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Spacelike graphs of prescribed mean curvature in the steady state space

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Abstract

We study the Dirichlet problem of the constant mean curvature equation in the steady state space over a bounded domain of a slice. Under suitable conditions on the convexity of the domain, we prove the existence of a spacelike graph with constant mean curvature H with $-1 \leq H < 0$ and constant boundary values.

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1 Introduction and statement of results

The steady state space $\mathcal{H}^{n+1}$ is a model for the universe proposed by Bondi and Gold [4] and Hoyle [12] which is homogeneous and isotropic, that is, it looks the same not only at all points and in all directions, but also at all times [10, Sect. 5.2]. If $\mathbb{R}^{n+2}_1 = (\mathbb{R}^{n+2}, \langle , \rangle = dx_1^2 + \ldots + dx_{n+1}^2 - dx_{n+2}^2)$ denotes the $n+2$-dimensional Minkowski space, the steady state space $\mathcal{H}^{n+1}$ is the half of the de Sitter space $\mathbb{S}^{n+1}_1 = \{p \in \mathbb{R}^{n+2}_1 : \langle p, p \rangle = 1\}$ given by $\mathcal{H}^{n+1} = \{p \in \mathbb{S}^{n+1}_1 : \langle p, a \rangle > 0\}$ being $a \in \mathbb{R}^{n+1}_1$ a non-zero null vector in the past half of the null cone. This space is

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a non-complete manifold foliated by a uniparametric family of totally umbilical hypersurfaces $\{L_\tau : \tau \in (0,\infty)\}$ called slices, where $L_\tau = \{p \in \mathcal{H}^{n+1} : \langle p,a\rangle = \tau\}$ and all have constant mean curvature $H=-1$. The boundary of $\mathcal{H}^{n+1}$ is the null hypersurface $L_0 = \{p \in \mathbb{S}_1^{n+1} : \langle p,a\rangle = 0\}$ that represents the past infinity and $L_\infty = \{p \in \mathbb{S}_1^{n+1} : \langle p,a\rangle = \infty\}$ is the limit boundary that stands as the future infinity.
An equivalent model of $\mathcal{H}^{n+1}$ is $(\mathbb{R}_+^{n+1}, g)$ where $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times \mathbb{R}_+$ is the upper half-space of vector space $\mathbb{R}^{n+1}$ endowed with the metric

$$g = \frac{1}{x_{n+1}^2}(dx_1^2 + \ldots + dx_n^2 - dx_{n+1}^2), \tag{1.1}$$

being $x=(x_1,\ldots,x_{n+1})$ the canonical coordinates of $\mathbb{R}^{n+1}$. By the expression of the metric g in (1.1), the steady state space $\mathcal{H}^{n+1}$ is the Lorentzian analogue to the upper half-space model of hyperbolic space viewed as a subset of the $(n+1)$-dimensional Lorentz-Minkowski space $\mathbb{R}_1^{n+1}$. In this model a slice is a horizontal hyperplane $L(h) = \{x \in \mathbb{R}^{n+1} : x_{n+1} = h\}$ for $h>0$.
In this paper we study the Dirichlet problem for the prescribed mean curvature equation on a domain of a slice: if $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, given $\varphi \in C^0(\partial\Omega)$ and $H \in \mathbb{R}$, find a solution $u \in C^2(\Omega)\cap C^0(\overline{\Omega})$ of

$$\begin{cases} \mathrm{div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right) = \frac{n}{u}\left(H + \frac{1}{\sqrt{1-|Du|^2}}\right) & \text{in } \Omega \\ |Du| < 1 & \text{in } \overline{\Omega} \\ u = \varphi & \text{along } \partial\Omega. \end{cases} \tag{1.2}$$

Here div and D are the Euclidean divergence and gradient operators of $\mathbb{R}^n$. If u is a solution of (1.2), its graph $\Sigma_u = \{(x_1,\ldots,x_n,u(x_1,\ldots,x_n)) : (x_1,\ldots,x_n) \in \Omega\}$ is a spacelike hypersurface in $\mathcal{H}^{n+1}$ of constant mean curvature H with respect to the upwards orientation and its boundary is the graph of φ. Spacelike hypersurfaces with constant mean curvature have interest in a spacetime, as for example $\mathcal{H}^{n+1}$, because they are used as convenient initial hypersurfaces for the Cauchy problem corresponding to the Einstein equations [15]. Also it is interesting to have foliations of the spacetime by hypersurfaces with constant mean curvature because all points of each leaf of the foliation are instantaneous observers (or normal observers) and the (timelike) unit normal vector of the hypersurfaces measures how the observers get away with respect to the next ones. Thus the graph of a solution of (1.2) can be viewed as a local result (on the domain Ω) of prescribing the behavior of normal observers. In the simplest spacetime, the Lorentz-Minkowski space, the first remarkable result on the Dirichlet problem is due to Bartnik and Simon who proved the solvability for almost any domain Ω and boundary values φ [3].
In the literature, the first existence result of the Dirichlet problem (1.2) appears in [16]. More recently, it is shown in [8] the existence of radially symmetric spacelike hypersurfaces in $\mathcal{H}^{n+1}$ with constant mean curvature. The methods employed in [8] were the classical Schauder fixed point theorem, and the results were extended to a wider class of spacetimes, namely, the Robertson-Walker spaces. However,

