INTERPOLATION SETS AND FUNCTION SPACES ON A LOCALLY COMPACT GROUP

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(JOINT WORK WITH JORGE GALINDO)

1. Function spaces

G is a locally compact group.

$\ell_\infty(G)$: bounded, scalar-valued fncts on G.

$\mathcal{CB}(G)$: continuous bounded scalar-valued fncts on G.

$C_0(G)$: continuous functions vanishing at infinity on G.

$\mathcal{LUC}(G)$: right uniformly continuous bounded fncts on G.
$f \in \mathcal{LUC}(G)$ when

$$\forall \epsilon > 0 \ \exists U \in \mathcal{N}(e) \ \text{ s.t. } st^{-1} \in U \Rightarrow |f(s) - f(t)| < \epsilon.$$

iff

$$s \mapsto f_s : G \to \mathcal{CB}(G) \ \text{ is continuous,}$$

where $f_s(t) = f(st)$.

$\mathcal{RUC}(G)$: left uniformly continuous.

$\mathcal{UC}(G) = \mathcal{LUC}(G) \cap \mathcal{RUC}(G)$.

$\mathcal{WAP}(G)$: weakly almost periodic functions.
$f \in \mathcal{WAP}(G)$ if $\{f_s : s \in G\}$ is a rel. weakly compact.

If μ is the unique invariant mean on $\mathcal{WAP}(G)$, put

$$\mathcal{WAP}_0(G) = \{f \in \mathcal{WAP}(G) : \mu(|f|) = 0\}.$$

Date: May 20, 2013.

$\mathcal{AP}(G)$: almost periodic functions on G.
$f \in \mathcal{AP}(G)$ if $\{f_s : s \in G\}$ is a rel. norm compact subset.

The *Fourier-Stieltjes algebra* $B(G)$ is the space of coefficients of unitary representations of G. Equivalently, $B(G)$ is the linear span of the set of all continuous positive definite functions on G.
The Eberlein algebra $\mathcal{B}(G) = \overline{B(G)}^{\|\cdot\|_\infty}$.

$$\begin{aligned} C_0(G) \oplus \mathcal{AP}(G) &\subseteq \mathcal{B}(G) \subseteq \mathcal{WAP}(G) = \mathcal{AP}(G) \oplus \mathcal{WAP}_0(G) \\ &\subseteq \mathcal{LUC}(G) \cap \mathcal{RUC}(G) \subseteq \mathcal{LUC}(G) \subseteq \mathcal{CB}(G) \\ &\subseteq L^\infty(G). \end{aligned}$$

When G is finite, the diagram is trivial. When G is infinite and compact, the diagram reduces to $\mathcal{CB}(G) \subseteq L^\infty(G)$.

2. A brief historical review:

κ is the *compact covering number* of G.

Comparing $L^\infty(G)$ with its subspaces.

Civin and Yood (1961): $L^\infty(G)/\mathcal{CB}(G)$ is infinite-dimensional for any non-discrete lca G.
The radical of the Banach algebra $L^\infty(G)^*$ (with one of the Arens products) is also infinite-dimensional.

Gulick (1966): The quotient is not separable.

Granirer (1973): for any non-discrete locally compact group.

Young (1973): for any infinite lc group G, $L^\infty(G) \neq \mathcal{WAP}(G)$, proving the non-Arens regularity of $L^1(G)$.

Bouziad-Filali (2011): $\mathcal{LUC}(G)/\mathcal{WAP}(G)$ contains a linear isometric copy of $\ell_\infty(\kappa(G))$.
A fortiori, $L^\infty(G)/\mathcal{WAP}(G)$ contains the same copy.
$L^1(G)$ is extremely non-Arens regular (enAr) in the sense of Granirer, whenever κ is larger than or equal to $w(G)$, the minimal cardinal of a basis of neighbourhoods at the identity.

