Wavelet coorbit spaces over general dilation groups

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Outline

1. Introduction: Nice wavelets in dimension one

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Overview

Wavelet orthonormal bases

Definition

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$$(\psi_{j,k})_{j,k\in\mathbb{Z}} \subset \mathrm{L}^2(\mathbb{R}) \ , \psi_{j,k} = 2^{j/2}\psi(2^j x - k) \ , \psi \text{ fixed}$$

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- For sufficiently nice wavelets ψ, the wavelet expansion

$$f = \sum_{j,k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle \psi_{j,k}$$

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- There exist arbitrarily nice compactly supported wavelets. (Daubechies)

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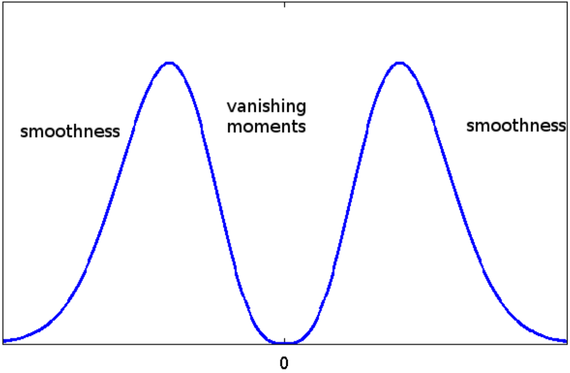
(Note: Frequency-side localization is understood away from zero.)

Cartoon: Fourier side decay of wavelets



Plot of $|\widehat{\psi}|$.

Vanishing moments and wavelet coefficient decay

Assumptions on nice wavelet ψ guarantee fast decay of $|\langle \psi, \psi_{j,k} \rangle|$:

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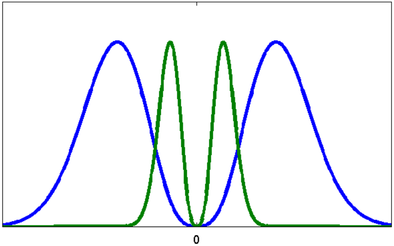
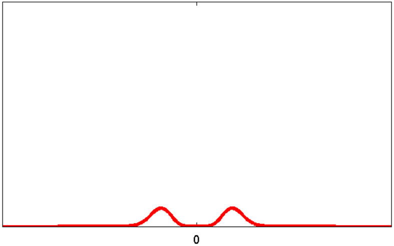


Plot of $\widehat{\psi}$ and $\widehat{\psi}(3\cdot)$



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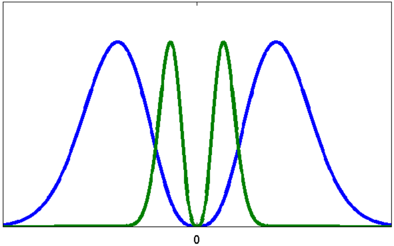

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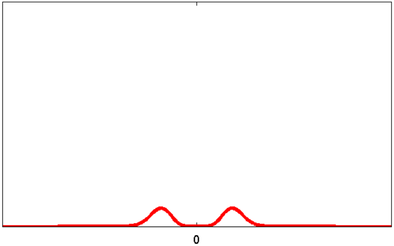

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$\Rightarrow$ vanishing moments, smoothness govern decay of overlap, as $j \to \pm\infty$

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Overview

1. Introduction: Nice wavelets in dimension one
2. Square-integrability over general dilation groups
3. Outline of coorbit theory: Analyzing vectors and frame atoms
4. Wavelet coorbit spaces over general dilation groups
5. Vanishing moment conditions and coorbit spaces

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- Dual action of H on $\mathbb{R}^d$, defined by

$$H \times \mathbb{R}^d \ni (h,\xi) \mapsto h^T \xi \ .$$

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Theorem (HF, 2010)

The quasiregular representation π is a discrete series representation iff there exists a single open orbit $\mathcal{O}$ under the dual action, with the additional property that, for some (equivalently: any) $\xi_0 \in \mathcal{O}$, the associated dual stabilizer

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- Key idea of coorbit theory: Use properties of the reproducing kernel $\mathcal{W}_\psi \psi$, and the fact that Y is a Banach convolution module over the algebra $\mathrm{L}^1_{v_0}(G)$.

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- Moreover, for all $f \in CoY$, there exist coefficients $(c_\gamma)_{\gamma\in\Gamma}$ such that

$$f = \sum_{\gamma\in\Gamma} c_\gamma \pi(\gamma)\psi \quad , \ \|f\|_{CoY} \asymp \|(c_\gamma)_{\gamma\in\Gamma}\|_{Y_d}$$

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Overview

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- Note: There is a control weight v_0 for $L_v^{p,q}(G)$ of the same type as v.