our approach for the solvability of (1.2) is the method of continuity following [3, 16]. Proving the existence of graphs with prescribed mean curvature and boundary requires establishing a priori C^1 estimates and these estimates are accomplished by the use of hyperbolic planes as barriers. For the question of uniqueness of (1.2), standard arguments of elliptic PDE's do not provide a complete answer (see Remark 2.1). On the other hand, the Dirichlet problem of the Gauss curvature equation was studied in [17]. Recently the author has proved that planar discs and hyperbolic caps are the only compact spacelike surfaces with constant mean curvature in $\mathcal{H}^3$ spanning a circle [13].
The solvability of the Dirichlet problem (1.2) depends if H is less than -1 or greater than -1, just the value of the mean curvature of slices. For example, and for $\varphi = h$, the graph Σ_u lies in one side of the slice $L(h)$, namely, Σ_u above $L(h)$ (or $u > h$) if $H < -1$ and below $L(h)$ (or $u < h$) if $H > -1$. Notice that the space is not symmetric with respect to $L(h)$ and one can not expect similar results if $H < -1$ or $H > -1$: see also this difference about the uniqueness in Remark 2.1. In the mentioned paper [16], Montiel showed the existence of solutions of (1.2) for $\varphi = h$ and $H < -1$ assuming that Ω is strictly mean convex. This result allowed to prove the existence of non-compact spacelike hypersurfaces with boundary in the infinity: by the maximum principle, the value of H for these hypersurfaces must be less than -1 and after letting $h \to 0$ together a suitable control of the C^1 estimates, he established successfully the existence of complete spacelike hypersurfaces with boundary at the past infinity. The goal of the present paper is considering the Dirichlet problem for $H > -1$ where getting apriori estimates is not a consequence of the case $H < -1$ and the technical difficulties are hard. Indeed, the existence result that we will prove holds when $-1 \leq H < 0$ and Ω satisfies stronger convexity assumptions. Here we recall that if $\kappa > 0$, a domain $\Omega \subset L(h)$ is said to be κ-convex if the principal curvatures $\kappa_1, \ldots, \kappa_{n-1}$ of $\partial\Omega$ with respect to the inward normal vector satisfy $\kappa_i \geq \kappa$. The main result that we prove is:

Theorem 1.1 *Let $-1 \leq H < 0$. Let $\Omega \subset L(h)$ be a κ-convex domain strictly contained in a ball of $L(h)$ radius 1. If*

$$\kappa \geq \sqrt{1 - H^2}, \tag{1.3}$$

then there exists a solution of the Dirichlet problem (1.2) for $\varphi = h$.

Let us point out that some kind of smallness assumption on the domain Ω is needed because the following result was proved by the author in [13]:

Theorem 1.2 *Let $-1 < H < 0$ and let $\Omega \subset L(h)$ be a bounded domain. If Ω contains a ball of radius*

$$r_0 = \sqrt{\frac{1 - H}{1 + H}}, \tag{1.4}$$

then there does not exist a spacelike graph on Ω of constant mean curvature H and with boundary $\partial\Omega$.

The proof of this result uses an argument of comparison of the graph of a prospective solution with a uniparametric family of hyperbolic planes that have the role of barrier hypersurfaces. The value of r_0 in (1.4) implies that $r_0 > 1$ and this is the reason that we suppose in Theorem 1.1 that the domain Ω is contained in a ball of radius 1. On the other hand, as $L(h)$ is isometric to the Euclidean hyperplane of equation $x_n = h$, then the κ-convexity of Ω means that Ω is also κ/h-convex as a submanifold of the Euclidean hyperplane $x_n = h$.

This paper is organized as follows. In Sect. 2 we recall some basics of the steady state space and we obtain the prescribed mean curvature equation. The next section is devoted to give the notion of a graph on a slice. The height and gradient estimates needed in the continuity method are obtained in Section 4. Finally in Sect. 5 we prove Theorem 1.1. As a consequence of the methods employed in the previous sections, in Sect. 6 we obtain a priori gradient estimates for a solution of the Dirichlet problem for other values of H and boundary conditions φ.

2 Preliminaries

Let Σ be a connected hypersurface. A smooth immersion $\psi : \Sigma \to \mathcal{H}^{n+1}$ is said to be a spacelike hypersurface if the induced metric via ψ is a Riemannian metric on Σ. A spacelike hypersurface is always orientable because the causal character of the ambient space allows to choose a unique unit timelike normal vector field N globally defined on Σ which is future-directed. As the metric of $\mathcal{H}^{n+1}$ is conformal to the Minkowski metric, the causal character of $\mathcal{H}^{n+1}$ is the same than the upper half-space viewed as an open set of the Minkowski space $\mathbb{R}^{n+1}_1$. Recall that there exists an explicit isometry $\Phi : \mathcal{H}^{n+1} \subset \mathbb{S}^{n+1}_1 \to \mathcal{H}^{n+1} \subset \mathbb{R}^{n+1}_+$ between both models which inverts the orientation [16]. We will also use the terminology horizontal and vertical in the Euclidean sense considering the target ambient space $\mathbb{R}^{n+1}_+$. In this model of $\mathcal{H}^{n+1}$, the isometries are the conformal transformations of the Minkowski space $\mathbb{R}^{n+1}_1$ that preserve the upper half-space $\mathbb{R}^{n+1}_+$. For example, the rotations about a vertical straight line are isometries of $\mathcal{H}^{n+1}$ as well as the horizontal translations or homotheties from any point of $\mathbb{R}^{n-1} \times \{0\}$.

If ∇ stands for the Levi-Civita connection on Σ, the mean curvature H is defined as
$$H = -\frac{\mathrm{trace}(W)}{n} = -\frac{\kappa_1 + \ldots + \kappa_n}{n}, \tag{2.5}$$
where W is the shape operator and κ_i are the principal curvatures of ψ. When H is constant we say that Σ is a cmc hypersurface or a H-hypersurface if we want to emphasize the value of the mean curvature. We point out that the choice of the minus sign in (2.5) follows [1, 5] and it is of opposite sign to the one adopted in [16].

According to (2.5), a slice L_τ has constant mean curvature $H = -1$ with respect to the future unit normal vector.

In view of the metric (1.1), we can consider on Σ the Minkowski metric $\langle , \rangle$ and the metric g. If N is the Gauss map of $\psi : \Sigma \to \mathcal{H}^{n+1}$, then $N' = N/x_n$ is the Gauss map of the immersion $\psi : \Sigma \to \mathbb{R}^{n+1}_1$. Since g and $\langle , \rangle$ are conformal metrics by