$L^\infty(G)/\mathcal{CB}(G)$ always contains a copy of ℓ_∞, so $L^1(G)$ is enAr for compact metrizable groups.

Filali-Galindo (2012): For any compact group G, $L^\infty(G)/\mathcal{CB}(G)$ contains a copy of $L^\infty(G)$.

$L^1(G)$ is enAr for any infinite locally compact group.

Comparing $\mathcal{CB}(G)$ with its subspaces.

Comfort and Ross (1966): $\mathcal{CB}(G) = \mathcal{AP}(G)$ for a topo. group iff G is pseudocompact.

Burckel (1970): $\mathcal{CB}(G) = \mathcal{WAP}(G)$ for lc groups iff G is compact.

Baker and Butcher (1976): $\mathcal{CB}(G) = \mathcal{LUC}(G)$ for lc group iff G is either discrete or compact.

Filali-Vedenjuoksu (2010): If G is a topological group which is not a P-group, then $\mathcal{CB}(G) = \mathcal{LUC}(G)$ if and only if G is pseudocompact.

Dzinotyiweyi (1982): $\mathcal{CB}(G)/\mathcal{LUC}(G)$ is non-separable if G is a non-compact, non-discrete, lc group.

Bouziad-Filali (2010 and 2012): $\mathcal{CB}(G)/\mathcal{LUC}(G))$ contains a linear isometric copy of ℓ_∞ whenever G is a non-precompact, non-P-group, topo. group.

For non-discrete, P-groups, the quotient $\mathcal{CB}(G)/\mathcal{LUC}(G)$ may be trivial as it is the case when G is a Lindelöf P-group but may also contain a linear isometric copy of ℓ_∞ for some other P-groups.

$\mathcal{CB}(G)/\mathcal{LUC}(G)$ contains a linear isometric copy of ℓ_∞ whenever G is a non-SIN topo. group.

Comparing $\mathcal{LUC}(G)$ with $\mathcal{WAP}(G)$.

Granirer (1972): $\mathcal{LUC}(G) = \mathcal{WAP}(G)$ if and only if G is compact.

Lau and Pym (1995): Granirer's thm from their main theorem on the topological centre of $G^{\mathcal{LUC}}$ being G.

Lau and Ülger (1996): Granirer's thm from the topological centre of $L^1(G)^{**}$ being $L^1(G)$.

Granirer (1972): If G is non-compact and amenable, then $\mathcal{LUC}(G)/\mathcal{WAP}(G)$ contains a linear isometric copy of ℓ_∞.

This result was extended by Chou (1975) to E-groups then by Dzinotyiweyi (1982) to all non-compact lc groups, and generalized by Bouziad and Filali (2011) to all non-precompact topological groups.

Bouziad-Filali (2011): There is a copy of $\ell_\infty(\kappa)$ in $\mathcal{LUC}(G)/\mathcal{WAP}(G)$ when G is a non-compact lc group.

COMPARING $\mathcal{WAP}(G)$ WITH ITS SUBSPACES.

Chou 1990, Veech 1979, Ruppert 1984: $\mathcal{WAP}(G) = \mathcal{B}(G) = \mathcal{WAP}(G) = \mathcal{AP}(G) \oplus C_0(G)$ when G is minimally weakly almost periodic group.

Rudin (1959): $\mathcal{B}(G) \subsetneq \mathcal{WAP}(G)$ if G is a lca group and contains a closed discrete subgroup which is not of bounded order.

Ramirez (1968): Rudin's result to any non-compact, lca group.

Chou (1990): $\mathcal{WAP}(G)/\mathcal{B}(G)$ contains a linear isometric copy of ℓ_∞ when G is a non-compact, IN-group or nilpotent group.

Burckel (1970): $C_0(G) \subsetneq \mathcal{WAP}_0(G)$ when G is a non-compact, lca group.

Chou (1975): $\mathcal{WAP}_0(G)/C_0(G)$ contains a linear isometric copy of ℓ_∞ when G is an E-group.

