# SPECTRAL SYNTHESIS
# FOR
# ABSOLUTELY CONTINUOUS
# FUNCTIONS OF HIGHER ORDER

joint work with María Martínez
& Pedro J. Miana
(Still in progress).

• CAUCHY EQ. $\begin{cases} u'(t) = A\,u(t), & t \geq 0. \\ u(0) = x \in D(A). \end{cases}$

$u : (0,\infty) \to D(A) \subseteq X$, $u(t) = e^{tA}x$.

Ill-posed ? $\cancel{e^{tA}}$

• Integrated semigroup, generator $A$ (Arend)

Strongly cts $(T_n(t))_{t>0} \subseteq \mathcal{B}(X)$ s.t.

$$(\lambda - A)^{-1} = \lambda^{\alpha} \int_0^{\infty} e^{-\lambda t}\, T_n(t)\, dt,$$

$\operatorname{Re} \lambda > \omega_A \in \mathbb{R}$

• (Tempered) distribution semig.

$\pi_n : \mathcal{T}_+^{(n)}(|t|^n) \to \mathcal{B}(X)$ bded. homom.

$\shortparallel$

$\mathcal{S}(\mathbb{R})^-$ wrt $\displaystyle\int_{-\infty,0}^{\infty} |f^{(n)}(t)|\, |t|^n\, dt$

Arendt $\simeq$ 80'

$\|T_n(t)\| \leq C|t|^n \left( \begin{smallmatrix} t \in \mathbb{R} \\ \text{or} \\ t > 0 \end{smallmatrix} \right) \longleftrightarrow \pi_n$

Miana 2002 $\mathcal{T}^{(\alpha)}$ SOBOLEV ALGEBRAS

- $\mathcal{L} : \mathcal{T}_+^{(\alpha)}(t^\alpha) \to \mathcal{A}_0^{(\alpha)}(\mathbb{C}^+)$  G-Miana-Royo JAT 2012

Radical version on $[0,1]$  G-Sánchez L. J. Austr. 2012

Nyman's th. in $\mathcal{T}_+^{(\alpha)}(t^\alpha)$  G-Miana-Royo Rev. Compl. 2012

Standard ideals in $\mathcal{A}_0^{(n)}(\mathbb{C}^+)$, $\mathcal{T}_+^{(n)}(t^n)$  G-Wawrzyńczyk Math. Scand., Colloq. 2011

- APPLICATIONS

Caffarelli-Silvestre (Comm. PDE 2007)
$$\begin{cases} u_{yy} + \dfrac{1-2\sigma}{y} u_y = -\Delta u & (y \geq 0) \\ u(\cdot, 0) = v \, ; & \underline{0 < \sigma < 1} \end{cases}$$

$A = -\Delta$,

(*) $$A^\sigma v = c_\sigma \lim_{y \to 0^+} y^{1-2\sigma} u_y(y) \;\&\; u := \text{"Poisson"}[v]$$

Stinga-Torrea (Comm. PDE 2010)  $A = L$ (diff.) on $L^2(\Omega)$
$\equiv \Delta + \|x\|^2$ harmonic oscillator

G-Miana-Stinga (J. Ev. Eq. 2013):

(*) valid for <u>any</u> $C_0$-semigroup

$A = \dfrac{\partial^3}{\partial x^3}$, $\dfrac{\partial^{2n-1}}{\partial x^{2n-1}}$, ..., $A = i\mathcal{L}$ (manifolds)

(Korteweg-de Vries), 
$$u(y) = (-1)^n \int_0^\infty (b^{\sigma,y})^{(n)}(t) \, T_n(t) v \, dt,$$

$b^{\sigma,y} \in \mathcal{T}_+^{(n)}(t^n)$

# STABILITY OF ORBITS

- $\mathfrak{A}$ regular B-algebra, $S$ closed $\subseteq M_{\mathfrak{A}}$

$\underline{a \in \mathfrak{A}}$ of <u>spectral synthesis</u> for $S$

if $\exists\, a_n \in \mathfrak{A}$, $\hat{a}_n = 0$ on


$U_n$ open $\supseteq S$

s.t. $\lim\limits_{n\to\infty} \|a_n - a\|_{\mathfrak{A}} = 0$.

$\underline{S}$ of <u>spectral synthesis</u> if

a <u>is</u> for $S$ $\forall a \in \mathfrak{A} \ni \hat{a}|_S \equiv 0$.

- Katznelson–Tzafrini JFA 1986:

$T \in \mathcal{B}(X) \ni \sup\limits_{n \geq 0} \|T^n\|_{op} < \infty$ &

$f$ analytic in $W(\mathbb{T}) := \left\{ \sum_{-\infty}^{\infty} a_n e^{int} ; \sum_{-\infty}^{\infty} |a_n| < \infty \right\}$

s.t. $f$ of spectral synth. for $S := \sigma(T) \cap \mathbb{T}$

$$\Longrightarrow \lim_{n\to\infty} \|T^n f(T)\|_{op} = 0$$

R.- $f(T) = \sum\limits_{n \geq 0} a_n T^n$.