(1.1), then $\kappa_i = x_n\kappa_i' + (x_n \circ N')$, where κ_i and κ_i' are the principal curvatures of (Σ, g) and $(\Sigma, \langle,\rangle)$, respectively. From the definition of H in (2.5), we conclude

$$H = x_n H' - (x_n \circ N'), \tag{2.6}$$

being H and H' the mean curvatures of $\Sigma \subset \mathcal{H}^{n+1}$ and $\Sigma \subset \mathbb{R}_1^{n+1}$, respectively.
Convention. In what follows, the orientation N of a spacelike hypersurface Σ in both models of $\mathcal{H}^{n+1}$ will be future directed. If $\Sigma \subset \mathbb{S}_1^{n+1}$, and as the vector a lies in the past half of the null cone, we have $\langle N, a\rangle > 0$. If $\Sigma \subset \mathbb{R}_1^{n+1}$, then N lies in the same time-orientation than $\mathbf{e}_{n+1} = (0,\ldots,0,1)$, that is, $g(N, \mathbf{e}_{n+1}) < 0$, or equivalently, $\langle N, \mathbf{e}_{n+1}\rangle < 0$. According this choice of future directed orientations, if H is the mean curvature of $\Sigma \subset \mathbb{S}_1^{n+1}$, then $-H$ is the mean curvature of $\Sigma \subset (\mathbb{R}_+^{n+1}, g)$.
The first examples of cmc hypersurfaces of $\mathcal{H}^{n+1}$ are the totally umbilical hypersurfaces and among them, we stand out in slices and in upper hyperbolic planes. A *slice* $L(h) = \{x \in \mathbb{R}^{n+1} : x_{n+1} = h\}$ $(h > 0)$ has mean curvature $H = -1$ and by the isometry Φ, we have $\Phi(L_\tau) = L(h)$, where $h = 1/\tau$.
An (upper) *hyperbolic plane* of radius $r > 0$ and center $c = (c_1,\ldots,c_{n+1}) \in \mathbb{R}_1^{n+1}$ is

$$\mathbb{H}^n(r;c) = \{p \in \mathbb{R}_1^{n+1} : \langle p-c, p-c\rangle = -r^2, \langle p-c, \mathbf{e}_{n+1}\rangle < 0\}.$$

This hypersurface has constant mean curvature $H = c_{n+1}/r$ for $N'(p) = (p-c)/r$. In what follows, if the center of a hyperbolic plane is $(0,\ldots,0,t)$, we only write $\mathbb{H}^n(r;t)$.
We finish this section obtaining the expression of H in local coordinates. A spacelike hypersurface of $\mathcal{H}^{n+1}$ (so of $\mathbb{R}_1^{n+1}$) is locally a graph Σ of a function u defined in the $(x_1,\ldots,x_n)$-plane with $|Du| < 1$. The mean curvature H' of Σ as a hypersurface of Minkowski space $\mathbb{R}_1^{n+1}$ satisfies

$$\operatorname{div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right) = nH' \tag{2.7}$$

with respect to the future directed unit normal vector field

$$N' = \frac{1}{\sqrt{1-|Du|^2}}\left(u_{x_1},\ldots,u_{x_n},1\right). \tag{2.8}$$

For the Gauss map $N = uN'$, the expression (2.6) combined with equation (2.7) gives the partial differential equation for H of $\Sigma \subset \mathcal{H}^{n+1}$:

$$Q_H[u] := \operatorname{div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right) - \frac{n}{u}\left(H + \frac{1}{\sqrt{1-|Du|^2}}\right) = 0. \tag{2.9}$$

Equation (2.9) is of quasilinear elliptic type with the property that the difference function of two solutions satisfies a linear equation. Then the strong maximum principle of Hopf applies ([9, Th. 3.5]) and this allows extending the usual tangency principle of cmc hypersurfaces in Euclidean space.

Proposition 2.1 (The tangency principle) *Let Σ_1 and Σ_2 be two spacelike hypersurfaces in $\mathcal{H}^{n+1}$ with an interior or boundary tangent point p and both hypersurfaces have the same constant mean curvature. If Σ_1 lies in one side of Σ_2 around p, then Σ_1 coincides with Σ_2 in an open set around p.*

Applying successively the tangency principle everywhere Σ_1 and Σ_2 coincide, we find that $\Sigma_1 \cap \Sigma_2$ is an open set in Σ_i for $i = 1, 2$.

Remark 2.1 *The uniqueness of the Dirichlet problem associated to (1.2) is not assured. If we write Eq. (2.9) as*

$$\sum_{i,j=1}^{n} a_{ij}(x, u, Du)\frac{\partial^2 u}{\partial x_i \partial x_j} + b(x, u, Du) = 0, \quad a_{ij} = a_{ji},$$

the term

$$b(x, u, Du) = -\frac{n}{u}\left(H + \frac{1}{\sqrt{1 - |Du|^2}}\right)$$

is increasing in the variable u and we cannot apply the standard theory ([9, Th. 10.1]). However, when the domain Ω is star-shaped, there is uniqueness of solutions ([13] and [16, Cor. 12]).

3 Graphs in the steady state space

The notion of a graph for the Dirichlet problem (1.2) coincides with the graph on a domain of a slice of $\mathcal{H}^{n+1}$. We need to link this concept in both models to obtain in Section 4 the height and gradient estimates needed in the solvability of (1.2). Let $h > 0$. Given a domain $\Omega \subset \mathbb{R}^n$, take $\Omega \times \{h\} \subset L(h)$. If f is a function on Ω, we associate to each $q \in \Omega \times \{h\}$ the point of the geodesic of $\mathcal{H}^{n+1}$ passing by q and orthogonal to $L(h)$ at distance $f(q)$ from $L(h)$. In the upper half-space model, this type of geodesic is a vertical straight line, so a graph on $\Omega \times \{h\}$ is an Euclidean graph $x_{n+1} = u(x_1, \ldots, x_n)$ on Ω where $u = he^f$. In the model of $\mathcal{H}^{n+1}$ as a subset of $\mathbb{S}^{n+1}_1$, the orthogonal geodesic to $\Omega \subset L_\tau$ at $q \in \Omega$ is $\gamma(t) = \cosh(t)q + \sinh(t)(-q + a/\tau)$. If f is a function on Ω, the graph Σ_f of f is $\Sigma_f = \{X(q) = \cosh(f(q))q + \sinh(f(q))(-q + a/\tau) : q \in \Omega\}$. Notice that

$$\langle X, a\rangle = \tau e^{-f}. \tag{3.10}$$

The tangent space of Σ at $X(q)$ is

$$T_{X(q)}\Sigma_f = \{-\langle \nabla f, v\rangle e^{-f} q + e^{-f} v + \frac{\cosh(f)\langle \nabla f, v\rangle}{\tau} a : v \in T_q L_\tau\},$$

where ∇ is the gradient operator in L_τ. The unit normal vector field N pointing to the future is

$$N(X(f)) = \frac{1}{\sqrt{e^{-2f} - |\nabla f|^2}}\left(e^{-2f} q - \frac{1}{\tau} e^{-f}\cosh(f) a - \nabla f\right).$$