3. Interpolation sets

-Interpolation sets help to construct functions on infinite discrete or, more generally, locally compact groups G.

-They have the crucial property that any function defined on them extends to the whole group as a function of the required type.

- Almost periodic functions: I_0-sets, introduced by Hartman and Ryll-Nardzewsky [1964].
  Galindo, Graham, Hare, Hernández, and Körner, [1999-2008].

- Fourier-Stieltjes functions: Sidon sets when G is discrete Abelian and weak Sidon sets in general. Lopez and Ross [1975] and Picardello [1973].
  A Sidon set T is in fact uniformly approximable (Drury [1970]): in addition of being interpolation set, its characteristic function $1_T \in \mathcal{B}(G)$. This is the key in the proof of Drury's union theorem: the union of two Sidon sets remains Sidon.

- Weakly almost periodic functions on infinite discrete groups: Ruppert [1985] and Chou [1990] considered interpolation sets T with the extra condition that 1_T is also weakly almost periodic. Translation-finite sets by Ruppert and R_W-sets by Chou.

- Right uniformly continuous functions: right uniformly discrete sets are used.
- Weakly almost periodic on locally compact E-groups: Recent work with Jorge Galindo. Interpolation sets with an additional condition analogue to the one above. Translation compact-sets.

STRATEGY

Appro. interpolation sets for $\mathcal{A}_2$ that are not interpolation sets for $\mathcal{A}_1$ give a copy of $\ell_\infty(\kappa)$ in $\mathcal{A}_2/\mathcal{A}_1$.

Definition 3.1. *Let G be a topological group and $\mathcal{A} \subseteq \ell_\infty(G)$. A subset $T \subseteq G$ is said to be:*

(i) *an* $\mathcal{A}$-interpolation set *if every bounded function $f\colon T \to \mathbb{C}$ can be extended to a function $\tilde{f}\colon G \to \mathbb{C}$ such that $\tilde{f} \in \mathcal{A}$.*

(ii) *an* approximable $\mathcal{A}$-interpolation set *if it is an $\mathcal{A}$-interpolation set and for every $U \in \mathcal{N}(e)$, there are $V_1, V_2 \in \mathcal{N}(e)$ with $\overline{V_1} \subseteq V_2 \subseteq U$ such that, for each $T_1 \subseteq T$ there is $h \in \mathcal{A}$ with $h(V_1T_1) = \{1\}$ and $h(G \setminus (V_2T_1)) = \{0\}$.*

Definition 3.2. *Let G be a topological group, T be a subset of G and U be a neighbourhood of the identity. We say that T is* right U-uniformly discrete *if*

$$Us \cap Us' = \varnothing \quad \textit{for every} \quad s \neq s' \in T.$$

Definition 3.3. *Let G be a non-compact topological group. We say that a subset S of G is*

(i) *right translation-compact if every non-relatively compact subset $L \subseteq G$ contains a finite subset F such that*

$$\bigcap \{b^{-1}S \colon b \in F\}$$

is relatively compact,

(ii) *a* right t-set *if there exists a compact subset K of G containing e such that $gS \cap S$ is relatively compact for every $g \notin K$.*

We also need to establish the range of locally compact groups to which our methods apply, these are those locally compact groups for which the existence of a good supply of $\mathcal{WAP}$-functions is guaranteed.
Recall that G is an *$IN-$group* if it has an invariant neighbourhood of e. We recall also that G is an *E-group* if it contains a non-relatively compact set X such that for each neighbourhood U of e, the set

$$\bigcap \{x^{-1}Ux : x \in X \cup X^{-1}\}$$

is again a neighbourhood of e. The set X is called an *E-set.*

(F+Galindo 2013) Let G be a topological group and let $T \subset G$.

(i) If the underlying topological space of G is normal, then all discrete closed subsets of G are approximable $\mathcal{CB}(G)$-interpolation sets.

