International Conference
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On the L^p- Fourier transform norm of some locally compact groups.

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The L^p- Fourier transform on $\mathbb{R}^n$

- For $\mathbb{R}^n$ with a standard inner product $\langle \cdot, \cdot \rangle$, we identify $(\mathbb{R}^n)^*$ with $\mathbb{R}^n$ by $\mathbb{R}^n \ni x \mapsto l_x \in (\mathbb{R}^n)^*$ so that $l_x(y) = \langle y, x \rangle$, $y \in \mathbb{R}^n$.

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- Then we have the equality (the classical Plancherel formula)

$$\int_{\mathbb{R}^n} |\widehat{f}(y)|^2 dy = \int_{\mathbb{R}^n} |f(x)|^2 dx,$$

for $f \in (L^1 \cap L^2)(\mathbb{R}^n)$.

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- By the abstract Plancherel Theorem, there exists a unique Borel measure μ on $\widehat{G}$ such that

$$\int_G |f(g)|^2\, dg = \int_{\widehat{G}} \mathrm{Tr}\left(\pi(f)^* \pi(f)\right) d\mu(\pi), \quad \forall f \in (L^1 \cap L^2)(G).$$

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$$\|F\|_r := \left(\int_{\widehat{G}} \|F(\pi)\|_{C_r}^r \, d\mu(\pi) \right)^{\frac{1}{r}},$$

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- We denote by $L^r(\widehat{G})$ the Banach space defined by measurable fields F such that $\|F\|_r < \infty$ with norm $\|\cdot\|_r$.

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2. A restrictive class of exponential solvable Lie groups.
3. Arbitrary compact extensions of $\mathbb{R}^n$ (and some of their universal coverings).

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- J. Inoue, 1992 : In this context, let m be the dimension of generic coadjoint orbits of $\mathfrak{g}^*$.
  Suppose that there exists an open dense subset $\mathscr{U}$ of $\mathfrak{g}^*$, such that the ideal generated by $\bigcup_{\ell\in\mathscr{U}}\mathfrak{g}(\ell)$ is abelian.
  Then for $1 < p \leq 2$,

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- Let $\{X_1, ..., X_n\}$ be a strong Malcev basis of $\mathfrak{g}$, for any $j = 1, ..., n$, the space $\mathfrak{g}_j = \mathbb{R} - \text{span}\{X_1, ..., X_j\}$ is an ideal of $\mathfrak{g}$.

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- The composed map $\mathbb{R}^n \to \mathfrak{g} \to G$,

$$(x_1, ..., x_n) \mapsto \sum_{j=1}^{n} x_j X_j \mapsto \exp(\sum_{j=1}^{n} x_j X_j)$$

  is a diffeomorphism and maps Lebesgue measure on $\mathbb{R}^n$ to a Haar measure on G.

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- Let

$$e(\ell) = \{j : j \text{ is a jump index for } \ell\}.$$

This set contains exactly $\dim(G \cdot \ell)$ indices, which is necessarily an even number.

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Then $\mathfrak{g}^* = \mathscr{V}_T \oplus \mathscr{V}_S$, $\mathscr{V}_T$ meets $\mathscr{U}$ and $\mathscr{W} = \mathscr{U} \cap \mathscr{V}_T$ is a cross-section for coadjoint orbits through points in $\mathscr{U}$.

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- So, every G-orbit in $\mathscr{U}$ related to a representation π meets $\mathscr{W}$ in a single unique element. Define the Pfaffian $Pf(\ell)$ of the skew-symmetric matrix $M_S(\ell) = (\ell([X_i, X_j]))_{i,j \in S}$.

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$$\|\mathscr{F}^p(f)\|_q = \left(\int_{\mathscr{W}} \|\pi_\ell(f)\|_{C_q}^q d\mu(\ell) \right)^{\frac{1}{q}}.$$

- The problem now is how to get a sharp estimate (to a certain extent) of $\|\pi_\ell(f)\|_{C_q}$.

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- Fournier-Russo, 1977 : The Hausdorff-Young inequality for integral operators. Let X be a σ-finite measure space, k a measurable function on $X \times X$ and K an operator on $L^2(X)$ defined for $\phi \in L^2(X)$ by :

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- Then if $1 < p \leq 2$, we have

$$\|K\|_{C_q} \leq \|k\|_{p,q}^{\frac{1}{2}} \|k^*\|_{p,q}^{\frac{1}{2}},$$

where $k^*(x,y) = \overline{k(y,x)}$.

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  2. If H_{2n+1} is the $(2n+1)$-dimensional Heisenberg group, then accurately $\|\mathscr{F}^p(H_{2n+1})\| = A_p^{2n+1}$.

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2. If H_{2n+1} is the $(2n+1)$-dimensional Heisenberg group, then accurately $\|\mathscr{F}^p(H_{2n+1})\| = A_p^{2n+1}$.

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1. On connected simply connected nilpotent Lie groups

Remarks.

- This result is the most general so far in the case of connected and simply connected nilpotent Lie groups.
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- The problem of finding the exact norm for general cases is still open.

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- We denote by $\exp : \mathfrak{g} \to \tilde{G}$ the exponential mapping, and let $\Lambda \subset \mathfrak{g}$, such that $\exp \Lambda = \Gamma$. Then Λ is a discrete additive subgroup of the center of $\mathfrak{g}$.
- Let $\mathfrak{h} := \mathbb{R}\text{-span}(\Lambda)$, $\tilde{H} := \exp \mathfrak{h} \subset \tilde{G}$ and $H := \tilde{H}/\Gamma \subset G$, which is the compact maximal subgroup of G.