A computation on $\mathbb{S}_1^{n+1}$ gives then

$$\langle N, a\rangle = \frac{\tau e^{-f}}{\sqrt{1-e^{2f}|\nabla f|^2}}.$$

By the isometry Φ, let Σ_u denote the corresponding graph on the domain $\Phi(\Omega) \equiv \Omega \subset L(h)$ and we relate the functions f and u. If we let p stand for a point in $\mathbb{S}_1^{n+1}$ and $\Phi(p) = x \in \mathbb{R}_+^{n+1}$, it follows from (3.10) that

$$\langle p, a\rangle = \frac{1}{u} = \tau e^{-f}, \tag{3.11}$$

so $h = 1/\tau$. We also express N defined in $\Sigma \subset \mathbb{S}_1^{n+1}$ with the Gauss map N' given in (2.8). By the isometry Φ between both models we find $\langle N, a\rangle = (x_{n+1} \circ N')/x_{n+1}$ and thus

$$\langle N, a\rangle = \frac{1}{u\sqrt{1-|Du|^2}}. \tag{3.12}$$

From the expression of N' in (2.8) we have

$$\langle N', \mathbf{e}_{n+1}\rangle = \frac{-1}{\sqrt{1-|Du|^2}}. \tag{3.13}$$

Two examples of graphs are slices and hyperbolic planes. For a slice $L(h)$, we have $u = h$, $N' = \mathbf{e}_{n+1}$ and $\langle N', \mathbf{e}_{n+1}\rangle = -1$. For a hyperbolic plane $\mathbb{H}^n(r;t)$, $t \in \mathbb{R}$, we have $u(x_1,\ldots,x_n) = t + \sqrt{x_1^2+\ldots+x_n^2-r^2}$ and $\langle N', \mathbf{e}_{n+1}\rangle = -\sqrt{x_1^2+\ldots+x_n^2-r^2}/r$. The control of the functions $\langle p, a\rangle$ and $\langle N, a\rangle$ given in (3.11) and (3.12) respectively will be decisive in the proof of Theorem 1.1 when we establish a priori C^1 estimates of a solution u. These estimates will be based on the expressions of the Laplacians of $\langle p, a\rangle$ and $\langle N, a\rangle$. For the function $\langle p, a\rangle$, we are identifying p with its image $\psi(p)$ by the immersion $\psi : \Sigma \to \mathbb{S}_1^{n+1}$. With respect to the induced metric on Σ via ψ, the Laplacian of $\langle p, a\rangle$ (or $\langle \psi(p), a\rangle$) is

$$\Delta\langle p, a\rangle = -n\langle p, a\rangle + nH\langle N, a\rangle. \tag{3.14}$$

If we now suppose that the mean curvature is constant, the Laplacian of $\langle N, a\rangle$ is

$$\Delta\langle N, a\rangle = |\sigma|^2\langle N, a\rangle - nH\langle p, a\rangle, \tag{3.15}$$

where σ is the second fundamental form of ψ. See [16].

4 The method of continuity

The technique used in the proof of Theorem 1.1 is the method of continuity [9]; in the context of the prescribed mean curvature equation, see e.g., [2, 6, 14]. After an isometry of $\mathcal{H}^{n+1}$, we assume that $h = 1$, that is, Ω is included in the slice $L(1)$. In this slice, the curvature $\partial\Omega$ coincides with the Euclidean one as subset of $\mathbb{R}^n$. The method of continuity considers the family of Dirichlet problems

$$\begin{cases} Q_{H(t)}[u] = 0 & \text{in } \Omega \\ u = 1 & \text{along } \partial\Omega \end{cases} \tag{4.16}$$

where $H(t) = t(1+H) - 1$ and $t \in [0,1]$. We show that the subset of $[0,1]$ defined by

$$A = \{t \in [0,1] : \exists u_t \in C^{2,\alpha}(\Omega), \text{ such that } Q_{H(t)}(u_t) = 0 \text{ and } u_{t|\partial\Omega} = 1\}$$

is non-empty, closed and open in $[0,1]$. In such a case, $1 \in A$, proving the existence of a solution $u \in C^{2,\alpha}(\overline{\Omega})$. As H is constant and Ω is smooth, any $C^{2,\alpha}$ solution will be smooth on $\overline{\Omega}$ [9, Th. 6.17], proving Theorem 1.1.
Let us observe that if $H = -1$ then the solution of (1.2) for $\varphi = 1$ is the constant function $u = 1$. Moreover, this solution is unique as a consequence of the tangency principle comparing Σ_u with the slices $L(c)$, for $c > 0$.
The proof that A is closed follows once we establish a priori C^1 estimates of the prospective solutions of (1.2), that is, height and gradient estimates for every solution u of (1.2), [9] . The convexity condition (1.3) is required first to obtain an estimate of the C^0 norm of the solution comparing the graph with hyperbolic planes of type $\mathbb{H}^n(r;c)$ and, in addition, these hyperbolic planes will provide the boundary gradient estimates. Establishing gradient estimates $|Du|$ goes through to prevent that $|Du| \to 1$ at a point of $\overline{\Omega}$, or in the terminology of Marsden and Tipler, that the hypersurface cannot 'go null' [15, p. 124]. This is the hard part of the proof of Theorem 1.1 and it will be proved in the next subsections.

