal{A} -interpolation set for any $\eta < \kappa$.

$$\bigcup_{\eta<\kappa} T_{\eta} \text{ is an aproximable } \mathcal{B}\text{-interpolation set.}$$

Then, there is a linear isometry $\Psi: \ell_{\infty}(\kappa) \to \mathcal{B}/\mathcal{A}$. Back to ENAR

• We need a family $\{\mathbf{T}_{\eta} : \eta < \kappa\}$ of pairwise disjoint sets such that:

- None of the T_{η} 's is a $C_0(G)$ -interpolation set.
- $T = \bigcup_{\eta < \kappa} T_{\eta}$ is a uniformly discrete approximable $WAP_0(G)$ -interpolation
- Useful data:
 - A: G₀-interpolation set must be relatively compact.
 - 2.010.7. Is night: US-uniformly discrete and both UT set UT and all T-010T areas relatively compact for all a 6-16 \ K-with K-compact (UT-local level), then T-is an approximable W/D9[6] - interpolation and.
- Let G be SIN. Then construct T with |T| = κ(G) (κ(G)=compact covering number of G) such that UT is a t-set and T is right U²-uniformly discrete. Any partition T = ∪ T_η will do.

Theorem 2 (Chou for $\kappa = \omega$)

• We need a family $\{{\bf T}_\eta\colon \eta<\kappa\}$ of pairwise disjoint sets such that:

- None of the T_η's is a C₀(G)-interpolation set.
- $T = \bigcup_{\eta < \kappa} T_{\eta}$ is a uniformly discrete approximable $WAP_0(G)$ -interpolation set
- Useful data:
 - A: G_f-interpolation set must be relatively compact.
 - If T is night UP-uniformly discrete and both UTs O UT and all CO UT and all relatively compact for all a sc G \ K with K compact (UT-is a k-ac), then T is an approximable WOD(G) interpolation set.
- Let G be SIN. Then construct T with |T| = κ(G) (κ(G)=compact covering number of G) such that UT is a t-set and T is right U²-uniformly discrete. Any partition T = [] T_η will do.

Theorem 2 (Chou for $\kappa = \omega$)

- We need a family $\{{\bf T}_\eta\colon \eta<\kappa\}$ of pairwise disjoint sets such that:
 - None of the T_{η} 's is a $C_0(G)$ -interpolation set.
 - $T = \bigcup_{\eta < \kappa} T_{\eta}$ is a uniformly discrete approximable $\mathcal{WAP}_0(G)$ -interpolation

set.

- Useful data:
 - A C₀-interpolation set must be relatively compact.
 - If T is night UP-uniformly discrete and both UTs O UT and all CO UT and all relatively compact for all a sc G \ K with K compact (UT-is a k-ac), then T is an approximable WOD(G) interpolation set.
- Let G be SIN. Then construct T with |T| = κ(G) (κ(G)=compact covering number of G) such that UT is a t-set and T is right U²-uniformly discrete. Any partition T = | | T_n will do.

Theorem 2 (Chou for $\kappa = \omega$)

- We need a family $\{\mathbf{T}_{\eta}: \eta < \kappa\}$ of pairwise disjoint sets such that:
 - None of the T_{η} 's is a $C_0(G)$ -interpolation set.
 - $T = \bigcup T_{\eta}$ is a uniformly discrete approximable $\mathcal{WAP}_0(G)$ -interpolation $n < \kappa$ set.

- Useful data:

- We need a family $\{\mathbf{T}_{\eta}: \eta < \kappa\}$ of pairwise disjoint sets such that:
 - None of the T_{η} 's is a $C_0(G)$ -interpolation set.
 - $T = \bigcup T_{\eta}$ is a uniformly discrete approximable $\mathcal{WAP}_0(G)$ -interpolation $\eta < \kappa$ set.

- Useful data:
 - A C₀-interpolation set must be relatively compact.

- We need a family $\{{\bf T}_\eta\colon \eta<\kappa\}$ of pairwise disjoint sets such that:
 - None of the T_{η} 's is a $C_0(G)$ -interpolation set.
 - $T = \bigcup_{\eta < \kappa} T_{\eta}$ is a uniformly discrete approximable $WAP_0(G)$ -interpolation set.
- Useful data:
 - A C₀-interpolation set must be relatively compact.
 - If T is right U²-uniformly discrete and both UTs ∩ UT and sUT ∩ UT are relatively compact for all s ∈ G \ K with K compact (UT is a t-set), then T is an approximable WAP₀(G)-interpolation set.
- Let G be SIN. Then construct T with |T| = κ(G) (κ(G)=compact covering number of G) such that UT is a t-set and T is right U²-uniformly discrete. Any partition T = ⋃ T_η will do.

Theorem 2 (Chou for $\kappa = \omega$ **)**

- We need a family $\{{\bf T}_\eta\colon \eta<\kappa\}$ of pairwise disjoint sets such that:
 - None of the T_{η} 's is a $C_0(G)$ -interpolation set.
 - $T = \bigcup_{\eta < \kappa} T_{\eta}$ is a uniformly discrete approximable $WAP_0(G)$ -interpolation set.
- Useful data:
 - A C₀-interpolation set must be relatively compact.
 - If T is right U²-uniformly discrete and both UTs ∩ UT and sUT ∩ UT are relatively compact for all s ∈ G \ K with K compact (UT is a t-set), then T is an approximable WAP₀(G)-interpolation set.
- Let G be SIN. Then construct T with |T| = κ(G) (κ(G)=compact covering number of G) such that UT is a t-set and T is right U²-uniformly discrete. Any partition T = ⋃ T_η will do.

Theorem 2 (Chou for $\kappa = \omega$)

If G is a locally compact SIN group, then there is a linear isometry

 $n < \kappa$

- Useful data:
 - A C₀-interpolation set must be relatively compact.
 - If T is right U^2 -uniformly discrete and both $UTs \cap UT$ and $sUT \cap UT$ are relatively compact for all $s \in G \setminus K$ with K compact (UT is a t-set), then T is an approximable $\mathcal{WAP}_0(G)$ -interpolation set.
- Let G be SIN. Then construct T with |T| = κ(G) (κ(G)=compact covering number of G) such that UT is a t-set and T is right U²-uniformly discrete. Any partition T = ⋃_{η<κ} T_η will do.

Theorem 2 (Chou for $\kappa = \omega$ **)**

If G is a locally compact SIN group, then there is a linear isometry $\psi \colon \ell_{\infty}(\kappa(\mathbf{G})) \to \frac{\mathcal{WAP}_{0}(\mathbf{G})}{\mathbf{C}_{0}(\mathbf{G})}.$ This and more is valid if G is an E-group with an E-set X, replacing $\kappa(G)$ by $\kappa(X).$

- We need {T_η: η < κ} (pairwise disjoint) such that none of the T_η's is a B(G)-interpolation set and T = U_{η<κ} T_η is a uniformly discrete approximable WAP(G)-interpolation set.
- (Chou, 1990) B(G)-interpolation sets cannot contain *n*-squares for every *n*: an *n*-square is a set AB ⊂ G with |A| = |B| = n and |AB| = n².
- If G is discrete, we manufacture a collection of pairwise disjoint sets $\{T_{\eta}: \eta < |G|\}$, with: $T_{\eta} = \bigcup_{n} C_{\eta,n} D_{\eta,n}$, $|C_{\eta,n}| = |D_{\eta,n}| = n$, $|C_{\eta,n}D_{\eta,n}| = n^2$ such that T is a *t*-set.
- This can be extended to groups *G* with an open normal subgroup *H*. Then we obtain |G/H|-many right *H*-uniformly discrete subsets with the above properties.

- We need {T_η: η < κ} (pairwise disjoint) such that none of the T_η's is a B(G)-interpolation set and T = U_{η<κ} T_η is a uniformly discrete approximable WAP(G)-interpolation set.
- (Chou, 1990) $\mathcal{B}(G)$ -interpolation sets cannot contain *n*-squares for every *n*: an *n*-square is a set $AB \subset G$ with |A| = |B| = n and $|AB| = n^2$.
- If G is discrete, we manufacture a collection of pairwise disjoint sets $\{T_{\eta}: \eta < |G|\}$, with: $T_{\eta} = \bigcup_{n} C_{\eta,n} D_{\eta,n}$, $|C_{\eta,n}| = |D_{\eta,n}| = n$, $|C_{\eta,n}D_{\eta,n}| = n^{2}$ such that T is a *t*-set.
- This can be extended to groups G with an open normal subgroup H. Then we obtain |G/H|-many right H-uniformly discrete subsets with the above properties.