(ii) If T is right uniformly discrete (resp. left-uniformly discrete), then T is an approximable $\mathcal{LUC}$-interpolation set (resp. $\mathcal{RUC}$-interpolation set).

(iii) If G is assumed to be metrizable, then every $\mathcal{LUC}$-interpolation set is right uniformly discrete.

(iv) If G is an E-group and T is an E-set in G which is right (or left) uniformly discrete with respect to U^2 for some neighbourhood U of the identity such that UT is translation-compact, then T is an approximable $\mathcal{WAP}_0(G)$-interpolation set.

(v) If G is a metrizable E-group, $T \subset G$ is an approx. $\mathcal{WAP}(G)$-interpolation set if and only if UT is translation-compact for some compact neighbourhood U of the identity such that T is right (or left) uniformly discrete with respect to U^2.

4. Interpolation and quotient

Theorem 4.1 ((Chou, 1982))**.** *Let G be a discrete group. A subset $T \subseteq G$ fails to be a $B(G)$-interpolation set if and only if there is a bounded function $f \in \ell_\infty(G)$, with $\|f\|_\infty = 1$ such that*

$$f(G \backslash T) = \{0\} \quad \text{and} \quad \|\phi - f\|_T \geq 1 \ \text{for all}\ \phi \in B(G).$$

Lemma 4.2. *Let $\mathcal{A}$ be a C^*-subalgebra of $\mathcal{CB}(G)$ with $1 \in \mathcal{A}$ and $T \subseteq G$.*

(i) *T is an $\mathcal{A}$-interpolation set if and only if $\overline{T}^{\mathcal{A}}$ is homeomorphic to βT, where the homeomorphism leaves the points of T fixed.*

(ii) *T is an $\mathcal{A}$-interpolation set if and only if for every pair of subsets $T_1, T_2 \subset T$, $T_1 \cap T_2 = \varnothing$ implies $\overline{T_1}^{\mathcal{A}} \cap \overline{T_2}^{\mathcal{A}} = \varnothing$.*

(iii) *If T is an $\mathcal{A}$-interpolation set and $f\colon T \to \mathbb{C}$ is a bounded function, then f has an extension $f^{\mathcal{A}} \in \mathcal{A}$ with $\|f^{\mathcal{A}}\|_\infty = \|f\|_T$.*

(iv) *If T is an approximable $\mathcal{A}$-interpolation set, then for every bounded function $h\colon T \to \mathbb{C}$ and every ngh. U of e there is $f \in \mathcal{A}$ such that*

$$f\restriction_T = h, \quad f(G \setminus UT) = \{0\} \quad \text{and} \quad \|f\|_\infty = \|h\|_T.$$

Proof. (iii). Take $f\colon T \to \mathbb{C}$, extend it continuously by (i) to $\overline{T}^{\mathcal{A}}$, then by Tietze's extension theorem, to $G^{\mathcal{A}}$, then restrict to G.

(iv). Using (iii), we find $f_1 \in \mathcal{A}$ with $f_1 \restriction_T = h$ and $\|f_1\|_\infty = \|h\|_T$. Pick two nghs V_1, V_2 with $\overline{V_1} \subseteq V_2 \subseteq U$ and $f_2 \in \mathcal{A}$ such that

$$f_2(V_1T) = \{1\} \quad \text{and} \quad f_2(G \setminus V_2T) = \{0\}.$$

We may assume (taking the minimum of f_2 and the cte function 1) that $\|f_2\|_\infty = 1$. Then $f_1 \cdot f_2 = h$ on T and vanishes off V_2T. □

Lemma 4.3. *Let G be a topo. group, $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \mathcal{CB}(G)$ be C^*-subalgebras with $1 \in \mathcal{A}_1$, and $(T_\eta)_{\eta<\kappa}$ a family of disjoint subsets of G such that*