4.1 Height estimates

We introduce the next notation. For each $t \geq 0$, the light cone with apex $\xi = (0, \ldots, 0, t)$ is defined by $\mathcal{C}_t = \{x \in \mathbb{R}^{n+1}_+ : \langle x - \xi, x - \xi \rangle = 0\}$. The apex ξ separates the cone in two half-cones and let $\mathcal{C}_t^+$ denote the upper half-cone. Let $\text{int}(\mathcal{C}_t^+)$ be the convex domain of $\mathbb{R}^{n+1}_+$ bounded by $\mathcal{C}_t^+$.
Let $-1 < H < 0$. Let u be a solution of (1.2) for $u = 1$ along $\partial\Omega$. By the tangency principle comparing with slices $L(h)$ for values of h going from $h = \infty$ to $h = 1$, we have $u < 1$ in Ω. For any $t \in [0,1)$, let the upper light cone $\mathcal{C}_t^+$. Denote by $\Omega_\rho \subset \mathbb{R}^n$ the ball of radius $\rho > 0$ centered at the origin O and we suppose $\Omega \subset \Omega_1$. Since the inclusion $\Omega \subset \Omega_1$ is strict, let $\rho_0 < 1$ be a number such that $\overline{\Omega} \subset \Omega_{\rho_0}$. As $u = 1$ on $\partial\Omega$, the spacelike condition $|Du| < 1$ and the convexity of Ω imply that $|u(q) - u(q')| < |q - q'|$ whenever $q, q' \in \Omega$ for $q \neq q'$. This inequality together with $\partial\Sigma_u \subset \Omega_{\rho_0} \times \{1\}$ implies $\Sigma_u \subset \text{int}(\mathcal{C}^+_{1-\rho_0})$: see Fig. 1. Because the apex of $\mathcal{C}^+_{1-\rho_0}$ is $(1-\rho_0)p_0$, where $p_0 = (0,0,1)$, then we find

$$1 - \rho_0 < x_{n+1}(p) \leq 1 \tag{4.17}$$

for all $p \in \Sigma_u$, obtaining the uniform height estimate for u.

4.2 Boundary gradient estimates

We establish an a priori gradient estimate along the boundary $\partial\Omega$.

Proposition 4.1 *Under the assumptions of Theorem 1.1, we have*

$$\sup_{\partial\Omega}|Du|\leq\sqrt{1-H^2} \tag{4.18}$$

for every solution u of the Dirichlet problem (1.2).

Proof. If $H=-1$, we know $u=1$. Then $|Du|=0$ and (4.18) is trivial. Let $-1<H<0$. Consider the uniparametric family of hyperbolic planes $\{\mathbb{H}^n(r;Hr):r\in(0,r_1]\}$, where

$$r_1=\frac{H}{H^2-1}.$$

These hyperbolic planes satisfy the following properties:

(i) $\mathbb{H}^n(r;Hr)\cap L(1)=\partial\Omega_{R(r)}\times\{1\}$, where $R(r)=\sqrt{(1-Hr)^2-r^2}$. In the interval $(0,r_1]$, we have $R(r)>1$ and the function $R(r)$ is increasing in r taking all values between 1 and $1/\sqrt{1-H^2}$.

(ii) The vertex of $\mathbb{H}^n(r;Hr)$ is $V(r)=(H+1)rp_0$. Then $V(r)$ goes from O at $r=0$ until $V(r_1)=p_0/2$.

(iii) The intersection $\mathbb{H}^n(r;Hr)\cap\mathcal{C}_0$ is a sphere of radius

$$z(r)=\frac{r(H^2-1)}{2H}$$

included in the slice $L(z(r))$. The part of $\mathbb{H}^n(r;Hr)$ inside the open set $\mathrm{int}(\mathcal{C}_0^+)$ lies below $L(z(r))$. The function $z(r)$ is increasing on r, with $z(0)=0$.

We now proceed to prove the estimate (4.18). Let $r_0>0$ be sufficiently close to $r=0$ such that $z(r_0)<1-\rho_0$, where ρ_0 is the number obtained in (4.17). Because $\mathcal{C}^+_{1-\rho_0}\subset\mathrm{int}(\mathcal{C}_0^+)$, then the property (iii) implies that $\mathbb{H}^n(r_0;Hr_0)$ does not intersect Σ_u and that $\mathbb{H}^n(r_0;Hr_0)$ lies below Σ_u (with respect to the height coordinate x_{n+1}). Let us observe that $\mathbb{H}^n(r_0;Hr_0)\cap L(1)=\partial\Omega_{R(r_0)}\times\{1\}$ with $R(r_0)>1$. We increase r from the value $r=r_0$ to $r=r_1$. By the property (i) and as the radius $R(r)$ of the ball $\Omega_{R(r)}$ increases in r, the intersection $\mathbb{H}^n(r;Hr)\cap L(1)$ does not meet $\partial\Sigma_u$ because the assumption on $\partial\Sigma_u$ asserts that $\partial\Sigma_u\subset\Omega_1\times\{1\}$. See Fig. 1. Since the mean curvature of every hyperbolic plane $\mathbb{H}^n(r;Hr)$ is H, the tangency principle assures that there are no touching points between Σ_u and $\mathbb{H}^n(r;Hr)$ for all $r_0\leq r\leq r_1$. In particular, and once we arrive at $r=r_1$, the graph Σ_u lies above the hyperbolic plane $\mathbb{H}^n(r_1;Hr_1)$, finding a new lower height estimate for Σ_u, namely,

$$\frac{H}{H-1}<x_{n+1}(p),\ p\in\Sigma_u.$$

Let $\rho_1 = 1/\sqrt{1-H^2}$. We recall by the property (i) that the intersection of $\mathbb{H}^n(r_1; Hr_1)$ with the slice $L(1)$ is an Euclidean sphere of radius ρ_1. The assumption on the k-convexity of Ω says $\kappa \geq 1/\rho_1$, so the domain Ω has the following Blaschke's outer rolling sphere property: for every point in the boundary $\partial\Omega$ there exists a ball in $L(1)$ of radius ρ_1 touching that point and the interior of the ball includes Ω (e.g. [11]). Thus, for each $q \in \partial\Omega$, it is possible to horizontally move $\Omega_{\rho_1} \times \{1\}$ in the hyperplane $L(1)$ so in the new position, we have $q \in \partial\Omega_{\rho_1} \cap \partial\Omega$ and $\Omega \subset \Omega_{\rho_1}$. Consequently, and for each $q \in \partial\Omega$, we take the initial hyperbolic plane $\mathbb{H}^n(r_1; Hr_1)$ and let us move it horizontally until that touches $\Omega \times \{1\}$ at the point $q^* = (q, 1)$ such as it was described previously. Then there is a neighborhood of q in Ω such that the graph Σ_u of u lies locally sandwiched between the hyperbolic plane $\mathbb{H}^n(r_1; Hr_1)$ and the slice $L(1)$. This implies that at the point q^*, the value $\langle N'(q^*), \mathbf{e}_{n+1}\rangle$ is bounded from above and from below by the value at q^* of the scalar product of $\mathbf{e}_{n+1}$ with the Gauss maps of $L(1)$ and $\mathbb{H}^n(r_1; Hr_1)$, respectively. For the slice $L(1)$, this value is -1 and for $\mathbb{H}^n(r_1; Hr_1)$, it is $-\sqrt{R(r_1)^2 - r_1^2}/r_1 = 1/H$. Then we have the inequality $1/H < \langle N'(q^*), \mathbf{e}_{n+1}\rangle < -1$. It follows from (3.12) that $|Du|(q) < \sqrt{1-H^2}$ and thus

$$\sup_{\partial\Omega} |Du| < \sqrt{1-H^2}. \tag{4.19}$$

In particular, the gradient estimate for $|Du|$ does not depend on q.