- We need {T_η: η < κ} (pairwise disjoint) such that none of the T_η's is a B(G)-interpolation set and T = U_{η<κ} T_η is a uniformly discrete approximable WAP(G)-interpolation set.
- (Chou, 1990) $\mathcal{B}(G)$ -interpolation sets cannot contain *n*-squares for every *n*: an *n*-square is a set $AB \subset G$ with |A| = |B| = n and $|AB| = n^2$.
- If G is discrete, we manufacture a collection of pairwise disjoint sets $\{T_{\eta}: \eta < |G|\}$, with: $T_{\eta} = \bigcup_{n} C_{\eta,n} D_{\eta,n}$, $|C_{\eta,n}| = |D_{\eta,n}| = n$, $|C_{\eta,n}D_{\eta,n}| = n^2$ such that T is a *t*-set.
- This can be extended to groups *G* with an open normal subgroup *H*. Then we obtain |G/H|-many right *H*-uniformly discrete subsets with the above properties.

- We need {T_η: η < κ} (pairwise disjoint) such that none of the T_η's is a B(G)-interpolation set and T = U_{η<κ} T_η is a uniformly discrete approximable WAP(G)-interpolation set.
- (Chou, 1990) $\mathcal{B}(G)$ -interpolation sets cannot contain *n*-squares for every *n*: an *n*-square is a set $AB \subset G$ with |A| = |B| = n and $|AB| = n^2$.
- If G is discrete, we manufacture a collection of pairwise disjoint sets $\{T_{\eta}: \eta < |G|\}$, with: $T_{\eta} = \bigcup_{n} C_{\eta,n} D_{\eta,n}$, $|C_{\eta,n}| = |D_{\eta,n}| = n$, $|C_{\eta,n}D_{\eta,n}| = n^2$ such that T is a *t*-set.
- This can be extended to groups G with an open normal subgroup H. Then we obtain |G/H|-many right H-uniformly discrete subsets with the above properties.

A general theorem Some quotients One application: strong Arens irregularity of $L_1(G)$

Linearly isometric copies of $\ell_{\infty}(\kappa)$ in quotients: $\frac{\mathcal{WAP}(G)}{\mathcal{B}(G)}$ (II)

A t-set T in a central subgroup H of G, is a t-set in G. It follows that:

Theorem 3

Let G be any locally compact group. There is a linear isometry $\psi: L_{\infty}(\kappa(Z(G))) \rightarrow \frac{WAP(G)}{B(G)}$.

dding some functorial properties of $\mathcal{B}(G)$ and $\mathcal{WAP}(G)$ it follows that

Theorem 4 (Chou 1990, for $\kappa=\omega$)

Adding some structure theory (of locally compact groups):

Theorem 5 (Chou, 1990, for $\kappa = \omega$)

If G is an [IN] locally compact group and $\kappa = \kappa[G]$, then there is a linear

Linearly isometric copies of $\ell_{\infty}(\kappa)$ in quotients: $\frac{WAP(G)}{B(G)}$ (II)

A t-set T in a central subgroup H of G, is a t-set in G. It follows that:

Theorem 3

Let G be any locally compact group. There is a linear isometry $\psi: \ell_{\infty}(\kappa(\mathsf{Z}(\mathsf{G}))) \rightarrow \frac{\mathcal{WAP}(\mathsf{G})}{\mathcal{B}(\mathsf{G})}.$

dding some functorial properties of $\mathcal{B}(G)$ and $\mathcal{WAP}(G)$ it follows that

Theorem 4 (Chou 0.990, for $\kappa = \omega$)

If G is a nilpotent docally compact group and $\kappa = \kappa(G)$, then there is a linear isometry $\psi \in L_{2}(n)$.

Adding some structure theory (of locally compact groups):

Theorem 5 (Chou, 1990, for $\kappa = \omega$)

If G is an [N] locally compact group and $\kappa = \kappa(G)$, then there is a linear

Linearly isometric copies of $\ell_{\infty}(\kappa)$ in quotients: $\frac{WAP(G)}{B(G)}$ (II)

A t-set T in a central subgroup H of G, is a t-set in G. It follows that:

Theorem 3

Let G be any locally compact group. There is a linear isometry $\psi: \ell_{\infty}(\kappa(\mathsf{Z}(\mathsf{G}))) \rightarrow \frac{\mathcal{WAP}(\mathsf{G})}{\mathcal{B}(\mathsf{G})}.$

Adding some functorial properties of $\mathcal{B}(G)$ and $\mathcal{WAP}(G)$ it follows that

Theorem 4 (Chou 1990, for $\kappa=\omega$)

If G is a nilpotent locally compact group and $\kappa = \kappa(G)$, then there is a linear isometry: $\psi: \ell_{\infty}(\kappa) \to \frac{WAP(G)}{B(G)}$.

Adding some structure theory (of locally compact groups):

Theorem 5 (Chou, 1990, for $\kappa = \omega$)

If G is an |N| locally compact group and $\kappa = \kappa(G)$, then there is a linear

A t-set T in a central subgroup H of G, is a t-set in G. It follows that:

Theorem 3

Let G be any locally compact group. There is a linear isometry $\psi: \ell_{\infty}(\kappa(\mathsf{Z}(\mathsf{G}))) \rightarrow \frac{\mathcal{WAP}(\mathsf{G})}{\mathcal{B}(\mathsf{G})}.$

Adding some functorial properties of $\mathcal{B}(G)$ and $\mathcal{WAP}(G)$ it follows that

If G is a nilpotent locally compact group and $\kappa = \kappa(G)$, then there is a linear isometry: $\psi: \ell_{\infty}(\kappa) \to \frac{WA\mathcal{P}(G)}{mred}$.

Adding some structure theory (of locally compact groups):

Theorem 5 (Chou, 1990, for $\kappa = \omega$)

If G is an ||N| locally compact group and $\kappa = \kappa(G)$, then there is a linear

Linearly isometric copies of $\ell_{\infty}(\kappa)$ in quotients: $\frac{WAP(G)}{B(G)}$ (II)

A t-set T in a central subgroup H of G, is a t-set in G. It follows that:

Theorem 3

Let G be any locally compact group. There is a linear isometry $\psi: \ell_{\infty}(\kappa(\mathsf{Z}(\mathsf{G}))) \rightarrow \frac{\mathcal{WAP}(\mathsf{G})}{\mathcal{B}(\mathsf{G})}.$

Adding some functorial properties of $\mathcal{B}(G)$ and $\mathcal{WAP}(G)$ it follows that

Theorem 4 (Chou 1990, for $\kappa = \omega$)

If G is a nilpotent locally compact group and $\kappa = \kappa(G)$, then there is a linear isometry: $\psi \colon \ell_{\infty}(\kappa) \to \frac{\mathcal{WAP}(\mathbf{G})}{\mathcal{B}(\mathbf{G})}$.

Adding some structure theory (of locally compact groups):

Theorem 5 (Chou, 1990, for $\kappa = \omega$)

If G is an [IN] locally compact group and $\kappa = \kappa(G)$, then there is a linear

A general theorem Some quotients One application: strong Arens irregularity of $L_1(G)$

Linearly isometric copies of $\ell_{\infty}(\kappa)$ in quotients: $\frac{WAP(G)}{B(G)}$ (II)

Theorem 3

Let G be any locally compact group. There is a linear isometry $\psi: \ell_{\infty}(\kappa(\mathsf{Z}(\mathsf{G}))) \rightarrow \frac{\mathcal{WAP}(\mathsf{G})}{\mathcal{B}(\mathsf{G})}.$

dding some functorial properties of $\mathcal{B}(G)$ and $\mathcal{WAP}(G)$ it follows that

Theorem 4 (Chou 1990, for $\kappa = \omega$)

If G is a nilpotent locally compact group and $\kappa = \kappa(G)$, then there is a linear isometry: $\psi : \ell_{\infty}(\kappa) \to \frac{\mathcal{WAP}(G)}{\mathcal{B}(G)}$.