Removing the assumption of simply connectedness

- We have the following :
  Theorem 2 : (A. Bak and J. Inoue, 2011) *Let G be a connected nilpotent Lie group, $\tilde{G}$ its universal covering group and H the maximal compact subgroup of G. If m designates the maximal dimension of generic coadjoint orbits of $\tilde{G}$, then for $1 < p \leq 2$,*
  $\|\mathscr{F}^p(G)\| \leq A_p^{\dim(G/H)-\frac{m}{2}}$.

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  2. A measure μ on $\widehat{G}$ such that for $\mu-$almost all $\pi \in \widehat{G}$, the operator $\pi(f)K_\pi^{-\frac{1}{2}}$ extends to a Hilbert-Schmidt operator on the space of π for any $f \in (L^1 \cap L^2)(G)$.

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- All exponential Lie groups of dimension ≤ 4, except the so-called Leptin-Boidol Lie group, whose Lie algebra admits a basis $\{A, X, Y, Z\}$ for which $[A, X] = -X, [A, Y] = Y$ and $[X, Y] = Z$.

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  2. *We have the following inequality :*

$$\left(\int_{\widehat{G}} \|\pi^p(f)\|_{C_q}^q d\mu(\pi) \right)^{\frac{1}{q}} \leq A_p^{\frac{2 \dim G - m}{2}} \|f\|_p,$$

where m denotes the maximal dimension of coadjoint orbits.

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3. On compact extensions of $\mathbb{R}^n$

- Let $M(n) = \mathbb{R}^n \rtimes SO(n)$ be the Euclidean motion group. We denote each element of $M(n)$ by (a, k), where $a = (a_1, \cdots, a_n) \in \mathbb{R}^n$, and $k \in SO(n)$, which acts on $\mathbb{R}^n$ as a rotation.

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- Let $\chi \in \widehat{\mathbb{R}^n}$ defined by $\chi(x) := e^{i\langle x, e_n\rangle}$, where $e_n := (0, \cdots, 0, 1)$, $x \in \mathbb{R}^n$.

3. On compact extensions of $\mathbb{R}^n$

- Let now $M(n)$ act naturally on $\widehat{\mathbb{R}^n}$ as : for $(a,k) \in M(n)$ and $b \in \mathbb{R}^n$,

$$(a,k)\cdot\chi(b) := \chi(k^{-1}\cdot b) = e^{i\langle k^{-1}\cdot b, e_n\rangle} = e^{i\langle b, k\cdot e_n\rangle},$$

and the stabilizer $M(n)_\chi$ of χ is described by $M(n)_\chi = \mathbb{R}^n \rtimes K'$, where

$$K' := \left\{ \begin{pmatrix} k' & 0 \\ 0 & 1 \end{pmatrix} ;\ k' \in SO(n-1) \right\} \simeq SO(n-1).$$

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  1. $\phi(kk') = \sigma(k')^{-1}\phi(k), \ k \in SO(n)$ and $k' \in K'$,
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- For $\sigma \in \widehat{K'}$ and $r > 0$, define a representation $\pi_{r,\sigma}$ of $M(n)$ on $\mathscr{H}$ by

$$\pi_{r,\sigma}(a, k)\phi(h) := e^{ir\langle a,\, h\cdot e_n \rangle}\phi(k^{-1}h).$$

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- Then $(\pi_{r,\sigma}, \mathscr{H})$ is an irreducible unitary representation of $M(n)$ and we have the Plancherel formula for $f \in (L^1 \cap L^2)(M(n))$ by :

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$$\int_{M(n)} |f(a,k)|^2 \, da d\mu_n(k) = \int_0^\infty \sum_{\sigma \in \widehat{K'}} d_\sigma \|\pi_{r,\sigma}(f)\|_{HS}^2 \, (2\pi)^{-n} c_n r^{n-1} \, dr,$$

where dr is the Lebesgue measure on $\mathbb{R}$, and $c_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$, the volume of the unit sphere S^{n-1}.

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where dr is the Lebesgue measure on $\mathbb{R}$, and $c_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$, the volume of the unit sphere S^{n-1}.

- Thus for $f \in (L^1 \cap L^p)(M(n))$, we can describe

$$\|\mathscr{F}^p f\|_q = \left(\int_0^\infty \sum_{\sigma \in \widehat{K'}} d_\sigma \|\pi_{r,\sigma}(f)\|_{C_q}^q \, (2\pi)^{-n} c_n r^{n-1} \, dr \right)^{\frac{1}{q}}.$$

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  Theorem 4' : *Let G be the semi-direct product $K \ltimes \mathbb{R}^n$, where K designates a compact subgroup of $Aut(\mathbb{R}^n)$. Then for $1 < p \leq 2$,* $\|\mathscr{F}^p(G)\| = A_p^n$.

A glimpse on Spin extensions

- The universal covering group $\widetilde{M(n)} = \mathbb{R}^n \rtimes \mathrm{Spin}(n)$, where Spin($n$) designates the universal covering groups of the orthogonal groups. Here, the action of Spin(n) on $\mathbb{R}^n$ is merely the pullback of the action of $SO(n)$ on $\mathbb{R}^n$.

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- So, $\widetilde{M(2)} = \mathbb{R}^2 \rtimes \mathrm{Spin}(2) = \mathbb{R}^2 \rtimes \mathbb{R}$ is a solvable non-exponential Lie group with a trivial center.
- Here, $\mathbb{R}$ acts on $\mathbb{R}^2$ by

$$\theta \cdot (x, y) = (x\cos(2\pi\theta) + y\sin(2\pi\theta), -x\sin(2\pi\theta) + y\cos(2\pi\theta)),$$

$x, y, \theta \in \mathbb{R}$.

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A last remark : I can not compute so far the exact norm when $n = 2$.