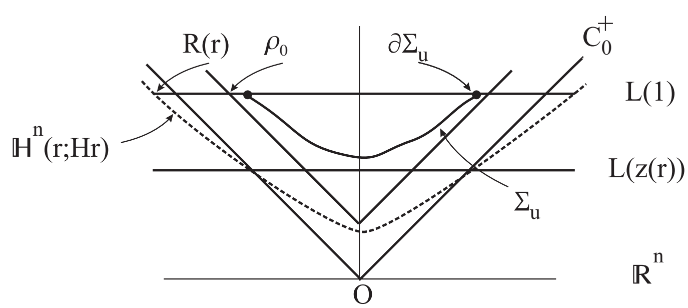

Figure 1: Proof of the boundary gradient estimates in Theorem 1.1

4.3 Gradient estimates

We will obtain a global a priori estimate for $|Du|$ on the domain Ω based on estimates of $|Du|$ along $\partial\Omega$. This is accomplished studying the Jacobi equation that satisfies the Gauss map of a spacelike cmc hypersurface. The next result holds for $H \leq 0$ and guarantees that the global estimates of the gradient reduce to get boundary gradient estimates. Here we consider a general case of boundary data in the Dirichlet problem assuming $u = \varphi$ along $\partial\Omega$, where $\varphi \in C^0(\partial\Omega)$. Let us introduce the next notation:

$$\varphi_m = \inf_{\partial\Omega} \varphi, \quad \varphi_M = \sup_{\partial\Omega} \varphi.$$

Proposition 4.2 *Let $H \leq 0$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let u be a bounded solution of the Dirichlet problem (1.2). Denote*

$$u_m = \inf_{\Omega} u, \quad u_M = \sup_{\Omega} u.$$

If there exists a positive constant $C_1 < 1$ such that

$$\sup_{\partial\Omega} |Du| \leq C_1,$$

then there is a constant $C = C(H, \varphi_m, u_m, u_M, C_1) < 1$ such that

$$\sup_{\Omega} |Du| \leq C. \tag{4.20}$$

Proof. Denote by Σ_u the graph of u in both models of $\mathcal{H}^{n+1}$. Since the isometry Φ inverts the orientations, $H \geq 0$ when $\langle N, a\rangle > 0$ in the model of $\mathcal{H}^{n+1}$ a subset of $\mathbb{S}_1^{n+1}$. Then (3.12) implies

$$\sup_{\partial\Sigma_u} \langle N, a\rangle = \sup_{\partial\Omega} \frac{1}{u\sqrt{1-|Du|^2}} \leq \frac{1}{\varphi_m\sqrt{1-C_1^2}} := C_2,$$

where $C_2 = C_2(\varphi_m, C_1)$. On the other hand, from the identities (3.14) and (3.15), we have

$$\Delta(-H\langle p, a\rangle + \langle N, a\rangle) = (|\sigma|^2 - nH^2)\langle N, a\rangle \geq 0$$

because $|\sigma|^2 \geq nH^2$. Then $-H\langle p, a\rangle + \langle N, a\rangle$ is a subharmonic function and the maximum principle yields

$$-H\langle p, a\rangle + \langle N, a\rangle \leq \sup_{\partial\Sigma_u}(-H\langle p, a\rangle + \langle N, a\rangle) \leq \sup_{\partial\Sigma_u}\langle N, a\rangle \leq C_2. \tag{4.21}$$

Hence (3.11) implies

$$\langle N, a\rangle \leq H\langle p, a\rangle + C_2 \leq \frac{H}{u_m} + C_2 := C_3, \tag{4.22}$$

where $C_3 = C_3(H, u_m, C_2)$. We replace this inequality into (3.12) obtaining

$$\frac{1}{u\sqrt{1-|Du|^2}} \leq C_3.$$

Thus

$$|Du| \leq \sqrt{1 - \frac{1}{C_3^2 u^2}} \leq \sqrt{1 - \frac{1}{C_3^2 u_M^2}} := C, \tag{4.23}$$

and this proves the result.

The interior gradient estimate needed for Theorem 1.1 is now a consequence of Propositions 4.1 and 4.2. However, we can give explicitly an expression of this estimate. From (4.19) and (4.21) (now $0 < H < 1$ and $h = 1$), we have

$$-H\langle p, a\rangle + \langle N, a\rangle \leq \sup_{\partial\Sigma_u}\left(-H + \frac{1}{\sqrt{1-|Du|^2}}\right) < \frac{1-H^2}{H}. \tag{4.24}$$

Hence we obtain $\langle N, a\rangle$ and using the value of $\langle p, a\rangle$ in (3.11), we have

$$\langle N, a\rangle < \frac{H}{u} + \frac{1-H^2}{H}.$$

Using (3.12) again, we conclude

$$\sup_{\Omega} |Du| < \sqrt{1-H^2}. \tag{4.25}$$

5 Proof of Theorem 1.1

Once obtained in the previous section the height and gradient estimates for a solution of the Dirichlet problem, we prove definitively Theorem 1.1. Again suppose $h = 1$ on the boundary values in (1.2). When $H = -1$, then $u = 1$ is a solution of (1.2). Suppose $-1 < H < 0$ and we use the notation of Section 4. In the continuity method, let u_t be a solution of the Dirichlet problem (4.16). First, we note that A is non-empty because $u = 1$ is a solution so $0 \in A$.

