Interpolation sets and quotients of function spaces on a locally compact group II

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International Conference on Abstract Harmonic Analysis, Granada
May 20-24, 2013

Based on Joint work with Mahmoud Filali (University of Oulu, Finland).

Linearly isometric copies of $\ell_\infty(\kappa)$ in quotients: a general theorem

Theorem 1

Let $\mathcal{A} \subset \mathcal{B} \subset \mathcal{LUC}(G)$, two admissible subalgebras of $\mathcal{LUC}(G)$. Let $U \in \mathcal{N}(e)$ be compact such that T is right U-uniformly discrete. If G contains a family of sets $\{\mathbf{T}_\eta \colon \eta < \kappa\}$ with:

1. *$T_\eta \cap T_{\eta'} = \emptyset$ for every $\eta < \eta' < \kappa$.*
2. *T_η fails to be an $\mathcal{A}$-interpolation set for any $\eta < \kappa$.*
3. *$\bigcup_{\eta<\kappa} T_\eta$ is an **aproximable** $\mathcal{B}$-interpolation set.*

Then, there is a linear isometry $\Psi \colon \ell_\infty(\kappa) \to \mathcal{B}/\mathcal{A}$.

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- We need a family $\{\mathbf{T}_\eta \colon \eta < \kappa\}$ of pairwise disjoint sets such that:
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  - $T = \bigcup_{\eta<\kappa} T_\eta$ is a uniformly discrete approximable $\mathcal{WAP}_0(G)$-interpolation set.
- Useful data:
  - A C_0-interpolation set must be relatively compact.
  - If T is right U^2-uniformly discrete and both $UTs \cap UT$ and $sUT \cap UT$ are relatively compact for all $s \in G \setminus K$ with K compact (UT is a t-set), then T is an approximable $\mathcal{WAP}_0(G)$-interpolation set.
- Let G be SIN. Then construct T with $|T| = \kappa(G)$ ($\kappa(G)$=*compact covering number* of G) such that UT is a t-set and T is right U^2-uniformly discrete. Any partition $T = \bigcup_{\eta<\kappa} T_\eta$ will do.

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If G is a locally compact SIN group, then there is a linear isometry $\psi\colon \ell_\infty(\kappa(\mathbf{G})) \to \frac{\mathcal{WAP}_0(\mathbf{G})}{\mathbf{C}_0(\mathbf{G})}$.

This and more is valid if G is an E-group with an E-set X, replacing $\kappa(G)$ by $\kappa(X)$.

Linearly isometric copies of $\ell_\infty(\kappa)$ in quotients: $\frac{\mathcal{WAP}(G)}{\mathcal{B}(G)}$

- We need $\{\mathbf{T}_\eta : \eta < \kappa\}$ (pairwise disjoint) such that none of the T_η's is a $\mathcal{B}(G)$-interpolation set and $T = \bigcup_{\eta<\kappa} T_\eta$ is a uniformly discrete approximable $\mathcal{WAP}(G)$-interpolation set.
- (Chou, 1990) $\mathcal{B}(G)$-interpolation sets cannot contain *n*-squares for every n: an n-square is a set $AB \subset G$ with $|A| = |B| = n$ and $|AB| = n^2$.
- If G is discrete, we manufacture a collection of pairwise disjoint sets $\{T_\eta : \eta < |G|\}$, with: $T_\eta = \bigcup_n C_{\eta,n} D_{\eta,n}$, $|C_{\eta,n}| = |D_{\eta,n}| = n$, $|C_{\eta,n} D_{\eta,n}| = n^2$ such that T is a *t-set.*
- This can be extended to groups G with an open normal subgroup H. Then we obtain $|G/H|$-many right H-uniformly discrete subsets with the above properties.