Adding some structure theory (of locally compact groups):

Theorem 5 (Chou, 1990, for $\kappa = \omega$)

If G is an [IN] locally compact group and $\kappa = \kappa(G)$, then there is a linear isometry: $\psi: \ell_{\infty}(\kappa) \to \frac{WAP(G)}{\sqrt{2}(G)}$.

A general theorem Some quotients One application: strong Arens irregularity of $L_1(G)$

Linearly isometric copies of $\ell_{\infty}(\kappa)$ in quotients: $\frac{WAP(G)}{B(G)}$ (II)

Theorem 3

Let G be any locally compact group. There is a linear isometry $\psi: \ell_{\infty}(\kappa(\mathbf{Z}(\mathbf{G}))) \rightarrow \frac{\mathcal{WAP}(\mathbf{G})}{\mathcal{B}(\mathbf{G})}.$

dding some functorial properties of $\mathcal{B}(G)$ and $\mathcal{WAP}(G)$ it follows that

Theorem 4 (Chou 1990, for $\kappa = \omega$)

If G is a nilpotent locally compact group and $\kappa = \kappa(G)$, then there is a linear isometry: $\psi : \ell_{\infty}(\kappa) \to \frac{\mathcal{WAP}(\mathbf{G})}{\mathcal{B}(\mathbf{G})}$.

Adding some structure theory (of locally compact groups):

Theorem 5 (Chou, 1990, for $\kappa = \omega$)

If G is an [IN] locally compact group and $\kappa = \kappa(G)$, then there is a linear isometry: $\psi \colon \ell_{\infty}(\kappa) \to \frac{\mathcal{WAP}(G)}{\mathcal{B}(G)}$.

- (Pym 1965) A Banach algebra A is Arens-regular when $A^* = \mathcal{WAP}(A)$.
- (Granirer 1996) A Banach algebra A is extremely Non-Arens Regular (ENAR) when A*/WAP(A) contains a closed subspace having A* as a continuous linear image
- All C'-algebras are Arens-regular.
- For infinite $G, L_1(G)$ is not regular (Young 1973).
- 1989).

- (Pym 1965) A Banach algebra A is Arens-regular when A* = WAP(A).
 WAP(A) = {λ ∈ A* : a ↦ a ⋅ λ is a weakly compact A → A'}.
 a ⋅ λ ∈ A' is defined as: ⟨b, a ⋅ λ⟩ = ⟨ab, λ⟩ for each b ∈ A (see, e.g., the recent Memoirs by Dales, Lau and Strauss & Dales and Lau).
 - (ENAR) when $A^*/WAP(A)$ contains a closed subspace having A^* as a continuous linear image
- All C*-algebras are Arens-regular.
- For infinite G; $L_1(G)$ is not regular (Young 1973).
- 10.03 (Law and Amerable, then 20(6) is not regular (Law and Wong) 1989).
- Since $\mathcal{MAP}(I_1(G)) = \mathcal{MAP}(G)$ (Uger, 1966), $I_1(G)$ is ENAR when the system the system $\mathcal{L}_{ab}(G)$

- (Pym 1965) A Banach algebra A is Arens-regular when $A^* = \mathcal{WAP}(A)$.
- (Granirer 1996) A Banach algebra A is extremely Non-Arens Regular
 (ENAR) when A*/WAP(A) contains a closed subspace having A* as a continuous linear image
- All *C**-algebras are Arens-regular.
- For infinite G, $L_1(G)$ is not regular (Young 1973).
- If G is infinite and amenable, then A(G) is not regular (Lau and Wong 1989).

- (Pym 1965) A Banach algebra A is Arens-regular when $A^* = \mathcal{WAP}(A)$.
- (Granirer 1996) A Banach algebra A is extremely Non-Arens Regular
 (ENAR) when A*/WAP(A) contains a closed subspace having A* as a continuous linear image (we could replace this by A*/WAP(A) contains an isometric copy of A*).
- All C*-algebras are Arens-regular.
- For infinite G, $L_1(G)$ is not regular (Young 1973).
- If G is infinite and amenable, then A(G) is not regular (Lau and Wong 1989).
- Since WAP(L₁(G)) = WAP(G) (Ülger, 1986), L₁(G) is ENAR when the quotient <u>L_∞(G)</u> contains a copy of L_∞(G).

- (Pym 1965) A Banach algebra A is Arens-regular when $A^* = WAP(A)$.
- (Granirer 1996) A Banach algebra A is extremely Non-Arens Regular
 (ENAR) when A*/WAP(A) contains a closed subspace having A* as a continuous linear image
- All C*-algebras are Arens-regular.
- For infinite G, $L_1(G)$ is not regular (Young 1973).
- If G is infinite and amenable, then A(G) is not regular (Lau and Wong 1989).
- Since WAP(L₁(G)) = WAP(G) (Ülger, 1986), L₁(G) is ENAR when the quotient L_∞(G) / WAP(G) contains a copy of L_∞(G).

- (Pym 1965) A Banach algebra A is Arens-regular when $A^* = WAP(A)$.
- (Granirer 1996) A Banach algebra A is extremely Non-Arens Regular
 (ENAR) when A*/WAP(A) contains a closed subspace having A* as a continuous linear image
- All C*-algebras are Arens-regular.
- For infinite G, $L_1(G)$ is not regular (Young 1973).
- If G is infinite and amenable, then A(G) is not regular (Lau and Wong 1989).
- Since $WAP(L_1(G)) = WAP(G)$ (Ülger, 1986), $L_1(G)$ is ENAR when the quotient $\frac{L_{\infty}(G)}{WAP(G)}$ contains a copy of $L_{\infty}(G)$.

- (Pym 1965) A Banach algebra A is Arens-regular when $A^* = WAP(A)$.
- (Granirer 1996) A Banach algebra A is extremely Non-Arens Regular
 (ENAR) when A*/WAP(A) contains a closed subspace having A* as a continuous linear image
- All C*-algebras are Arens-regular.
- For infinite G, $L_1(G)$ is not regular (Young 1973).
- If G is infinite and amenable, then A(G) is not regular (Lau and Wong 1989).
- Since $WAP(L_1(G)) = WAP(G)$ (Ülger, 1986), $L_1(G)$ is ENAR when the quotient $\frac{L_{\infty}(G)}{WAP(G)}$ contains a copy of $L_{\infty}(G)$.

A general theorem Some quotients One application: strong Arens irregularity of $L_1(G)$

When $\kappa(G) \ge w(G)$: using $\frac{\mathbb{CB}(G)}{\mathcal{LUC}(G)}$

• There is an obvious isometry
$$\frac{\mathcal{CB}(G)}{\mathcal{LUC}(G)} \rightarrow \frac{L_{\infty}(G)}{\mathcal{WAP}(G)}$$
.

- If κ = máx{κ(G), χ(G)}, then there is a linear isometry
 Ψ₁: L_∞(G) → ℓ_∞(κ)
- $T_{0} \in \mathcal{C}(\mathcal{O}) \to \mathcal{C}(\mathcal{O}) = \bigcup_{0 \leq i \leq n} T_{0}$ such that: the T_{0} 's are not $\mathcal{C}(\mathcal{O}) \to \mathcal{C}(\mathcal{O}) = \mathcal{C}(\mathcal{O})$. Such that: the T_{0} 's are not $\mathcal{C}(\mathcal{O}) \to \mathcal{C}(\mathcal{O})$. Such that: the T_{0} 's an approximable $\mathcal{C}(\mathcal{O})$ interpolation sets and T_{0} is an approximable $\mathcal{C}(\mathcal{O})$ interpolation sets.
- Consequence: There is a linear isometry: $\psi: L_{\alpha}(n) \rightarrow \frac{\Delta u(n)}{C(1+C)}$

- First difficulty: What happens with compact groups?
- Obstacle to pursue this approach (Resenthal, 1970): If K is a compact group and $L_{\alpha}(\kappa)$ is isomorphic to a subspace of $L_{\alpha}(\kappa)$, then $\kappa \leq \omega$.