In order to prove that A is a closed subset of $[0, 1]$, we have to find C^1 estimates of the solutions u_t for $t \in A$. The height estimates (4.17) hold for all values of the mean curvature between -1 and 0. From the global gradient estimate (4.25), we have

$$|Du_t| < \sqrt{1-H(t)^2} \leq \sqrt{1-H(1)^2} = \sqrt{1-H^2}. \tag{5.26}$$

The proof that A is open of $[0, 1]$ is a consequence of the implicit function theorem. The linearization of the mean curvature is the Jacobi operator corresponding to the second variation of the area functional and it is given by $L = \Delta - |\sigma|^2 + n$ ([16]). In this context, if we show that the kernel of L is trivial, then L is a self-adjoint Fredholm operator of index 0, hence L is invertible. It follows from the implicit function theorem for Banach spaces that if the Dirichlet problem (1.2) can solve for the value $H = H_0$, then it can also solve in an open interval around H_0. However we point out that we cannot apply the standard theory ([9, Th. 10.1]).

We prove the openness of A by considering the eigenvalue problem for L

$$\left\{ \begin{array}{ll} L[f] + \lambda f = 0 & \text{on } \Sigma \\ f = 0 & \text{on } \partial\Sigma. \end{array} \right. \tag{5.27}$$

Associated to L there is a quadratic form Q acting on the subspace $C_0^\infty(\Sigma)$ of smooth functions on Σ satisfying the condition $f = 0$ on $\partial\Sigma$. Here Q is $Q(f) = -\int_\Sigma f \cdot L[f]\, d\Sigma$. We say that Σ is strongly stable if $Q(f) \geq 0$ for all $f \in C_0^\infty(\Sigma)$ which, in terms of the spectrum of L, is equivalent to the first eigenvalue $\lambda_1(L)$ of L is non-negative so the kernel of L is zero. The next result closely follows [7, Th. 1].

Lemma 5.1 *Let Σ be a compact cmc spacelike hypersurface in $\mathcal{H}^{n+1}$ and assume that there exists a function g on Σ such that $g > 0$ on Σ and $L[g] \leq 0$. Then Σ is strongly stable.*

Proof. Let $f \in C^\infty(\Sigma)$ with $f = 0$ on $\partial\Sigma$. Define $h = \log(g)$. Since $L[g] \leq 0$, then $\Delta h \leq |\sigma|^2 - n - |\nabla h|^2$. Multiplying by f^2 and integrating on Σ, we have

$$\int_\Sigma (|\sigma|^2 f^2 - nf^2)\, d\Sigma - \int_\Sigma f^2 |\nabla h|^2\, d\Sigma \geq \int_\Sigma f^2 \Delta h\, d\Sigma. \tag{5.28}$$

As $\mathrm{div}(f^2\nabla h) = f^2\Delta h + 2f\langle \nabla f, \nabla h\rangle$, the divergence theorem yields

$$\begin{aligned} -\int_\Sigma f^2\Delta h\, d\Sigma &= 2\int_\Sigma f\langle \nabla f, \nabla h\rangle\, d\Sigma \leq 2\int_\Sigma |f||\nabla h||\nabla f|\, d\Sigma \\ &\leq \int_\Sigma f^2|\nabla h|^2\, d\Sigma + \int_\Sigma |\nabla f|^2\, d\Sigma. \end{aligned}$$

Combining this inequality with (5.28), we get $Q(f) \geq 0$. In fact, if $f \neq 0$, then $Q(f) > 0$ because in case of $Q(f) = 0$, we find that f is proportional to h, contradicting that $h \neq 0$ along ∂M.

Lemma 5.2 *The subset A is open in* $[0, 1]$.

Proof. Let $t_0 \in A$ and set $H_0 = H(t_0)$. Denote Σ the graph of u_{t_0}. Using Lemma 5.1, the kernel of the Jacobi operator L is zero provided we find a function $g > 0$ on Σ such that $L[g] \leq 0$. Consider the model of $\mathcal{H}^{n+1}$ as a subset of $\mathbb{S}^{n+1}_1$ where we now have $0 < H_0 \leq 1$ and $\langle N, a\rangle > 0$. Define

$$g = \langle p, a\rangle - H_0\langle N, a\rangle.$$

By making use of (3.14) and (3.15) we obtain

$$L[g] = (nH_0^2 - |\sigma|^2)\langle p, a\rangle \leq 0$$

since $|\sigma|^2 \geq nH_0^2$. It remains to prove that $g > 0$ on Σ. From inequality (4.24), we have

$$H_0\langle N, a\rangle < H_0^2\langle p, a\rangle + 1 - H_0^2.$$

Hence we deduce

$$g = \langle p, a\rangle - H_0\langle N, a\rangle > (1 - H_0^2)(\langle p, a\rangle - 1) > 0,$$

where we have used that $u \leq 1$ and the relation between u and $\langle p, a\rangle$ given in (3.11).

Finally, this lemma concludes the proof of Theorem 1.1.

Remark 5.1 *One may use a similar argument to give an alternative proof that the analogous subset A in [16, p. 931] is open in* $[0, 1]$. *Recall that it was proved in [16, Cor. 8] that a uniform gradient estimate holds for a solution u of the Dirichlet problem (1.2) when $\varphi = h$ and $H > 1$ in the de Sitter model for $\mathcal{H}^{n+1}$, namely,*

$$\sup_\Omega |Du| \leq \frac{\sqrt{H^2 - 1}}{H}. \tag{5.29}$$

Now Lemma 5.1 applies by letting $g = \langle N, a\rangle$. So we have $g > 0$ and taking into account (3.11), (3.12) and (3.15), we obtain

$$L[g] = -nH\langle p, a\rangle + n\langle N, a\rangle = \frac{n}{u}\left(-H + \frac{1}{\sqrt{1-|Du|^2}}\right) \leq \frac{n}{u}(-H + H) = 0,$$

where we have used the gradient estimate (5.29) in the last inequality.