(i) *each T_η fails to be an $\mathcal{A}_1$-interpolation, but*

(ii) *$T = \bigcup T_\eta$ is an appro. $\mathcal{A}_2$-interpolation set.*

Then for each open ngh. U of e, there is $f \in \mathcal{A}_2$ with $\|f\|_\infty = 1$ s.t

$$f(G \setminus UT) = \{0\} \text{ and } \|f - \phi\|_{T_\eta} \geq 1 \text{ for all } \eta < \kappa \text{ and } \phi \in \mathcal{A}_1.$$

Proof. By (ii) of Lemma 4.2, each T_η contains disjoint subsets $T_{1,\eta}, T_{2,\eta}$ such that $\overline{T_{1,\eta}}^{\mathcal{A}_1} \cap \overline{T_{2,\eta}}^{\mathcal{A}_1} \neq \varnothing$.
Define for each $\eta < \kappa$, a function $h_\eta \colon G \to [-1,1]$ supported on T_η with

$$h_\eta(T_{1,\eta}) = \{1\} \quad \text{and} \quad h_\eta(T_{2,\eta}) = \{-1\}.$$

Then consider the function $h \colon G \to [-1,1]$ supported on T and given by

$$h(t) = h_\eta(t) \ \text{ if } t \in T_\eta \text{ for some } \eta < \kappa.$$

By (iv) of Lemma 4.2, there is $f \in \mathcal{A}_2$ such that

$$f(G \setminus UT) = 0, \quad f\restriction_T = h \ \text{ and } \|f\|_\infty = \|h\|_T = 1.$$

Let now ϕ be any function in $\mathcal{A}_1$, and take $\varepsilon > 0$.
Fix $\eta < \kappa$. Take $p_\eta \in \overline{T_{1,\eta}}^{\mathcal{A}_1} \cap \overline{T_{2,\eta}}^{\mathcal{A}_1}$, pick $t_{1,\eta} \in T_{1,\eta}$ and $t_{2,\eta} \in T_{2,\eta}$ with

$$|\phi(t_{1,\eta}) - \phi^{\mathcal{A}_1}(p_\eta)| < \varepsilon \quad \text{and} \quad |\phi(t_{2,\eta}) - \phi^{\mathcal{A}_1}(p_\eta)| < \varepsilon,$$

where $\phi^{\mathcal{A}_1}$ denotes the extension of ϕ to $G^{\mathcal{A}_1}$.
Then

$$\begin{aligned} 2 &= |h_\eta(t_{1,\eta}) - h_\eta(t_{2,\eta})| = |h(t_{1,\eta}) - h(t_{2,\eta})| = |f(t_{1,\eta}) - f(t_{2,\eta})| \\ &\leq |f(t_{1,\eta}) - \phi(t_{1,\eta})| + |\phi(t_{1,\eta}) - \phi^{\mathcal{A}_1}(p_\eta)| \\ &+ |\phi^{\mathcal{A}_1}(p_\eta) - \phi(t_{2,\eta})| + |\phi(t_{2,\eta}) - f(t_{2,\eta})|. \end{aligned}$$

So $|f(t_{1,\eta}) - \phi(t_{1,\eta})| \geq 1 - \varepsilon$ or $|f(t_{2,\eta}) - \phi(t_{2,\eta})| \geq 1 - \varepsilon$.