When $\kappa(G) \ge w(G)$: using $\frac{\mathbb{CB}(G)}{\mathcal{LUC}(G)}$

- There is an obvious isometry $\frac{\mathcal{CB}(G)}{\mathcal{LUC}(G)} \rightarrow \frac{L_{\infty}(G)}{\mathcal{WAP}(G)}$.
- Let κ(G)=compact covering number of G and χ(G)=smallest cardinality of a base of nbhds of the identity.
- If $\kappa = \max{\kappa(G), \chi(G)}$, then there is a linear isometry

 $\Psi_1\colon L_\infty(G)\to \ell_\infty(\kappa)$

- Construct $T = \bigcup_{n \in A} T_n$ such that the T_n 's are not C(0) interpolation case and T is an approximable C(0)(G)-interpolation case and T is an approximable C(0)(G)-interpolation case.
- Consequence: There is a linear isometry: $\psi: L_{\alpha}(n) \rightarrow \frac{1}{\alpha}$ in the sequence of the seque

When $\kappa(G) \ge w(G)$: using $\frac{\mathbb{CB}(G)}{\mathcal{LUC}(G)}$

- There is an obvious isometry $\frac{\mathcal{CB}(G)}{\mathcal{LUC}(G)} \rightarrow \frac{\mathcal{L}_{\infty}(G)}{\mathcal{WAP}(G)}$.
- Let κ(G)=compact covering number of G and χ(G)=smallest cardinality of a base of nbhds of the identity.
- If $\kappa = \max{\{\kappa(G), \chi(G)\}}$, then there is a linear isometry

 $\Psi_1 \colon L_\infty(G) \to \ell_\infty(\kappa)$

- Construct T = U_{n < κ(G)} T_n such that the T_n's are not
 LUC(G)-interpolation sets and T is an approximable CB(G)-interpolation set.
- Consequence: There is a linear isometry: ψ: L_i(s) → <u>use</u>
- There is an obvious isometry $\frac{\mathcal{CB}(G)}{\mathcal{LUC}(G)} \rightarrow \frac{\mathcal{L}_{\infty}(G)}{\mathcal{WAP}(G)}$.
- Let κ(G)=compact covering number of G and χ(G)=smallest cardinality of a base of nbhds of the identity.
- If $\kappa = \max{\{\kappa(G), \chi(G)\}}$, then there is a linear isometry

 $\Psi_1 \colon L_\infty(G) \to \ell_\infty(\kappa)$

- Construct T = U_{n < κ(G)} T_n such that the T_n's are not
 LUC(G)-interpolation sets and T is an approximable CB(G)-interpolation set.
- Consequence: There is a linear isometry: ψ: L_i(s) → <u>use</u>

- There is an obvious isometry $\frac{\mathcal{CB}(G)}{\mathcal{LUC}(G)} \rightarrow \frac{\mathcal{L}_{\infty}(G)}{\mathcal{WAP}(G)}$.
- If $\kappa = \max\{\kappa(G), \chi(G)\}$, then there is a linear isometry $\Psi_1: L_{\infty}(G) \to \ell_{\infty}(\kappa)$ Also: $\Psi_1: L_{\infty}(G \times H) \to \ell_{\infty}(\kappa, L_{\infty}(H))$. Back.
- Construct $T = \bigcup_{\eta < \kappa(G)} T_{\eta}$ such that the T_{η} 's are not $\mathcal{LUC}(G)$ -interpolation sets and T is an approximable $\mathcal{CB}(G)$ -interpolation set.
- **Consequence:** There is a linear isometry: $\psi: \ell_{\infty}(\kappa) \to \frac{CO(6)}{2\Pi(\mathcal{C}(6))}$

- First difficulty: What happens with compact groups?
- Obstacle to pursue this approach (Rosenthal, 1970): if K is a compactive group and $L_{ac}(\kappa)$ is isomorphic to a subspace of $L_{ac}(\kappa)$, then $\kappa \leq \omega$

- There is an obvious isometry $\frac{\mathcal{CB}(G)}{\mathcal{LUC}(G)} \rightarrow \frac{L_{\infty}(G)}{\mathcal{WAP}(G)}$.
- If $\kappa = \max\{\kappa(G), \chi(G)\}$, then there is a linear isometry $\Psi_1: L_{\infty}(G) \to \ell_{\infty}(\kappa)$
- Construct T = U_{η<κ(G)} T_η such that the T_η's are not
 LUC(G)-interpolation sets and T is an approximable CB(G)-interpolation set.

This can be done by perturbing a family of uniformly discrete subsets $X_{\eta} = \{x_{\eta,n} : n \in \mathbb{N}\}, \ \eta < \kappa(G) \text{ with a convergent sequence } \{s_n : n \in \mathbb{N}\}.$ Then the sets $T_{\eta} = \bigcup_n \{s_j x_{\eta,n} : 1 \le j \le n\}$ do the job. The sets T_{η} are not $\mathcal{LUC}(G)$ -interpolation sets and T is closed and discrete.

- There is an obvious isometry $\frac{\mathcal{CB}(G)}{\mathcal{LUC}(G)} \rightarrow \frac{L_{\infty}(G)}{\mathcal{WAP}(G)}$.
- If $\kappa = \max\{\kappa(G), \chi(G)\}$, then there is a linear isometry $\Psi_1: L_{\infty}(G) \to \ell_{\infty}(\kappa)$
- Construct T = U_{η<κ(G)} T_η such that the T_η's are not
 LUC(G)-interpolation sets and T is an approximable CB(G)-interpolation set.

This can be done by perturbing a family of uniformly discrete subsets $X_{\eta} = \{x_{\eta,n} : n \in \mathbb{N}\}, \ \eta < \kappa(G)$ with a convergent sequence $\{s_n : n \in \mathbb{N}\}$. Then the sets $T_{\eta} = \bigcup_n \{s_j x_{\eta,n} : 1 \le j \le n\}$ do the job. The sets T_{η} are not $\mathcal{LUC}(G)$ -interpolation sets and T is closed and discrete.

• **Consequence:** There is a linear isometry: $\psi \colon \ell_{\infty}(\kappa) \to \frac{\mathcal{CB}(\mathbf{G})}{\mathcal{CUC}(\mathbf{G})}$.

Proposition 1 (Bouziad and Filali, 2010; Fong and Neufang, 2006)

If $\kappa(\mathbf{G}) \geq \mathbf{w}(\mathbf{G})$, then $L_1(G)$ is ENAR.

- There is an obvious isometry $\frac{\mathcal{CB}(G)}{\mathcal{LUC}(G)} \rightarrow \frac{L_{\infty}(G)}{\mathcal{WAP}(G)}$.
- If κ = máx{κ(G), χ(G)}, then there is a linear isometry
 Ψ₁: L_∞(G) → ℓ_∞(κ)
- Construct $T = \bigcup_{\eta < \kappa(G)} T_{\eta}$ such that the T_{η} 's are not $\mathcal{LUC}(G)$ -interpolation sets and T is an approximable $\mathcal{CB}(G)$ -interpolation set.
- **Consequence:** There is a linear isometry: $\psi: \ell_{\infty}(\kappa) \to \frac{\mathcal{CB}(\mathbf{G})}{\mathcal{CUC}(\mathbf{G})}$.

Proposition 1 (Bouziad and Filali, 2010; Fong and Neufang, 2006)

If $\kappa(\mathbf{G}) \geq w(\mathbf{G})$, then $L_1(G)$ is ENAR.

- First difficulty: What happens with compact groups?
- Obstacle to pursue this approach (Rosenthal, 1970): If K is a compact group and ℓ_∞(κ) is isomorphic to a subspace of L_∞(K), then κ ≤ ω.

- There is an obvious isometry $\frac{\mathcal{CB}(G)}{\mathcal{LUC}(G)} \rightarrow \frac{L_{\infty}(G)}{\mathcal{WAP}(G)}$.
- If κ = máx{κ(G), χ(G)}, then there is a linear isometry
 Ψ₁: L_∞(G) → ℓ_∞(κ)
- Construct $T = \bigcup_{\eta < \kappa(G)} T_{\eta}$ such that the T_{η} 's are not $\mathcal{LUC}(G)$ -interpolation sets and T is an approximable $\mathcal{CB}(G)$ -interpolation set.
- **Consequence:** There is a linear isometry: $\psi: \ell_{\infty}(\kappa) \to \frac{\mathcal{CB}(\mathbf{G})}{\mathcal{CUC}(\mathbf{G})}$.

