Separable C^*-Algebras and weak*-fixed point property of their Banach space dual.

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joint work with
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Contents

Introduction

It is a well known theorem in harmonic analysis that a locally compact group G is compact if and only if its dual $\widehat{G}$ is discrete. This dual is just the spectrum of the full C^*-algebra $C^*(G)$ of G.
There is a bunch of properties of the weak* topology for the Fourier–Stieltjes algebra $B(G)$ of G, which are equivalent to the compactness of the group.
Some of them, which can be formulated in purely C^*-algebraic terms are the topic of this note.

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Denote $\mathcal{S}$ a semitopological left reversible semigroup acting by non-expansive mappings separately continuously on a non-empty weak* compact convex set $K \subset E$.
(left-reversible: $a\overline{\mathcal{S}} \cap b\overline{\mathcal{S}} \neq \emptyset, \ \forall a, b \in \mathcal{S}$.)

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A a C^*-algebra , $\pi : A \to B(H)$ a *-representation.
Denote $\pi^{'} \simeq \pi$ unitary equivalence of $*$-representations $\pi^{'}$ and π.
$\widehat{A}$ – the set of unitary equivalence classes with topology induced from the Jacobson topology on the kernels – the spectrum of A.

Theorem

A separable C^*-algebra has a discrete spectrum if and only if its Banach space dual has the weak* fixed point property.

Why separable C^*-algebras only?

H a Hilbert space $\mathcal{K} = \mathcal{K}(H)$ compact operators on H.

Remark (Naimark '48)

$\widehat{\mathcal{K}}$ is a point. If $\widehat{A}$ is a point, is $A = \mathcal{K}(H)$, for some H?

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Weak* Fixed Point Property

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Lemma (Glimm '61)

If $\varphi \in A^*$ is a state $\varphi_{|A \cap \mathcal{K}} = 0$ then $\varphi \in \overline{\{\omega_\xi : \xi \in H\}}^{w*}$.

Theorem (Anderson '77)

If A is separable $\varphi \in A^*$ is a state $\varphi_{|A \cap \mathcal{K}} = 0$ then there is an orthonormal sequence ξ_n, such that $\varphi = w^* - \lim \omega_{\xi_n}$.

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Let $\pi' \not\simeq \pi \in \widehat{A}$ with $\pi' \in \overline{\{\pi\}}$ be given and assume that φ is a state of A associated with π'. Then there is an orthonormal sequence (ξ_n) in H_π with $(\pi(.)\xi_n|\xi_n) \to \varphi$ weakly*.

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By assumption, $\ker \pi \subset \ker \pi'$ so there is a representation π° of $\pi(A)$ such that $\pi' = \pi^\circ \circ \pi$. We may therefore assume that π is the identical representation.

(i) Suppose $\varphi_{|\mathcal{K}(H_\pi)\cap A} \neq \emptyset$. Then $\pi' = \pi_\varphi$ does not annihilate $\mathcal{K}(H_\pi) \cap A$. Hence $\mathcal{K}(H_\pi) \subset A$ and it is a two sided ideal. Moreover π' is faithful on $\mathcal{K}(H_\pi)$ and $\pi'_{|\mathcal{K}(H_\pi)}$ is an irreducible representation. Therefore it is equivalent to the identical representation of $\mathcal{K}(H_\pi)$. Then π' is equivalent to the identical representation of A. This contradicts the assumption, so this case can not happen.

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If A^* has the weak fpp then points in $\widehat{A}$ are closed.*

Assume A^* has w*fpp. Proof a contrario.
If $\{\pi\}$ is non–closed then there is $\pi' \not\simeq \pi$ contained in $\overline{\{\pi\}}$. For φ, a pure state associated with π', there exists (ξ_n) in H_π an orthonormal sequence such that $\varphi_n := < \pi(.)\xi_n | \xi_n > \to \varphi$ weakly*.
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(i) $\implies$ (iii)

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If $\widehat{A}$ is discrete, then $A = \oplus_{c_0} A_n$, where $\widehat{A_n} = \{\text{one point}\}$, hence $A_n = \mathcal{K}(H_n)$.

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If $\widehat{A}$ is not discrete: $\pi_0 \in \overline{M}^{w*}$, $\pi_0 \notin M$ and M must be infinite. So, since A is separable, for any state φ_0 associated to π_0 there is a sequence of states (φ_n) associated to pairwise non-equivalent representations π_n with $\varphi_n \to \varphi_0$ weakly*. The support projections (of the states φ_n) in the universal representation of A are mutually orthogonal. As above the set

$$C = \{\sum_0^\infty \alpha_i \varphi_i : \alpha_i \geq 0, \sum \alpha_i = 1\}$$

is convex and weak* compact. The map $T : C \to C$ defined like there is well defined and isometric because of the orthogonality of the supports, and it has no fixed point in C.