Thus $\|f - \phi\|_{T_\eta} \geq 1$.
Since $\|f\|_\infty = 1$ and $f(G \setminus UT) = \{0\}$, we see that f is the required function. $\square$

Lemma 4.4. *Let $\mathcal{A}$ be a left invariant, unital C^*-subalgebra of $\mathcal{LUC}(G)$, U a compact ngh. of e, and T an approximable $\mathcal{A}$-interpolation, right U-uniformly discrete set. Partition T into $(T_\eta)_{\eta<\kappa}$.*
Then there is a compact ngh. V of e with $V^2 \subseteq U$ such that whenever functions $f, g \in \ell_\infty(G)$ supported in VT and a function $\mathbf{c} \in \ell_\infty(\kappa)$ are such that

$$f\restriction_{VT_\eta} = \mathbf{c}(\eta) g\restriction_{VT_\eta} \quad \textit{for each} \;\; \eta < \kappa,$$

we have:

$$g \in \mathcal{A} \Longrightarrow f \in \mathcal{A}.$$

Proof. f is well defined follows from $UT_\eta \cap UT_{\eta'} = \varnothing$.
Let V_1 and V_2 be two nghs provided by the definition of approximable $\mathcal{A}$-interpolation set for the ngh. U. We take the set V as V_1, we can obviously assume that $V^2 \subseteq U$.
Define for every pair $(s, x) \in G \times G^{\mathcal{A}}$, the functional sx on $\mathcal{A}$ by $sx(f) = x(f_s)$.
Then $sx \in G^{\mathcal{A}}$.
Now define φ on T by $\varphi(t) = \mathbf{c}(\eta)$ for every $t \in T_\eta$.
Extend φ to a function $f_0 \in \mathcal{A}$.
Let $g^{\mathcal{A}}$ and $f_0^{\mathcal{A}}$ be the respective extensions of g and f_0 to $G^{\mathcal{A}}$.
Define $f^{\mathcal{A}} \colon G^{\mathcal{A}} \to \mathbb{C}$ by

$$f^{\mathcal{A}}(vp) = f_0^{\mathcal{A}}(p) \cdot g^{\mathcal{A}}(vp) \text{ if } v \in V \text{ and } p \in \overline{T}$$
$$f^{\mathcal{A}}(x) = 0 \text{ if } x \notin V\overline{T}.$$

We check that $f^{\mathcal{A}}$ is a well-defined, continuous extension of f to $G^{\mathcal{A}}$.

(1) $f^{\mathcal{A}}$ is well defined. $f^{\mathcal{A}}$ does not depend of the choice of the decomposition of vp.

Let $v_1p_1 = v_2p_2$ with $v_1, v_2 \in V$ and $p_1, p_2 \in \overline{T}$.

If $p_1 \neq p_2$, by (ii) of Lemma 4.2, we may choose $T_1, T_2 \subset T$ such that

$$\overline{T_1} \cap \overline{T_2} = \varnothing,\ p_1 \in \overline{T_1} \quad \text{and} \quad p_2 \in \overline{T_2}$$

Pick $h \in \mathcal{A}$ such that

$$h(VT_1) = \{1\} \quad \text{and} \quad h(G \setminus V_2T_1) = \{0\}.$$

By Ellis-Lawson's Theorem on joint continuity, the map

$$G \times G^{\mathcal{A}} \to G^{\mathcal{A}} : (s, x) \mapsto sx$$

is jointly continuous. So $h^{\mathcal{A}}(v_1p_1) = 1$.

By the same reason, and since T is right U-uniformly discrete, $h^{\mathcal{A}}(v_2p_2)$ must be zero.

This contradiction shows that $v_1p_1 = v_2p_2$ implies $p_2 = p_1$. This shows already that $f^{\mathcal{A}}$ is well defined, since $v_1p_1 = v_2p_2$ and $p_1 = p_2$ give us

$$f^{\mathcal{A}}(v_1p_1) = f_0^{\mathcal{A}}(p_1)g^{\mathcal{A}}(v_1p_1) = f^{\mathcal{A}}(v_2p_2).$$

(In fact, v_1 and v_2 must be also equal, but this is enough for our purposes.)

(2) $f^{\mathcal{A}}$ is continuous.

Using the continuity of

$$G \times G^{\mathcal{A}} \to G^{\mathcal{A}} : (s, x) \mapsto sx,$$

we see that $V\overline{T}$ is closed in $G^{\mathcal{A}}$. So the continuity of $f^{\mathcal{A}}$ at thee points outside of $V\overline{T}$ is clear.