Proposition 1 (Bouziad and Filali, 2010; Fong and Neufang, 2006) If $\kappa(\mathbf{G}) \geq w(\mathbf{G})$, then $L_1(G)$ is ENAR.

- First difficulty: What happens with compact groups?
- Obstacle to pursue this approach (Rosenthal, 1970): If K is a compact group and ℓ_∞(κ) is isomorphic to a subspace of L_∞(K), then κ ≤ ω.

- There is an obvious isometry $\frac{\mathcal{CB}(G)}{\mathcal{LUC}(G)} \rightarrow \frac{L_{\infty}(G)}{\mathcal{WAP}(G)}$.
- If κ = máx{κ(G), χ(G)}, then there is a linear isometry
 Ψ₁: L_∞(G) → ℓ_∞(κ)
- Construct $T = \bigcup_{\eta < \kappa(G)} T_{\eta}$ such that the T_{η} 's are not $\mathcal{LUC}(G)$ -interpolation sets and T is an approximable $\mathcal{CB}(G)$ -interpolation set.
- **Consequence:** There is a linear isometry: $\psi \colon \ell_{\infty}(\kappa) \to \frac{\mathcal{CB}(\mathbf{G})}{\mathcal{LUC}(\mathbf{G})}$.

Proposition 1 (Bouziad and Filali, 2010; Fong and Neufang, 2006) If $\kappa(\mathbf{G}) \geq \mathbf{w}(\mathbf{G})$, then $L_1(G)$ is ENAR.

- First difficulty: What happens with compact groups?
- Obstacle to pursue this approach (Rosenthal, 1970): If K is a compact group and ℓ_∞(κ) is isomorphic to a subspace of L_∞(K), then κ ≤ ω.

Theorem 6

(Bouziad and Filali, 2010) If G is compact, then there is an isometry $\psi \colon \ell_{\infty} \to \frac{\mathsf{L}_{\infty}(\mathsf{G})}{\mathfrak{CB}(\mathsf{G})} = \frac{\mathfrak{L}_{\infty}(G)}{W\mathcal{AP}(G)}.$

- For the proof, an infinite disjoint collection of open sets of *G* is needed. Uncountable such families do not exist, no matter how large the compact group is.
- (Again, Rosenthal, 1970): If K is a compact group and $\ell_{\infty}(\kappa)$ is isomorphic to a subspace of $L_{\infty}(K)$, then $\kappa \leq \omega$.
- Our strategy:
 - \sim Compact groups look very much like products $[-M_{\ell}$ of metrizable groups.
 - Locally, compact groups have open subgroups of the form R² × R₁, K compacts and Leg [G] to for [g, Leg [2]] when R is open R. G and [G, R] to s

Theorem 6

(Bouziad and Filali, 2010) If G is compact, then there is an isometry $\psi \colon \ell_{\infty} \to \frac{\mathsf{L}_{\infty}(\mathsf{G})}{\mathfrak{CB}(\mathsf{G})} = \frac{\mathfrak{L}_{\infty}(G)}{W\mathcal{AP}(G)}.$

- For the proof, an infinite disjoint collection of open sets of *G* is needed. Uncountable such families do not exist, no matter how large the compact group is.
- (Again, Rosenthal, 1970): If K is a compact group and ℓ_∞(κ) is isomorphic to a subspace of L_∞(K), then κ ≤ ω.
- Our strategy:
 - \sim Compact groups look very much like products $M_{\rm i}$ of metrizable groups.
 - Locally, compact groups have open subgroups of the form R² > R₁, R compacts
 And Lec(G) as for [1: deg(C)] (when R) is open in G and [G = I] is a

Theorem 6

(Bouziad and Filali, 2010) If G is compact, then there is an isometry $\psi \colon \ell_{\infty} \to \frac{\mathsf{L}_{\infty}(\mathsf{G})}{\mathfrak{CB}(\mathsf{G})} = \frac{\mathfrak{L}_{\infty}(G)}{W\mathcal{AP}(G)}.$

- For the proof, an infinite disjoint collection of open sets of *G* is needed. Uncountable such families do not exist, no matter how large the compact group is.
- (Again, Rosenthal, 1970): If K is a compact group and ℓ_∞(κ) is isomorphic to a subspace of L_∞(K), then κ ≤ ω.
- Our strategy:
 - Compact groups look very much like products [] M, of metrizable groups.
 A compact groups look very much like products [] A compact groups look very much like
 - And $|G_{i}(f)| = |G_{i}(f_{i})| = |G_{i}(f_{i})|$ when |f| is seen in $|G_{i}(f_{i})| = 0$.

Theorem 6

(Bouziad and Filali, 2010) If G is compact, then there is an isometry $\psi: \ell_{\infty} \rightarrow \frac{\mathsf{L}_{\infty}(\mathsf{G})}{\mathfrak{CB}(\mathsf{G})} = \frac{L_{\infty}(G)}{\mathcal{WAP}(G)}.$

- For the proof, an infinite disjoint collection of open sets of *G* is needed. Uncountable such families do not exist, no matter how large the compact group is.
- (Again, Rosenthal, 1970): If K is a compact group and $\ell_{\infty}(\kappa)$ is isomorphic to a subspace of $L_{\infty}(K)$, then $\kappa \leq \omega$.
- Our strategy:
 - Compact groups look very much like products M_i of metrizable groups.
 - Locally compact groups have open subgroups of the form ℝⁿ × K, K compact. And L_∞(G) = ℓ_∞ (α, L_∞(H)) when H is open in G and |G : H| = α.
 - When $G := M \times H_c$ even if H is not discrete, $L_{co}(M \times H)$ looks very much like $L_{co}(M \times H)$ looks very much like $L_{co}(H)$ (or $L_{co}(H)$) (or $L_{co}(H)$).

Theorem 6

(Bouziad and Filali, 2010) If G is compact, then there is an isometry $\psi: \ell_{\infty} \rightarrow \frac{\mathbf{L}_{\infty}(\mathbf{G})}{\mathcal{CB}(\mathbf{G})} = \frac{L_{\infty}(G)}{\mathcal{WAP}(G)}.$

- For the proof, an infinite disjoint collection of open sets of *G* is needed. Uncountable such families do not exist, no matter how large the compact group is.
- (Again, Rosenthal, 1970): If K is a compact group and ℓ_∞(κ) is isomorphic to a subspace of L_∞(K), then κ ≤ ω.
- Our strategy:
 - Compact groups look very much like products ∏ M_i of metrizable groups (only topology and measure needs to be cared of).
 - Locally compact groups have open subgroups of the form ℝⁿ × K, K compact.
 And L_∞(G) = ℓ_∞ (α, L_∞(H)) when H is open in G and |G : H| = α.
 - When $G = M \times H$, even if H is not discrete, $L_{\infty}(M \times H)$ looks very much like

Theorem 6

(Bouziad and Filali, 2010) If G is compact, then there is an isometry $\psi: \ell_{\infty} \to \frac{\mathsf{L}_{\infty}(\mathsf{G})}{\mathfrak{CB}(\mathsf{G})} = \frac{\mathcal{L}_{\infty}(G)}{\mathcal{WAP}(G)}.$

- For the proof, an infinite disjoint collection of open sets of *G* is needed. Uncountable such families do not exist, no matter how large the compact group is.
- (Again, Rosenthal, 1970): If K is a compact group and ℓ_∞(κ) is isomorphic to a subspace of L_∞(K), then κ ≤ ω.
- Our strategy:
 - Compact groups look very much like products $\prod M_i$ of metrizable groups.
 - Locally compact groups have open subgroups of the form $\mathbb{R}^n \times K$, K compact. And $L_{\infty}(G) = \ell_{\infty}(\alpha, L_{\infty}(H))$ when H is open in G and $|G:H| = \alpha$.