- Esterle-Strousse-Zouakia JOT 1992 & Vũ P. JFA 1992:

$T(t)_{t\geq 0} \subseteq B(X)$ $C_0$-semigroup s.t. $\sup\limits_{t\geq 0} \|T(t)\|_{op} < \infty$ , $T(t)=e^{tA}$,

$f \in L^1(\mathbb{R}^+)$ spectral synth. in $L^1(\mathbb{R})$ for $S := i\sigma(A) \cap \mathbb{R}$

$$\Longrightarrow \lim_{t\to\infty} \|T(t)\pi_0(f)\|_{op} = 0, \quad \pi_0(f) := \int_0^\infty f(t)T(t)\,dt$$

- Arendt-Batty TAMS 1988 & Lyubich-Vũ Studia 1988

$S := i\sigma(A) \cap \mathbb{R}$ countable

& $\sigma_p(A^*) \cap i\mathbb{R} = \emptyset$

$$\Longrightarrow \lim_{t\to\infty} \|T(t)x\|_X = 0 \quad (x \in X)$$

- ESZ: ¡ $S$ countable $\Rightarrow$ of spectral synth. !

ESZ-V $\Rightarrow$ AB-LV:

$\|T(t)\pi_0(f)x\| \to 0$ $\forall f \in J(S)$: $\hat{f}$ $s$

by $\overline{\pi_0(J(S))X} \underset{\text{Th.}}{=} X$

• Q.- What about $T_n(t)$ & $\mathcal{T}_+^{(n)}(|t|^n)$ ?

THEOREM (G-Martínez-Miana JOT 2013).-

$T_n(t)$ $C_n$-integrated semig. $\left[\!\left[ n!\, t^{-n} T_n(t)x \underset{t\to 0^+}{\to} x \in X \right]\!\right]$

s.t. $\sup_{t>0} t^{-n} \|T_n(t)\|_{op} < \infty$

&

$f \in \mathcal{T}_+^{(n)}(t^n)$ of spectral synth. in $\mathcal{T}^{(n)}(|t|^n)$

for $S = i\,\sigma(A) \cap \mathbb{R}$

$$\Longrightarrow \lim_{t\to\infty} t^{-n} \|T_n(t)\, \pi_n(f)\|_{op} = 0$$

where $\pi_n(f) := (-1)^n \int_0^\infty f^{(n)}(t)\, T_n(t)\, dt$.

R.- Also valid for $n \rightsquigarrow \alpha > 0$

Proof.- Duality in $\mathcal{T}_+^{(n)}(t^n)$ & $\mathcal{T}^{(n)}(|t|^n)$ for convolution ■

• Spectral synthesis in $\mathcal{T}^{(n)}(|t|^n)$

countable closed $S \subseteq \mathcal{M}_{\mathcal{T}^{(n)}(|t|^n)}$ ?

$\shortparallel$

$\mathbb{R}$

- Follow Reiter or Reiter–Stegman's books

DEF.– $C_0^{(n)}(\mathbb{R}; \xi^n)$ :

$\varphi \in C_0(\mathbb{R}) \cap C^{(n)}(\mathbb{R}\setminus\{0\})$ s.t. $\xi \mapsto \xi^k \varphi^{(k)}(\xi) \in C_0(\mathbb{R})$

$$\|\varphi\|_{\infty,(n)} := \sum_{k=0}^{n} \|\xi^k \varphi^{(k)}\|_\infty$$

Banach algebra

PROPOSITION.– $\mathcal{F}: \mathcal{I}^{(n)}(|t|^n) \to C_0^{(n)}(\mathbb{R}; \xi^n)$ is a bounded B-alg. homomorphism

Proof.–

$\forall h \in \mathcal{S}(\mathbb{R})$, $\xi \in \mathbb{R}$

$$\mathcal{F}(\underset{\substack{\uparrow \\ L^1}}{t^n h^{(n)}})(\xi) = (-1)^n \sum_{k=0}^{n} c_k \underset{\substack{\uparrow \\ C_0^n(\xi^n)}}{\xi^k \hat{h}^{(k)}(\xi)}$$

- Notation For $S$ cerrado $\subseteq \mathbb{R}$,

$J(S) := \{ f \in \mathcal{T}^{(n)} : \hat{f} = 0$ on $U_f$ open $\supseteq S \}$

$I(S) := \{ f \in \mathcal{T}^{(n)} : \hat{f}(S) = 0,\ \hat{f}^{(k)}(S\setminus\{0\}) = 0,$ $k = 1, 2, \dots, n \}$

- $\forall a \in \mathbb{R}$, $J_a = J(\{a\})$, $I_a = I(\{a\})$.