So let $x = vp \in V\overline{T}$.
Let P be a ngh. of p in $G^{\mathcal{A}}$ such that $|g^{\mathcal{A}}(x) - g^{\mathcal{A}}(y)| < \epsilon$ whenever $y \in vP$.
We may choose P such that $|f_0^{\mathcal{A}}(p) - f_0^{\mathcal{A}}(q)| < \epsilon$ for every $q \in P$.
Then, for every $y = vq \in vP$, we have

$$\begin{aligned}|f^{\mathcal{A}}(x) - f^{\mathcal{A}}(y)| &= |f_0^{\mathcal{A}}(p)g^{\mathcal{A}}(x) - f_0^{\mathcal{A}}(q)g^{\mathcal{A}}(y)| \\ &\leq |f_0^{\mathcal{A}}(p)||g^{\mathcal{A}}(x) - g^{\mathcal{A}}(y)| + |g^{\mathcal{A}}(y)||f_0^{\mathcal{A}}(p) - f_0^{\mathcal{A}}(q)|,\end{aligned}$$

which yields the continuity of $f^{\mathcal{A}}$.

(3) $f^{\mathcal{A}}$ coincides with f on G. Easy.

From (1), (2), (3) we conclude that $f \in \mathcal{A}$. □

Theorem 4.5. *Let G be a locally compact group, $\mathcal{A}_1 \subset \mathcal{A}_2 \subseteq \mathcal{LUC}(G)$ be unital C^*-subalgebras with $\mathcal{A}_2$ left invariant. Suppose that G contains a family of sets $(T_\eta)_{\eta<\kappa}$ such that*

(i) *each T_η fails to be an $\mathcal{A}_1$-interpolation set,*
(ii) *$T = \bigcup_{\eta<\kappa} T_\eta$ is an appro. $\mathcal{A}_2$-interpolation set.*
(iii) *T is right U-uniformly discrete for some compact ngh. of e.*

Then there is a linear isometry $\Psi\colon \ell_\infty(\kappa) \to \mathcal{A}_2/\mathcal{A}_1$.

Proof. Let V be the ngh. of e provided by Lemma 4.4.
Pick by Lemma 4.3 a function $f \in \mathcal{A}_2$ with $\|f\|_\infty = 1$ such that

$$f(G\backslash VT) = \{0\} \text{ and } \|f - \phi\|_{T_\eta} \geq 1 \text{ for all } \phi \in \mathcal{A}_1 \text{ and } \eta < \kappa.$$

For each $\mathbf{c} \in \ell_\infty(\kappa)$, define $f_{\mathbf{c}} : G \to \mathbb{C}$ supported in VT with

$$f_{\mathbf{c}} \restriction_{VT_\eta} = \mathbf{c}(\eta) f \restriction_{VT_\eta} .$$

Then $f_{\mathbf{c}} \in \mathcal{A}_2$ by Lemma 4.4. Obviously, the map $\Psi \colon \ell_\infty(\kappa) \to \mathcal{A}_2/\mathcal{A}_1$ given by

$$\Psi(\mathbf{c}) = f_{\mathbf{c}} + \mathcal{A}_1 \quad \text{for every} \quad \mathbf{c} \in \ell_\infty(\kappa)$$

is linear. We next check that it is isometric.