Wavelet coorbit spaces

Theorem (Kaniuth/Taylor '96,HF '12)

The quasiregular representation is v_0-integrable: If $\psi \in \mathcal{F}^{-1} C_c^\infty(\mathcal{O})$, then $\mathcal{W}_\psi \psi \in L^1_{v_0}(G)$.

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Theorem (HF, '12)

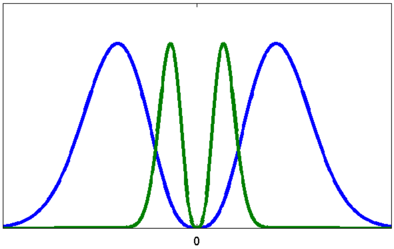
For all control weights v_0 satisfying $v_0(x,h) \le (1+|x|)^t w_0(h)$, with suitable $t > 0$ and continuous weights w_0 on H, we have

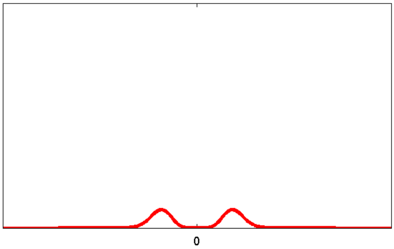
$$\mathcal{F}^{-1} C_c^\infty(\mathcal{O}) \subset \mathcal{B}_{v_0} \ .$$

Overview

Chief problem: Measuring and controlling overlap

Recall: Wavelet coefficient decay is related to overlap on the Fourier transform side





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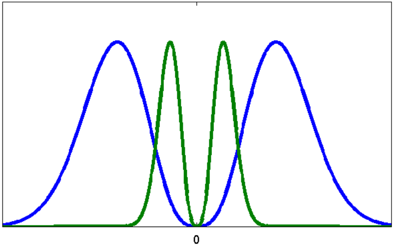

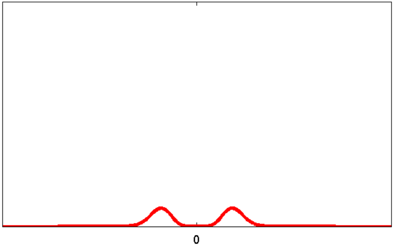

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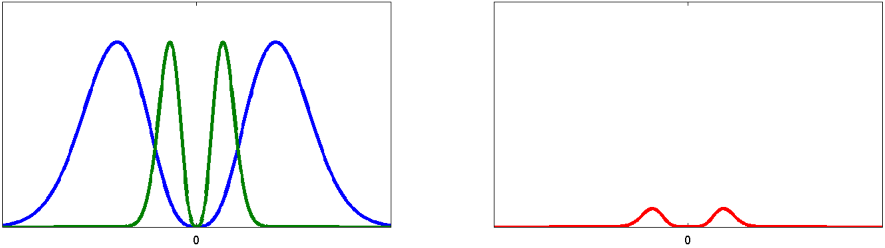

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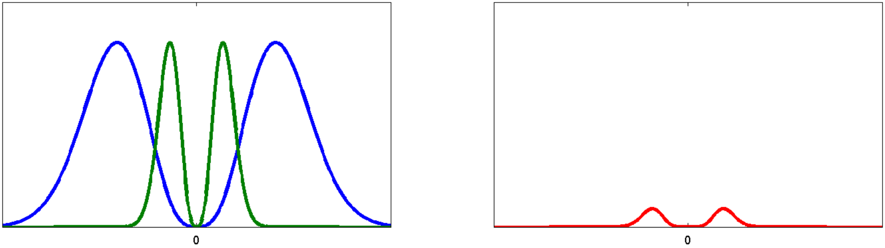

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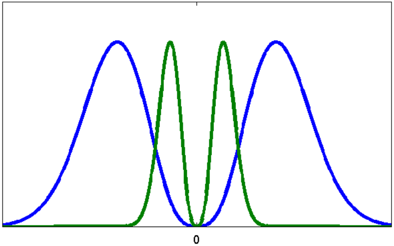

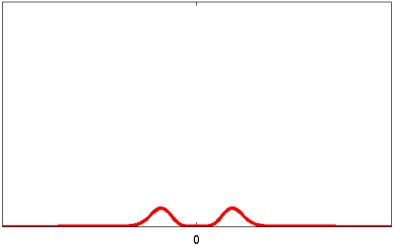

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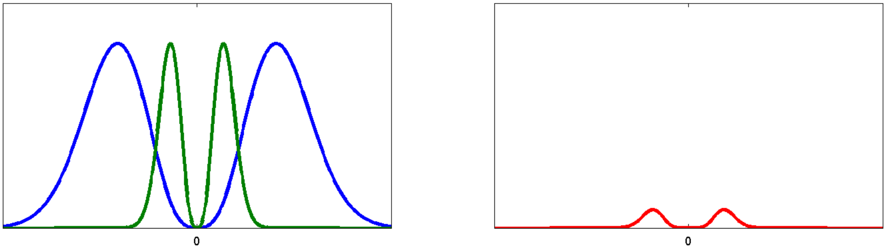