Uniform Weak* Kadec-Klee

$K \subset E$ – a closed convex bounded subset of a Banach space E.
$x \in K$ is a *diametral* point if $\sup\{\| x - y \| : y \in K\} = \operatorname{diam}(K)$.

Definition

1. The set K is said to have *normal structure* if every convex non-trivial (i.e. containing at least two different points) subset $H \subset K$ contains a non-diametral point of H.
2. E has *weak normal structure* if every convex weakly compact subset has normal structure.

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1. A dual Banach space E has *weak* Kadec-Klee property* (KK*) if weak* and norm convergence coincide on sequences of its unit sphere.
2. A dual Banach space E has *uniform weak* Kadec-Klee property* (UKK*) if for $\epsilon > 0$ there is $0 < \delta < 1$ such that for any subset C of its closed unit ball containing an infinite sequence $(x_i)_{i \in \mathbb{N}}$ with separation $\mathrm{sep}((x_i)_i) := \inf\{\| x_i - x_j \| : i \neq j\} > \epsilon$, there is an x in the weak*-closure of C with $\| x \| < \delta$.

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Well known fact:

Proposition

Let E be a dual Banach space.

(i) *The uniform weak* Kadec-Klee property implies the weak* Kadec-Klee property.*

(ii) *If E is the dual of a separable Banach space E_* and has the uniform weak* Kadec-Klee property then the weak* topology and the norm topology coincide on the unit sphere of E.*

Theorem

For a separable C^*-algebra A the following are equivalent

(i) *The dual $\widehat{A}$ is discrete,*

(ii) *The Banach space dual A^* has the UKK^* property,*

(iii) *On the unit sphere of A^* the weak* and the norm topology coincide,*

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(iv) *On the set of states $\mathcal{S}(A)$ of A the weak* and the norm topology coincide,*

(v) *On the set of pure states $\mathcal{P}(A)$ of A the weak* and the norm topology coincide.*

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(iii) *On the unit sphere of A^* the weak* and the norm topology coincide,*

(iv) *On the set of states $\mathcal{S}(A)$ of A the weak* and the norm topology coincide,*

(v) *On the set of pure states $\mathcal{P}(A)$ of A the weak* and the norm topology coincide.*

(vi) *A^* has weak* normal structure.*

Theorem

For a separable C^*-algebra A the following are equivalent

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(vii) *A^* has the weak* fixed point property for non-expansive mappings.*

(viii) *A^* has the weak* fixed point property for left reversible semigroups.*

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Remark

The C^*-algebras fulfilling the equivalent conditions of the theorem are known as separable dual C^* algebras. This follows from the fact that separable dual C^*-algebras are characterised by the property that their spectrum is discrete.

[Proof of the theorem]

- (i) $\Longrightarrow$ (ii)
  $A^* = \oplus_1 \mathcal{T}(H_l)$ a countable l^1-direct sumthe canonical dual to $A = \oplus_0(\mathcal{K}(H_l)$ a c_0-direct sum of compact operators. Reordering $A^* \subset (\mathcal{T}(\oplus_2 H_l)$, weak* closed and we obtain the UKK* property of A^* from the UKK* property of the trace class operators (Lennard '90).
- (ii) $\Longrightarrow$ (iii)
  by the above Proposition.
- (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v)
  by restriction.

- (v) $\Longrightarrow$ (i)
  (Argument due to Bekka, Kaniuth, Lau and Schlichting '98, based on an argument of Glimm and Kadison)
  For $\varphi \in \mathcal{S}(A)$ corresponds π_φ by GNS-construction. Extreme points of $\mathcal{P}(A)$ corresponds $\widehat{A}$. If $\mathcal{P}(A)$ is endowed with the weak* topology then the mapping $q : \varphi \to \pi_\varphi$ is open. For $\varphi, \psi \in \mathcal{P}(A)$ the two representations π_φ and π_ψ are equivalent if $\| \varphi - \psi \| < 2$. Hence, assuming (v), the (norm open) set $\{\psi \in \mathcal{P}(A) : \| \varphi - \psi \| < 2\}$ contains a weak* open neighbourhood of $\varphi \in \mathcal{P}(A)$. Its image under q is open but just reduces to the point π_φ. This shows that points in $\widehat{A}$ are open.
- (ii) $\Longrightarrow$ (vi) Lau and Mah '88,
- (vi) $\Longrightarrow$ (vii) Lim '80.
- (vii) $\Longrightarrow$ (i) holds true by Theorem.
- From this theorem we have (viii) $\Longleftrightarrow$ (i) too.

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Tank You for Your Attention!