The same argument of Chou shows now that, for every $\eta_0 < \kappa$,

$$\begin{aligned} \|\Psi\,(\mathbf{c}))\,\|_{\mathcal{A}_2/\mathcal{A}_1} &= \inf\{\|f_{\mathbf{c}} - \phi\|_\infty : \phi \in \mathcal{A}_1\} \\ &\geq \inf\{\|f_{\mathbf{c}} - \phi\|_{T_{\eta_0}} : \phi \in \mathcal{A}_1\} \\ &= \inf\{\|\mathbf{c}(\eta_0) f - \phi\|_{T_{\eta_0}} : \phi \in \mathcal{A}_1\} \\ &= |\mathbf{c}(\eta_0)| \inf\{\|f - \phi\|_{T_{\eta_0}} : \phi \in \mathcal{A}_1\} \\ &\geq |\mathbf{c}(\eta_0)|, \end{aligned}$$

where the last inequality follows from the choice of f. Since, obviously,

$\|\Psi(\mathbf{c})\|_{\mathcal{A}_2/\mathcal{A}_1} \leq \|f_{\mathbf{c}}\|_\infty = \|\mathbf{c}\|,$ for every $\mathbf{c} = (c_\eta)_{\eta<\kappa} \in \ell_\infty(\kappa)$, we see that Ψ is the required isometry. □

Corollary 4.6. *If in the above theorem $\mathcal{A}_2 = \mathcal{CB}(G)$ and T is not assumed to be right U-uniformly discrete but still $UT_\eta \cap UT_{\eta'} = \varnothing$, then the quotient $\mathcal{CB}(G)/\mathcal{A}_1$ contains a linearly isometric copy of $\ell_\infty(\kappa)$.*

Remark 4.7. *Two C^*-subalgebras of $\ell_\infty(G)$ may be different, and yet produce a small quotient (i.e., separable), for example if G is a minimally weakly almost* *periodic group (Chou 1990, Ruppert 1984, Veech 1979) then* $\mathcal{WAP}(G)/\mathcal{AP}(G) = C_0(G)$. *If* $G = SL(2,\mathbb{R})$, *then* $\mathcal{WAP}(G) = C_0(G) \oplus \mathbb{C}1$, *and so* $\mathcal{WAP}(G)/C_0(G) = \mathbb{C}$. *In the theorem above, we have just met conditions under which this is not so.*

Corollary 4.8. *Under the hypotheses of Theorem 4.5, the quotient space* $\mathcal{A}_2/\mathcal{A}_1$ *is non-separable.*

5. Application

Theorem 5.1. *Let G be a non-compact locally compact E-group with an E-set X having a compact covering number κ. Then $\mathcal{WAP}(G)/(\mathcal{AP}(G) \oplus C_0(G)$ contains a linear isometric copy of $\ell_\infty(\kappa)$.*

Theorem 5.2. *Let G be a non-compact locally compact E-group with an E-set X having a compact covering number κ. Then the quotient space $\mathcal{WAP}_0(G)/C_0(G)$ contains a linear isometric copy of $\ell_\infty(\kappa)$.*

Theorem 5.3. *Let G be a locally compact group and $\kappa = \kappa(Z(G))$. There is always a linear isometry $\ell_\infty(\kappa)$ in $\mathcal{WAP}(G)/\mathcal{B}(G)$.*

Theorem 5.4. *Let G be a a non-compact, locally compact, IN-group and put $\kappa = \kappa(G)$. Then there is a linear isometry copy of $\ell_\infty(\kappa)$ in $\mathcal{WAP}(G)/\mathcal{B}(G)$.*

Corollary 5.5. *Let G be a non-compact IN-group with compact covering κ. Then $\mathcal{WAP}_0(G)/C_0(G)$ contains a linear isometric copy of $\ell_\infty(\kappa)$.*

Corollary 5.6. *Let G be a a non-compact, locally compact, nilpotent group and put $\kappa = \kappa(G)$. Then $\mathcal{WAP}(G)/\mathcal{B}(G)$ contains a linear isometric copy of $\ell_\infty(\kappa)$.*

Theorem 5.7. *Let G be a locally compact group. Then $\mathcal{CB}(G)/\mathcal{LUC}(G)$ contains a linear isometric copy of $\ell_\infty(\kappa(G))$ if and only if G is neither compact nor discrete.*