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Linearly isometric copies of $\ell_\infty(\kappa)$ in quotients: $\frac{\mathcal{WAP}(G)}{\mathcal{B}(G)}$ (II)

A t-set T in a central subgroup H of G, is a t-set in G. It follows that:

Theorem 3

Let G be any locally compact group. There is a linear isometry

$$\psi\colon \ell_\infty(\kappa(\mathbf{Z}(\mathbf{G}))) \to \frac{\mathcal{WAP}(\mathbf{G})}{\mathcal{B}(\mathbf{G})}.$$

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Arens regularity

- (Pym 1965) A Banach algebra A is **Arens-regular** when $A^* = \mathcal{WAP}(A)$.
- (Granirer 1996) A Banach algebra A is *extremely Non-Arens Regular* **(ENAR)** when $A^*/\mathcal{WAP}(A)$ contains a closed subspace having A^* as a continuous linear image
- All C^*-algebras are Arens-regular.
- For infinite G, $L_1(G)$ is not regular (Young 1973).
- If G is infinite and amenable, then $A(G)$ is not regular (Lau and Wong 1989).
- Since $\mathcal{WAP}(L_1(G)) = \mathcal{WAP}(G)$ (Ülger, 1986), $L_1(G)$ is ENAR when the quotient $\dfrac{L_\infty(G)}{\mathcal{WAP}(G)}$ contains a copy of $L_\infty(G)$.

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  $\mathcal{WAP}(A) = \{\lambda \in A^* \colon a \mapsto a \cdot \lambda$ is a weakly compact $A \to A'\}$.
  $a \cdot \lambda \in A'$ is defined as: $\langle b, a \cdot \lambda \rangle = \langle ab, \lambda \rangle$ for each $b \in A$ (see, e.g., the recent Memoirs by Dales, Lau and Strauss & Dales and Lau).
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- (Pym 1965) A Banach algebra A is **Arens-regular** when $A^* = \mathcal{WAP}(A)$.
- (Granirer 1996) A Banach algebra A is *extremely Non-Arens Regular* **(ENAR)** when $A^*/\mathcal{WAP}(A)$ contains a closed subspace having A^* as a continuous linear image
- All C^*-algebras are Arens-regular.
- For infinite G, $L_1(G)$ is not regular (Young 1973).
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When $\kappa(G) \geq w(G)$: using $\dfrac{\mathcal{CB}(G)}{\mathcal{LUC}(G)}$

- There is an obvious isometry $\dfrac{\mathcal{CB}(G)}{\mathcal{LUC}(G)} \to \dfrac{L_\infty(G)}{\mathcal{WAP}(G)}$.
- If $\kappa = \text{máx}\{\kappa(G), \chi(G)\}$, then there is a linear isometry $\Psi_1 \colon L_\infty(G) \to \ell_\infty(\kappa)$
- Construct $T = \bigcup_{\eta < \kappa(G)} T_\eta$ such that the T_η's are not $\mathcal{LUC}(G)$-interpolation sets and T is an approximable $\mathcal{CB}(G)$-interpolation set.
- Consequence: There is a linear isometry $\psi \colon \ell_\infty(\kappa) \to \dfrac{\mathcal{CB}(G)}{\mathcal{LUC}(G)}$.
- First difficulty: What happens with compact groups?
- Obstacle to pursue this approach (Rosenthal, 1970): If K is a compact group and $\ell_\infty(\kappa)$ is isomorphic to a subspace of $L_\infty(K)$, then $\kappa \leq \omega$

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  This can be done by perturbing a family of uniformly discrete subsets $X_\eta = \{x_{\eta,n} \colon n \in \mathbb{N}\}$, $\eta < \kappa(G)$ with a convergent sequence $\{s_n \colon n \in \mathbb{N}\}$. Then the sets $T_\eta = \bigcup_n \{s_j x_{\eta,n} \colon 1 \leq j \leq n\}$ do the job. The sets T_η are not $\mathcal{LUC}(G)$-interpolation sets and T is closed and discrete.
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How to proceed

Theorem 6

(Bouziad and Filali, 2010) If G is compact, then there is an isometry
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- For the proof, an infinite disjoint collection of open sets of G is needed. Uncountable such families do not exist, no matter how large the compact group is.
- (Again, Rosenthal, 1970): If K is a compact group and $\ell_\infty(\kappa)$ is isomorphic to a subspace of $L_\infty(K)$, then $\kappa \le \omega$.
- Our strategy:
  - Compact groups look very much like products $\prod_{i\in I} M_i$ of metrizable groups.
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First step: the basic isometry