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Controlling overlap: Fourier envelopes

Definition

$| \cdot | : \mathbb{R}^d \to \mathbb{R}_0^+$ denotes the euclidean norm. For $r, m \geq 0$ and $f : \mathbb{R}^d \to \mathbb{C}$, let

$$|f|_{r,m} = \sup_{x \in \mathbb{R}^d, |\alpha| \leq r} (1 + |x|)^m |\partial^\alpha f(x)| \ .$$

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Definition (Fourier envelope function)

Let $\mathcal{O} \subset \mathbb{R}^d$ denote the dual orbit. Given $\xi \in \mathcal{O}$, let $\mathrm{dist}(\xi, \mathcal{O}^c)$ denote the euclidean distance of ξ to $\mathcal{O}^c$. Let

$$A(\xi) = \min \left(\frac{\mathrm{dist}(\xi, \mathcal{O}^c)}{1 + \sqrt{|\xi|^2 - \mathrm{dist}(\xi, \mathcal{O}^c)^2}}, \frac{1}{1 + |\xi|} \right) \ .$$

Vanishing moment conditions and wavelet coefficient decay

Definition

Let $r \in \mathbb{N}$ be given. $f \in \mathrm{L}^1(\mathbb{R}^d)$ has vanishing moments in $\mathcal{O}^c$ of order r if all distributional derivatives $\partial^\alpha \widehat{f}$ with $|\alpha| < r$ are continuous functions, identically vanishing on $\mathcal{O}^c$.

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Lemma

Let α be a multiindex with $|\alpha| < r$. Assume that $f, \psi \in \mathrm{L}^1(\mathbb{R}^d)$ have vanishing moments of order r in $\mathcal{O}^c$, and fulfill $|\widehat{f}|_{r,r-|\alpha|} < \infty, |\widehat{\psi}|_{r,r-|\alpha|} < \infty$. Then there exists a constant $C > 0$, independent of f and ψ, such that

$$\begin{aligned} & |\partial^\alpha(\widehat{f} \cdot D_h \widehat{\psi})(\xi)| \\ \leq \quad & C |\widehat{f}|_{r,r-|\alpha|} |\widehat{\psi}|_{r,r-|\alpha|} |\det(h)|^{1/2} (1 + \|h\|)^{|\alpha|} A(\xi)^{r-|\alpha|} A(h^T \xi)^{r-|\alpha|} \end{aligned}$$

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Lemma (Wavelet coefficient decay)

Let $0 < m < r$, and let $\psi \in \mathrm{L}^1(\mathbb{R}^d)$ denote a function with vanishing moments of order r in $\mathcal{O}^c$ and $|\widehat{\psi}|_{r,r} < \infty$. Then

$$|\mathcal{W}_\psi \psi(x,h)| \prec |\widehat{\psi}|^2_{r,r}(1+|x|)^{-m} |\det(h)|^{1/2}(1+\|h\|_\infty)^m \Phi_{r-m}(h) \ .$$

Main result: Vanishing moment criteria for atoms

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Theorem (HF '13)

Assume that $\mathcal{O}$ is strongly temperately (s, w_0)-embedded with index ℓ. Then any function $\psi \in \mathrm{L}^1(\mathbb{R}^d) \cap C^{\ell+d+1}(\mathbb{R}^d)$ with vanishing moments in $\mathcal{O}^c$ of order $t > \ell + s + d$ and $|\widehat{\psi}|_{t,t} < \infty$ is contained in $\mathcal{B}_{v_0}$, for any weight v_0 satisfying $v_0(x, h) \leq (1 + |x|)^s w_0(h)$.
There exists $\psi \in C_c^\infty(\mathbb{R}^d)$ satisfying this condition.

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$$\|\mathcal{W}_\psi \psi\|_{W^R(C^0, \mathrm{L}^1_{v_0})} \preceq \int_H \int_{\mathbb{R}^d} \left(\sup_{y \in V} (1 + |x + hy|)^{-k} \right) (1 + |x|)^s dx \left(\sup_{g \in W} \Psi(hg) \right) w_0(h) \frac{dh}{|\det(h)|}$$

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- Using that V is the unit ball, we find

$$\sup_{y \in V} (1 + |x + hy|)^{-k} \le (1 + \max(0, |x| - \|h\|_\infty))^{-k} \ .$$

Sketch of proof, cont'd

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The last expression is finite by assumption.

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- For temperately embedded dual orbits: Besov-type coorbit spaces embed naturally into quotient spaces of tempered distributions; compare homogeneous Besov spaces as spaces of tempered distributions mod polynomials.

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References

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