Remark 5.2 *The solutions obtained in Theorem 1.1, as well as the Montiel solutions in [16], are strongly stable.*

Remark 5.3 *We also have the following result: If $H \leq 0$ in the de Sitter space model of $\mathcal{H}^{n+1}$, every spacelike H-graph on a bounded domain of a slice is strong stable. The proof follows taking the function $g = \langle p, a\rangle$ which is positive and observing that $L[g] = nH\langle N, a\rangle - |\sigma|^2\langle p, a\rangle \leq 0$. Notice that this result holds for every domain Ω and boundary data φ along $\partial\Omega$, in contrast to the graphs that appear in [16] and in Theorem 1.1 where the boundary of the graph is a closed curve contained in a slice ($\varphi = h$).*

6 Other gradient estimates results

We have seen in Proposition 4.2 that boundary gradient estimates transfer into the interior of the domain. In this section we will extend these estimates for other assumptions on the value of H and boundary data φ. The following results have its own interest although they have not been used in this paper. The first one improves the estimate $|Du|$ given in (4.20) when $H \leq -1$.

Proposition 6.1 *Let $H \leq -1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let u be a bounded solution of the Dirichlet problem (1.2). If there exists a positive constant $C_1 < 1$ such that*

$$\sup_{\partial\Omega} |Du| \leq C_1,$$

then there is a constant $C = C(H, \varphi_m, u_M, C_1) < 1$ such that

$$\sup_{\Omega} |Du| \leq C.$$

Proof. We consider the model of $\mathcal{H}^{n+1}$ as subset of $\mathbb{S}^{n+1}_1$ where we know that $H \geq 1$. Using that $|\sigma|^2 \geq nH^2$, we have from (3.15), (3.11) and (3.12)

$$\Delta\langle N, a\rangle \geq nH(H\langle N, a\rangle - \langle p, a\rangle) = \frac{nH}{u\sqrt{1-|Du|^2}}(H - \sqrt{1-|Du|^2}) \geq 0.$$

The maximum principle yields $\langle N, a\rangle \leq \sup_{\partial\Sigma_u}\langle N, a\rangle$. As $u \leq u_M$, we have from (3.12) that

$$\frac{1}{u_M\sqrt{1-|Du|^2}} \leq \frac{1}{u\sqrt{1-|Du|^2}} \leq \sup_{\partial\Omega}\frac{1}{u\sqrt{1-|Du|^2}} = \frac{1}{\varphi_m\sqrt{1-C_1^2}}$$

obtaining definitively

$$|Du|^2 \leq 1 - \frac{\varphi_m^2}{u_M^2}(1 - C_1^2). \tag{6.30}$$

Let us compare the estimates obtained in this result as well as Proposition 4.2 with the ones in [16]. Montiel gets directly the interior gradient estimates (5.29) from the hypothesis of mean convexity of the domain Ω. Now our estimates are derived once we have boundary gradient estimates and they hold for *any* bounded domain and *any* boundary condition φ. When the value of the mean curvature is non-negative, it is possible to obtain the same gradient estimate in the interior and in the boundary of Ω. Exactly we have:

Proposition 6.2 *Let $H \geq 0$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let u be a bounded solution of the Dirichlet problem (1.2). If there exists a positive constant $C_1 < 1$ such that*

$$\sup_{\partial\Omega} |Du| \leq C_1,$$

then there is a constant $C = C(H, \varphi_m, \varphi_M, C_1) < 1$ such that

$$\sup_{\Omega} |Du| \leq C.$$

In the particular case that $\varphi = h$ on $\partial\Omega$, we have

$$\sup_{\Omega} |Du| = \sup_{\partial\Omega} |Du|.$$

Proof. As $H \geq 0$, the tangency principle comparing Σ_u with the slices $L(h)$ gives $u \leq \varphi_M$. On the other hand, following the same steps as in the above proof, and as $H \leq 0$ in the model of $\mathbb{S}_1^{n+1}$, then $\Delta\langle N, a\rangle \geq 0$. The maximum principle gives the same inequality (6.30) but now $u_M = \varphi_M$. The second part is immediate because $\varphi_m = \varphi_M$.

7 Discussion and conclusions

In this paper we have studied the Dirichlet problem for the prescribed mean curvature equation on a domain Ω of a slice in the steady state space $\mathcal{H}^{n+1}$. This space can foliate by slices which are spacelike hypersurfaces with constant mean curvature $H = -1$. The model for $\mathcal{H}^{n+1}$ usually utilized in the literature has been as a subset of the De Sitter space $\mathbb{S}_1^{n+1}$. However, here we have worked with the upper half-space model of $\mathcal{H}^{n+1}$ which is analogue to the hyperbolic space in Lorentz-Minkowski space $\mathbb{R}_1^{n+1}$. In this model a slice $L(h)$ is just a horizontal hyperplane of equation $x_n = h$. Notice that the space is not symmetric with respect to a slice $L(h)$ and the value $H = -1$ is critical for the geometrical behavior of a spacelike constant mean curvature hypersurface of $\mathcal{H}^{n+1}$. In this sense, Montiel showed the existence of solutions when $H < -1$ assuming that Ω is strictly convex. In our work we studied the solvability of the Dirichlet problem for values H with $H > -1$, a scenario that has not been previously considered. Now the graph of a solution lies in the halfspace of $\mathcal{H}^{n+1}$ determined by $L(h)$ pointing to the past and in contrast to the case $H < -1$, there are restrictions on the size of the domain Ω, indeed, Ω can not be very large. We used the continuity method to solve the Dirichlet problem and we needed to get C^1 a priori estimates of the prospective solutions. We utilized a rolling argument of comparison with hyperbolic planes as barriers to establish the boundary gradient estimates. Here we required that the domain is κ-convex and $-1 \leq H < 0$. Once obtained the boundary gradient estimates, we proved the existence of gradient estimates in the interior of Ω. This argument is typical in elliptic PDE's theory but it contrasts to the one utilized by Montiel when $H < -1$ where the mere mean convexity assumption of Ω gives directly the needed gradient estimates. The case $H > -1$ is harder and this was reflected, for instance, on the proof of the openness step in the continuity method where the previous C^1 estimates were suitably utilized. It is desirable that the κ-convexity assumption can be replaced by a weaker hypothesis, as for example, that $\partial\Omega$ were mean convex. This is expectable when $H < -1$ but it seems more difficult when $H > -1$. We discussed and initiated this process because we obtained interior gradient estimates once that previously we deduced gradient estimates along the boundary $\partial\Omega$.

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