Theorem 6

(Bouziad and Filali, 2010) If G is compact, then there is an isometry $\psi: \ell_{\infty} \to \frac{\mathsf{L}_{\infty}(\mathsf{G})}{\mathfrak{CB}(\mathsf{G})} = \frac{\mathcal{L}_{\infty}(G)}{\mathcal{WAP}(G)}.$

- For the proof, an infinite disjoint collection of open sets of *G* is needed. Uncountable such families do not exist, no matter how large the compact group is.
- (Again, Rosenthal, 1970): If K is a compact group and ℓ_∞(κ) is isomorphic to a subspace of L_∞(K), then κ ≤ ω.
- Our strategy:
 - Compact groups look very much like products $\prod M_i$ of metrizable groups.
 - Locally compact groups have open subgroups of the form $\mathbb{R}^n \times K$, K compact. And $L_{\infty}(G) = \ell_{\infty}(\alpha, L_{\infty}(H))$ when H is open in G and $|G:H| = \alpha$.

Theorem 6

(Bouziad and Filali, 2010) If G is compact, then there is an isometry $\psi \colon \ell_{\infty} \to \frac{\mathsf{L}_{\infty}(\mathsf{G})}{\mathfrak{CB}(\mathsf{G})} = \frac{\mathfrak{L}_{\infty}(G)}{W\mathcal{AP}(G)}.$

- For the proof, an infinite disjoint collection of open sets of *G* is needed. Uncountable such families do not exist, no matter how large the compact group is.
- (Again, Rosenthal, 1970): If K is a compact group and ℓ_∞(κ) is isomorphic to a subspace of L_∞(K), then κ ≤ ω.
- Our strategy:
 - Compact groups look very much like products M_i of metrizable groups.
 - Locally compact groups have open subgroups of the form $\mathbb{R}^n \times K$, K compact. And $L_{\infty}(G) = \ell_{\infty}(\alpha, L_{\infty}(H))$ when H is open in G and $|G: H| = \alpha$.
 - When $G = M \times H$, even if H is not discrete, $L_{\infty}(M \times H)$ looks very much like $L_{\infty}(M, L_{\infty}(H))$ (or $L_{\infty}(M) \otimes L_{\infty}(H)$).

We can adapt our general theorem to work for algebras such as $L_{\infty}(G)$ and get:

Theorem 7

Let X be a Banach space and $A \subset CB(G, X) \subset L_{\infty}(G, X)$ be a C*-subalgebra of $L_{\infty}(G, X)$. Let in addition $\{U_n : n < \omega\}$ be pairwise disjoint open subsets of G. If G contains a family of sets $\{T_n : n < \omega\}$ such that:

- $U_n T_n \cap U_m T_m = \emptyset \text{ for every } n \neq m < \omega.$
- \bigcirc T_n contains a nontrivial sequence converging to the identity.

Then $L_{\infty}(G,X)/\mathcal{A}$ contains an isometric copy of $\ell_{\infty}(X)$.

Theorem 8

b) the second s second seco

We can adapt our general theorem to work for algebras such as $L_{\infty}(G)$ and get:

Theorem 7

Let X be a Banach space and $A \subset C\mathbb{B}(G, X) \subset L_{\infty}(G, X)$ be a C*-subalgebra of $L_{\infty}(G, X)$. Let in addition $\{U_n : n < \omega\}$ be pairwise disjoint open subsets of G. If G contains a family of sets $\{T_n : n < \omega\}$ such that:

In contains a nontrivial sequence converging to the identity.

Then $L_{\infty}(G,X)/\mathcal{A}$ contains an isometric copy of $\ell_{\infty}(X)$.

r maarmen in 1. maarmen in 2. de daard M. Instantige communité : et sconnende geronge welft M. maarmense menseering menseering menseering

 $L_{\mu}(M \times M) \longrightarrow \frac{L_{\mu}(M \times M)}{C(M \times M)}$

We can adapt our general theorem to work for algebras such as $L_{\infty}(G)$ and get:

Theorem 7

Let X be a Banach space and $A \subset C\mathbb{B}(G, X) \subset L_{\infty}(G, X)$ be a C*-subalgebra of $L_{\infty}(G, X)$. Let in addition $\{U_n : n < \omega\}$ be pairwise disjoint open subsets of G. If G contains a family of sets $\{T_n : n < \omega\}$ such that:

- \bigcirc T_n contains a nontrivial sequence converging to the identity.

Then $L_{\infty}(G,X)/A$ contains an isometric copy of $\ell_{\infty}(X)$.

in the second second

$\xrightarrow{L_{2}(M \times H)} \longrightarrow \xrightarrow{L_{2}(M \times H)} \cdots \longrightarrow$

We can adapt our general theorem to work for algebras such as $L_{\infty}(G)$ and get:

Theorem 7

Let X be a Banach space and $A \subset C\mathbb{B}(G, X) \subset L_{\infty}(G, X)$ be a C*-subalgebra of $L_{\infty}(G, X)$. Let in addition $\{U_n : n < \omega\}$ be pairwise disjoint open subsets of G. If G contains a family of sets $\{T_n : n < \omega\}$ such that:

- \bigcirc T_n contains a nontrivial sequence converging to the identity.

Then $L_{\infty}(G,X)/\mathcal{A}$ contains an isometric copy of $\ell_{\infty}(X)$.

Theorem 8

Let H and M be locally compact a compact groups with M nondiscrete and metrizable. Then there exists a linear isometry

$$\Psi_0: \ell_\infty(\mathsf{L}_\infty(\mathsf{H})) \longrightarrow \frac{\mathsf{L}_\infty(\mathsf{M} \times \mathsf{H})}{\mathfrak{CB}(\mathsf{M} \times \mathsf{H})}$$

We can adapt our general theorem to work for algebras such as $L_{\infty}(G)$ and get:

Theorem 7

Let X be a Banach space and $A \subset C\mathbb{B}(G, X) \subset L_{\infty}(G, X)$ be a C*-subalgebra of $L_{\infty}(G, X)$. Let in addition $\{U_n : n < \omega\}$ be pairwise disjoint open subsets of G. If G contains a family of sets $\{T_n : n < \omega\}$ such that:

- \bigcirc T_n contains a nontrivial sequence converging to the identity.

Then $L_{\infty}(G,X)/\mathcal{A}$ contains an isometric copy of $\ell_{\infty}(X)$.

Let H and M be locally compact: a-compact, groups with M nondiscrete an metrizable. Then there exists a linear isometry

$$\Psi_0 \colon \ell_{\infty}(\mathsf{L}_{\infty}(\mathsf{H})) \longrightarrow \frac{\mathsf{L}_{\infty}(\mathsf{M} \times \mathsf{H})}{\mathfrak{CB}(\mathsf{M} \times \mathsf{H})}$$

Theorem 7

Let X be a Banach space and $A \subset C\mathbb{B}(G, X) \subset L_{\infty}(G, X)$ be a C*-subalgebra of $L_{\infty}(G, X)$. Let in addition $\{U_n : n < \omega\}$ be pairwise disjoint open subsets of G. If G contains a family of sets $\{T_n : n < \omega\}$ such that:

- 2 T_n contains a nontrivial sequence converging to the identity.

Then $L_{\infty}(G,X)/\mathcal{A}$ contains an isometric copy of $\ell_{\infty}(X)$.

Theorem 8

Let H and M be locally compact σ -compact groups with M nondiscrete and metrizable. Then there exists a linear isometry

$$\Psi_0\colon \ell_\infty(\mathsf{L}_\infty(\mathsf{H})) \longrightarrow \frac{\mathcal{L}_\infty(M\times H)}{\mathfrak{CB}(M\times H)}.$$

Theorem 7

Let X be a Banach space and $A \subset C\mathbb{B}(G, X) \subset L_{\infty}(G, X)$ be a C*-subalgebra of $L_{\infty}(G, X)$. Let in addition $\{U_n : n < \omega\}$ be pairwise disjoint open subsets of G. If G contains a family of sets $\{T_n : n < \omega\}$ such that:

- 2 T_n contains a nontrivial sequence converging to the identity.

Then $L_{\infty}(G,X)/\mathcal{A}$ contains an isometric copy of $\ell_{\infty}(X)$.