We can adapt our general theorem to work for algebras such as $L_\infty(G)$ and get:

Theorem 7

Let X be a Banach space and $\mathcal{A} \subset \mathcal{CB}(G,X) \subset L_\infty(G,X)$ be a C^*-subalgebra of $L_\infty(G,X)$. Let in addition $\{U_n\colon n<\omega\}$ be pairwise disjoint open subsets of G. If G contains a family of sets $\{T_n\colon n<\omega\}$ such that:

1. *$U_nT_n \cap U_mT_m = \emptyset$ for every $n \neq m < \omega$.*
2. *T_n contains a nontrivial sequence converging to the identity.*

Then $L_\infty(G,X)/\mathcal{A}$ contains an isometric copy of $\ell_\infty(X)$.

Theorem 8

Let H and M be locally compact σ-compact groups with M nondiscrete and metrizable. Then there exists a linear isometry

$$\Psi_0\colon \ell_\infty(L_\infty(H)) \longrightarrow \frac{L_\infty(M\times H)}{\mathcal{CB}(M\times H)}$$

If $H=\{e\}$, this is Theorem 6.

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Let X be a Banach space and $\mathcal{A} \subset \mathcal{CB}(G,X) \subset L_\infty(G,X)$ be a C^*-subalgebra of $L_\infty(G,X)$. Let in addition $\{U_n \colon n < \omega\}$ be pairwise disjoint open subsets of G. If G contains a family of sets $\{T_n \colon n < \omega\}$ such that:

1. $U_n T_n \cap U_m T_m = \emptyset$ *for every* $n \neq m < \omega$.
2. T_n *contains a nontrivial sequence converging to the identity.*

Then $L_\infty(G,X)/\mathcal{A}$ contains an isometric copy of $\ell_\infty(X)$.

Theorem 8

Let H and M be locally compact σ-compact groups with M nondiscrete and metrizable. Then there exists a linear isometry

$$\Psi_0 \colon \ell_\infty(L_\infty(H)) \longrightarrow \frac{L_\infty(M \times H)}{\mathcal{CB}(M \times H)}$$

If $H = \{e\}$, this is Theorem 6.

First step: the basic isometry

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Towards the compact case: structure

- (From Grekas and Merkourakis, 1998) Let G be a compact group. One can find:
  - Two metrizable groups M_1 and M_2 and two compact groups K_1 and K_2.
  - Two *Haar measure preserving* quotient maps:

    $$\phi_2\colon M_2 \times K_2 \to G \quad \text{and} \quad \phi_3\colon G \to M_1 \times K_1.$$

  - A linear isometry $\Psi_4\colon L_\infty(K_2) \to L_\infty(K_1)$.
- The maps ϕ_2 and ϕ_3 induce linear isometries:

  $$\frac{L_\infty(M_1 \times K_1)}{\mathcal{CB}(M_1 \times K_1)} \xrightarrow{\Psi_3} \frac{L_\infty(G)}{\mathcal{CB}(G)} \qquad L_\infty(G) \xrightarrow{\Psi_2} L_\infty(M_2 \times K_2)$$

- Putting all the isometries together:

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The general case.

Theorem 9

If G is a compact group, there is a linear isometry $\psi\colon L_\infty(G) \to \frac{L_\infty(G)}{\mathcal{CB}(G)}$.
$L_1(G)$ is therefore ENAR.

- (Davis, Yamabe, 1950's) Every locally compact group contains an open subgroup H that admits a homeomorphism $\varphi\colon H \to \mathbb{R}^n \times K$ preserving the Haar measures.
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Theorem 10

If G is nondiscrete, then $\mathbf{L_1(G)}$ is **ENAR**.

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If $\kappa(\mathbf{G}) \geq \mathbf{w}(\mathbf{G})$, then $L_1(G)$ is ENAR.

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If G is a locally compact group, then $\mathbf{L_1(G)}$ *is* **ENAR**.

- Let G be a locally compact group. A subset $X \subset G$ is an E-set if it is not relatively compact and for each neighbourhood of the identity U:

$$\bigcap \left\{ x^{-1} U x \colon x \in X \cup X^{-1} \right\},$$

is again a neighbourhood of the identity.

Back