$I_0 = \{f : \hat{f}(0) = 0\}$, $I_a = \{f : \hat{f}^{(k)}(a) = 0 \ (k = 0, 1, \dots, n)\}$

**THEOREM** (GMM).-

(i) $\overline{J_0} = I_0$ [$\therefore$ $\{0\}$ of spectral synthesis]

(ii) $\overline{J_a} = I_a$, $\forall a \neq 0$.

**Pf**.- $h_\rho(\cdot) := \rho\, h(\rho\, \cdot)$, $\rho > 0$.

**Lemma**

REITER (i) $g, h \in L^1(\mathbb{R}) \Rightarrow \|h_\rho * g - \hat{h}(0) g\|_{L^1} \xrightarrow[\rho\to\infty]{} 0$.

(ii) $g, h \in \mathcal{S}(\mathbb{R})$, supp $h$ compact,

$$\Big\|h_\rho * g - \sum_{k=0}^{n} \frac{(-i)^k}{k!} \rho^{-k} \hat{h}^{(k)}(0)\, g^{(k)}\Big\|_{L^1} = o(\rho^{-n}),$$

as $\rho \to \infty$.

- Application: $u(x) = e^{-x}$, $x > 0$.

$\hat{e}(z) = \dfrac{1}{1+z}$ $\therefore$ $f := e - e * e \Rightarrow \hat{f}(0) = 0$.

$$\pi_n(f) = -A(1-A)^{-2} \ \therefore\ t^{-n} \int_0^t (t-s)^{n-1} T(s) A (1-A)^{-2} ds \xrightarrow[t\to\infty]{} 0$$

in $\|\cdot\|_{op}$

**THEOREM**.- $S$ countable, closed $\subseteq \mathbb{R}$

$$\Rightarrow \quad \overline{J(S)} = I(S)$$ / Standard

- Stability? ESZ?

**R**.- $\varphi(t)t^{-n} \in L^{\infty}(\mathbb{R})$ & $\operatorname{supp} \hat{\varphi}$ cpt (countable)

$$\Rightarrow \quad \operatorname{supp} \widehat{t^{-n}\varphi} = \text{convexhull}(\operatorname{supp} \hat{\varphi})$$

(non countable)

**THEOREM**.- $\forall a \in \mathbb{R}$,

$$\left.\begin{array}{l} i\sigma(A) \cap \mathbb{R} = \{a\} \\ \sigma_p(A^*) \cap i\mathbb{R} = \emptyset \end{array}\right\} \Rightarrow \lim_{t\to\infty} t^{-n} \|T_n(t)x\|_X = 0$$

$\forall x \in X$.

- Césaro means

$$\frac{1}{t^n}\int_0^t (t-s)^{n-1} T_0(t)x\, dt \underset{t\to\infty}{\longrightarrow} 0$$

(for $T_0(t) = e^{tA}$ non-unif$^{m'ly}$ bded. on $(0,\infty)$, $\sigma(A) \cap i\mathbb{R} = \{a\}$).

**LEMMA**.- $\varphi$ as above, $\operatorname{supp}\hat{\varphi}$ compact

$$\Longrightarrow \operatorname{supp}\widehat{t^{-n}\varphi} = \text{convex hull}(\operatorname{supp}\hat{\varphi})$$

**THEOREM**.-

$$S := i\sigma(A)\cap\mathbb{R} = \{a\}\ ,\ a\in\mathbb{R}$$

$$\&\ \sigma_p(A^*)\cap i\mathbb{R} = \emptyset$$

$$\Longrightarrow \lim_{t\to\infty} t^{-n}\|T_n(t)x\| = 0 \quad \forall x\in X$$

Pf.- $\operatorname{supp}\hat{\varphi} = \{a\} \longrightarrow \operatorname{supp}\widehat{t^{-1}\varphi} = \{a\}$ ∎

**COROLLARY**.- Césaro means of $C_0$-semigroups.

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**REMARK**.- $n=1$, $S=\{0,a\}$


Supp. $\ell, x$ s.t. $\langle T_1(t)x,\ell\rangle = \varphi(t) := c\,(e^{ita}-1)$

Then $\hat{\varphi} = c(\delta_a - \delta_0)$ so $\operatorname{supp}\hat{\varphi} = \{0,a\}$

BUT $\widehat{t^{-1}\varphi} = c\,\widehat{t^{-1}(e^{ita}-1)} = c\,\chi_{(0,a)}$ so

$$\operatorname{supp}\widehat{t^{-1}\varphi} = [0,a].$$

Notice

$$n!\,c\langle x,\ell\rangle\,\frac{e^{iat}-1}{t} = n!\,t^{-1}\langle T_1(t)x,\ell\rangle \xrightarrow[t\to 0^+]{} \langle x,\ell\rangle$$

$\overset{?}{\shortparallel}$

$0$