Theorem 8

Let H and M be locally compact σ -compact groups with M nondiscrete and metrizable. Then there exists a linear isometry

$$\Psi_0\colon \ell_\infty(\mathsf{L}_\infty(\mathsf{H})) \longrightarrow \frac{\mathcal{L}_\infty(M\times H)}{\mathfrak{CB}(M\times H)}.$$

Theorem 7

Let X be a Banach space and $A \subset C\mathbb{B}(G, X) \subset L_{\infty}(G, X)$ be a C*-subalgebra of $L_{\infty}(G, X)$. Let in addition $\{U_n : n < \omega\}$ be pairwise disjoint open subsets of G. If G contains a family of sets $\{T_n : n < \omega\}$ such that:

- **2** T_n contains a nontrivial sequence converging to the identity.

Then $L_{\infty}(G,X)/\mathcal{A}$ contains an isometric copy of $\ell_{\infty}(X)$.

Theorem 8

Let H and M be locally compact σ -compact groups with M nondiscrete and metrizable. Then there exists a linear isometry

$$\Psi_0\colon \ell_\infty(\mathsf{L}_\infty(\mathsf{H})) \longrightarrow \frac{\mathcal{L}_\infty(M\times H)}{\mathfrak{CB}(M\times H)}.$$

Theorem 7

Let X be a Banach space and $A \subset C\mathbb{B}(G, X) \subset L_{\infty}(G, X)$ be a C*-subalgebra of $L_{\infty}(G, X)$. Let in addition $\{U_n : n < \omega\}$ be pairwise disjoint open subsets of G. If G contains a family of sets $\{T_n : n < \omega\}$ such that:

- **2** T_n contains a nontrivial sequence converging to the identity.

Then $L_{\infty}(G,X)/\mathcal{A}$ contains an isometric copy of $\ell_{\infty}(X)$.

Theorem 8

Let H and M be locally compact σ -compact groups with M nondiscrete and metrizable. Then there exists a linear isometry

$$\Psi_0\colon \ell_\infty(\mathsf{L}_\infty(\mathsf{H})) \longrightarrow \frac{\mathcal{L}_\infty(M\times H)}{\mathfrak{CB}(M\times H)}.$$

Towards the compact case: structure

- (From Grekas and Merkourakis, 1998) Let G be a compact group. One can find:
 - Two metrizable groups M_1 and M_2 and two compact groups K_1 and K_2 .
 - Two Haar measure preserving quotient maps:

 $\phi_2: M_2 \times K_2 \to G$ and $\phi_3: G \to M_1 \times K_1.$

• A linear isometry $\Psi_4 \colon L_{\infty}(K_2) \to L_{\infty}(K_1)$.

• The maps ϕ_2 and ϕ_3 induce linear isometries:

$$\frac{L_{\infty}(M_1 \times K_1)}{\mathfrak{CB}(M_1 \times K_1)} \xrightarrow{\Psi_3} \frac{L_{\infty}(G)}{\mathfrak{CB}(G)} \qquad \qquad L_{\infty}(G) \xrightarrow{\Psi_2} L_{\infty}(M_2 \times K_2)$$

$$\frac{L_{\infty}(G)}{\mathcal{CB}(G)} \xleftarrow{\Psi_{3}} \frac{L_{\infty}(M_{1} \times K_{1})}{\mathcal{CB}(M_{1} \times K_{1})} \xleftarrow{\Psi_{0}} \ell_{\infty} (L_{\infty}(K_{1}))$$

$$\psi_{4}^{\uparrow}$$

$$L_{\infty}(G) \xrightarrow{\Psi_{2}} L_{\infty} (M_{2} \times K_{2}) \xrightarrow{\Psi_{1}} \ell_{\infty} (L_{\infty}(K_{2}))$$

Towards the compact case: structure

- (From Grekas and Merkourakis, 1998) Let G be a compact group. One can find:
 - Two metrizable groups M_1 and M_2 and two compact groups K_1 and K_2 .
 - Two Haar measure preserving quotient maps:

 $\phi_2: M_2 \times K_2 \to G$ and $\phi_3: G \to M_1 \times K_1$.

• A linear isometry $\Psi_4 \colon L_\infty(K_2) \to L_\infty(K_1)$.

• The maps ϕ_2 and ϕ_3 induce linear isometries

$$\begin{array}{ccc} L_{\infty}(M_1 \times K_1) & \xrightarrow{\Psi_3} & \frac{L_{\infty}(G)}{\mathbb{CB}(G)} & & L_{\infty}(G) & \xrightarrow{\Psi_2} & L_{\infty}(M_2 \times K_2) \end{array}$$

$$\frac{L_{\infty}(G)}{\mathcal{CB}(G)} \xleftarrow{\Psi_{3}} \frac{L_{\infty}(M_{1} \times K_{1})}{\mathcal{CB}(M_{1} \times K_{1})} \xleftarrow{\Psi_{0}} \ell_{\infty} (L_{\infty}(K_{1}))$$

$$\psi_{4}^{\uparrow}$$

$$L_{\infty}(G) \xrightarrow{\Psi_{2}} L_{\infty} (M_{2} \times K_{2}) \xrightarrow{\Psi_{1}} \ell_{\infty} (L_{\infty}(K_{2}))$$

Towards the compact case: structure

- (From Grekas and Merkourakis, 1998) Let G be a compact group. One can find:
 - Two metrizable groups M_1 and M_2 and two compact groups K_1 and K_2 .
 - Two Haar measure preserving quotient maps:

 $\phi_2 \colon M_2 \times K_2 \to G$ and $\phi_3 \colon G \to M_1 \times K_1$.

• A linear isometry $\Psi_4 \colon L_\infty(K_2) \to L_\infty(K_1)$.

• The maps ϕ_2 and ϕ_3 induce linear isometries

$$\begin{array}{ccc} L_{\infty}(M_{1} \times K_{1}) & \xrightarrow{\Psi_{3}} & \frac{L_{\infty}(G)}{\mathbb{CB}(G)} & & L_{\infty}(G) & \xrightarrow{\Psi_{2}} & L_{\infty}(M_{2} \times K_{2}) \end{array}$$

$$\frac{L_{\infty}(G)}{\mathcal{CB}(G)} \xleftarrow{\Psi_{3}} \frac{L_{\infty}(M_{1} \times K_{1})}{\mathcal{CB}(M_{1} \times K_{1})} \xleftarrow{\Psi_{0}} \ell_{\infty} (L_{\infty}(K_{1}))$$

$$\psi_{4}^{\uparrow}$$

$$L_{\infty}(G) \xrightarrow{\Psi_{2}} L_{\infty} (M_{2} \times K_{2}) \xrightarrow{\Psi_{1}} \ell_{\infty} (L_{\infty}(K_{2}))$$

Towards the compact case: structure

- (From Grekas and Merkourakis, 1998) Let G be a compact group. One can find:
 - Two metrizable groups M_1 and M_2 and two compact groups K_1 and K_2 .
 - Two Haar measure preserving quotient maps:

 $\phi_2 \colon M_2 \times K_2 \to G$ and $\phi_3 \colon G \to M_1 \times K_1$.

• A linear isometry $\Psi_4 \colon L_{\infty}(K_2) \to L_{\infty}(K_1)$.

• The maps ϕ_2 and ϕ_3 induce linear isometries

 $\begin{array}{ccc} L_{\infty}(M_1 \times K_1) & \xrightarrow{\Psi_3} & \underline{L_{\infty}(G)} \\ \mathbb{CB}(M_1 \times K_1) & \xrightarrow{\Psi_3} & \underline{L_{\infty}(G)} \\ \end{array} \qquad \qquad L_{\infty}(G) & \xrightarrow{\Psi_2} & L_{\infty}(M_2 \times K_2) \end{array}$

$$\frac{L_{\infty}(G)}{\mathcal{CB}(G)} \xleftarrow{\Psi_{3}} \frac{L_{\infty}(M_{1} \times K_{1})}{\mathcal{CB}(M_{1} \times K_{1})} \xleftarrow{\Psi_{0}} \ell_{\infty} (L_{\infty}(K_{1}))$$

$$\frac{\Psi_{4}}{\Psi_{4}} \uparrow$$

$$L_{\infty}(G) \xrightarrow{\Psi_{2}} L_{\infty} (M_{2} \times K_{2}) \xrightarrow{\Psi_{1}} \ell_{\infty} (L_{\infty}(K_{2}))$$

Towards the compact case: structure

- (From Grekas and Merkourakis, 1998) Let G be a compact group. One can find:
 - Two metrizable groups M_1 and M_2 and two compact groups K_1 and K_2 .
 - Two Haar measure preserving quotient maps:

$$\phi_2: M_2 \times K_2 \to G$$
 and $\phi_3: G \to M_1 \times K_1$.

• A linear isometry $\Psi_4 \colon L_{\infty}(K_2) \to L_{\infty}(K_1)$.

• The maps ϕ_2 and ϕ_3 induce linear isometries:

$$\frac{L_{\infty}(M_1 \times K_1)}{\mathfrak{CB}(M_1 \times K_1)} \xrightarrow{\Psi_3} \frac{L_{\infty}(G)}{\mathfrak{CB}(G)} \qquad \qquad L_{\infty}(G) \xrightarrow{\Psi_2} L_{\infty}(M_2 \times K_2)$$

$$\frac{L_{\infty}(G)}{\mathcal{CB}(G)} \xleftarrow{\Psi_{3}} \frac{L_{\infty}(M_{1} \times K_{1})}{\mathcal{CB}(M_{1} \times K_{1})} \xleftarrow{\Psi_{0}} \ell_{\infty} (L_{\infty}(K_{1}))$$
$$\overset{\Psi_{4}}{\downarrow}$$
$$L_{\infty}(G) \xrightarrow{\Psi_{2}} L_{\infty} (M_{2} \times K_{2}) \xrightarrow{\Psi_{1}} \ell_{\infty} (L_{\infty}(K_{2}))$$

Towards the compact case: structure

- (From Grekas and Merkourakis, 1998) Let G be a compact group. One can find:
 - Two metrizable groups M_1 and M_2 and two compact groups K_1 and K_2 .
 - Two Haar measure preserving quotient maps:

$$\phi_2 \colon M_2 \times K_2 \to G$$
 and $\phi_3 \colon G \to M_1 \times K_1$.

- A linear isometry $\Psi_4 \colon L_{\infty}(K_2) \to L_{\infty}(K_1)$.
- The maps ϕ_2 and ϕ_3 induce linear isometries:

$$\frac{L_{\infty}(M_1 \times K_1)}{\mathfrak{CB}(M_1 \times K_1)} \xrightarrow{\Psi_3} \frac{L_{\infty}(G)}{\mathfrak{CB}(G)} \qquad \qquad L_{\infty}(G) \xrightarrow{\Psi_2} L_{\infty}(M_2 \times K_2)$$

$$\begin{array}{c} \frac{L_{\infty}(G)}{\mathcal{CB}(G)} \xleftarrow{\Psi_{3}} & \frac{L_{\infty}(M_{1} \times K_{1})}{\mathcal{CB}(M_{1} \times K_{1})} \xleftarrow{\Psi_{0}} & \ell_{\infty}\left(L_{\infty}(K_{1})\right) \\ & & & & \\ & & & \\ & & & & \\ & & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\$$

Towards the compact case: structure

- (From Grekas and Merkourakis, 1998) Let G be a compact group. One can find:
 - Two metrizable groups M_1 and M_2 and two compact groups K_1 and K_2 .
 - Two Haar measure preserving quotient maps:

$$\phi_2 \colon M_2 \times K_2 \to G$$
 and $\phi_3 \colon G \to M_1 \times K_1$.

- A linear isometry $\Psi_4 \colon L_{\infty}(K_2) \to L_{\infty}(K_1)$.
- The maps ϕ_2 and ϕ_3 induce linear isometries:

$$\frac{L_{\infty}(M_1 \times K_1)}{\mathfrak{CB}(M_1 \times K_1)} \xrightarrow{\Psi_3} \frac{L_{\infty}(G)}{\mathfrak{CB}(G)} \qquad \qquad L_{\infty}(G) \xrightarrow{\Psi_2} L_{\infty}(M_2 \times K_2)$$

$$\begin{array}{c} \frac{L_{\infty}(G)}{\mathcal{CB}(G)} \xleftarrow{\Psi_{3}} & \frac{L_{\infty}(M_{1} \times K_{1})}{\mathcal{CB}(M_{1} \times K_{1})} \xleftarrow{\Psi_{0}} & \ell_{\infty}\left(L_{\infty}(K_{1})\right) \\ & & & & \\ & & & \\ & & & & \\ & & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\$$

The general case.

Theorem 9

If G is a compact group, there is a linear isometry $\psi \colon L_{\infty}(G) \to \frac{L_{\infty}(G)}{\mathbb{CB}(G)}$. $L_1(G)$ is therefore ENAR.

- (Davis, Yamabe, 1950's) Every locally compact group contains an open subgroup H that admits a homeomorphism φ: H → ℝⁿ × K preserving the Haar measures.
- If H is an open subgroup of G and |G: H| = α, then there are lineal isometries:

 $\Phi_1\colon L_\infty(G) o \ell_\infty(lpha,L_\infty(H))$ a

 $\Phi_2: \frac{\ell_{\infty}(\alpha, L_{\circ})}{\ell_{\infty}(\alpha, C_{\circ})}$

 $\frac{(\alpha, L_{\infty}(H))}{(\alpha, C\mathcal{B}(H))} \to \frac{L_{\infty}(H)}{C\mathcal{B}(H)}$

The general case.

Theorem 9

If G is a compact group, there is a linear isometry $\psi \colon L_{\infty}(G) \to \frac{L_{\infty}(G)}{\mathbb{CB}(G)}$. $L_1(G)$ is therefore ENAR.

- (Davis, Yamabe, 1950's) Every locally compact group contains an open subgroup H that admits a homeomorphism φ: H → ℝⁿ × K preserving the Haar measures.
- If H is an open subgroup of G and |G: H| = α, then there are lineal isometries:

 $\Phi_1 \colon L_\infty(G) o \ell_\infty(lpha, L_\infty(H))$ at

 $\Phi_2: \frac{\ell_{\infty}(\alpha, L_{\infty}(H))}{\ell_{\infty}(\alpha, \mathcal{CB}(H))}$

The general case.

Theorem 9

If G is a compact group, there is a linear isometry $\psi \colon L_{\infty}(G) \to \frac{L_{\infty}(G)}{\mathbb{CB}(G)}$. $L_1(G)$ is therefore ENAR.

- (Davis, Yamabe, 1950's) Every locally compact group contains an open subgroup H that admits a homeomorphism φ: H → ℝⁿ × K preserving the Haar measures.
- If H is an open subgroup of G and |G: H| = α, then there are lineal isometries:

$$\Phi_1 \colon L_{\infty}(G) \to \ell_{\infty}(\alpha, L_{\infty}(H)) \quad \text{ and } \quad \Phi_2 \colon \frac{\ell_{\infty}(\alpha, L_{\infty}(H))}{\ell_{\infty}(\alpha, \mathbb{CB}(H))} \to \frac{L_{\infty}(G)}{\mathbb{CB}(G)}$$

End

Theorem 10 If G is nondiscrete, then $L_1(G)$ is ENAR.

Theorem 11 (Bouziad and Filali, 2010) If $\kappa(\mathbf{G}) \geq w(\mathbf{G})$, then $L_1(G)$ is ENAR.

Theorem 12

If G is a locally compact group, then $L_1(G)$ is ENAR.

End

Theorem 10

If G is nondiscrete, then $L_1(G)$ is ENAR.

Theorem 11 (Bouziad and Filali, 2010) If $\kappa(\mathbf{G}) \geq w(\mathbf{G})$, then $L_1(G)$ is ENAR.

Theorem 12

If G is a locally compact group, then $L_1(G)$ is ENAR.
End

Theorem 10

If G is nondiscrete, then $L_1(G)$ is ENAR.

Theorem 11 (Bouziad and Filali, 2010) If $\kappa(\mathbf{G}) \geq w(\mathbf{G})$, then $L_1(G)$ is ENAR.

Theorem 12

If G is a locally compact group, then $L_1(G)$ is ENAR.

 Let G be a locally compact group. A subset X ⊂ G is an E-set if it is not relatively compact and for each neighbourhood of the identity U:

$$\bigcap\left\{x^{-1}Ux\colon x\in X\cup X^{-1}\right\},\,$$

is again a neighbourhood of the identity.

Back