

Convexity of the solutions to the constant mean curvature spacelike surface equation in the Lorentz-Minkowski space

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Abstract

We prove that a spacelike graph of constant mean curvature $H \neq 0$ in the 3-dimensional Lorentz-Minkowski space over a bounded domain with pseudo-elliptic boundary is strictly convex. By a pseudo-elliptic curve we mean a closed and planar curve which intersects any branch of any hyperbola at most at five points. We also provide an example that shows that we cannot remove the assumption on the boundary being a pseudo-elliptic curve.

Keywords: spacelike hypersurface, constant mean curvature, Dirichlet problem, convex curve, strictly convex surface

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1. Introduction

A hypersurface in the Lorentz-Minkowski space \mathbb{L}^n is called spacelike if its induced metric from \mathbb{L}^n is Riemannian. Spacelike hypersurfaces of constant mean curvature (CMC) are critical points of the area functional under a suitable volume constraint [1, 2]. Such hypersurfaces play an important role in general relativity, since they can be used as initial data where the constraint equations can be split into a linear system and a nonlinear elliptic equation (see [3] and

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references therein). A summary of other reasons justifying the study of CMC spacelike hypersurfaces can be found in [4].

Along this paper we will work with compact CMC spacelike surfaces immersed in \mathbb{L}^3 with (necessarily) non-empty smooth boundary, which will be supposed to be planar and simple. We will study the influence of the geometry of the boundary on the shape of the surface. All those surfaces are known to be graphs over a spacelike plane, see [5, Corollary 12.1.8], in contrast with what happens in the Euclidean space \mathbb{R}^3 . Up to an isometry, you can assume the plane to be the $z = 0$ plane. Specifically, if Ω is a domain of \mathbb{R}^2 , then every smooth function $u \in C^\infty(\Omega)$ determines a graph over Ω given by

$$\Sigma_u = \{(x, y, u(x, y)) : (x, y) \in \Omega\} \subset \mathbb{L}^3.$$

The graph Σ_u is a CMC spacelike surface with boundary $\partial\Omega$ if and only if the function u is the solution of the Dirichlet problem

$$\begin{aligned} \operatorname{div} \left(\frac{Du}{\sqrt{1 - |Du|^2}} \right) &= 2H, \quad |Du| < 1 && \text{in } \Omega, \\ u &= 0 && \text{along } \partial\Omega, \end{aligned}$$

where H is a constant and D , div and $|\cdot|$ stand for the gradient and divergence operators, and the norm in the Euclidean plane \mathbb{R}^2 , respectively (see Section 2 for the details).

The existence and uniqueness of a function $u \in C^2(\Omega) \cup C^0(\bar{\Omega})$ being the solution of the previous problem are always assured for any bounded domain Ω , see [6] and [5, Theorem 12.2.2]. There are two particular cases in which the solution is known. When $H = 0$, u must be zero too. When $H \neq 0$ and Ω is a round disc of radius R (which is supposed to be centered at the origin), then

$$u(x, y) = \sqrt{x^2 + y^2 + \frac{1}{H^2}} - \sqrt{R^2 + \frac{1}{H^2}},$$

that is, Σ_u is a hyperbolic cap [7].

In contrast with the Lorentzian case, the existence of the Dirichlet problem for the CMC equation in the Euclidean space is not assured and it depends on

the domain Ω . In this sense, a classical result due to Serrin [8] establishes the existence and uniqueness of a solution to the Dirichlet problem for any given boundary condition if and only if the curvature κ of $\partial\Omega$ satisfies $\kappa \geq 2H \geq 0$.

Another significant difference appears when estimating the height of the solution. While $1/|H|$ is a bound in the Euclidean case with zero boundary condition, see [9], the only possible bounds in the Lorentzian case involve either the diameter of Ω [5, Corollary 12.4.6], or the area of the graph [10].

As we have mentioned, we are interested in studying the influence of the geometry of the boundary on Σ_u . In particular, we ask if the convexity of $\partial\Omega$ is inherited by Σ_u . It is a classical problem to study if the convexity of the domain of a boundary value problem associated to an elliptic partial differential equation implies convexity of the solution [11, 12, 13, 14, 15, 16]. In this paper we consider bounded domains whose boundary curve is planar, with the added property that it intersects any branch of any hyperbola at most at five points. We call this type of curves *pseudo-elliptic curves* and we will prove them to be convex. Let us observe that ellipses are examples of such curves.

The paper is organized as follows. In Section 2, we present some basic preliminaries on spacelike hypersurfaces in \mathbb{L}^n . In particular, in Lemma 1 we prove that any compact spacelike non-planar hypersurface Σ in \mathbb{L}^n with boundary contained in a hyperplane Π , necessarily has an elliptic point. Section 3 is devoted to prove Theorem 3,

Let Σ be a spacelike compact surface in \mathbb{L}^3 with constant mean curvature $H \neq 0$, such that its boundary is a planar curve which is pseudo-elliptic. Then Σ has negative Gaussian curvature in all its interior points. In particular, Σ is a convex surface.

The proof of this result follows the ideas of Chen and Huang in [11] where, inspired by a previous argument by Alexandrov [17], they compare a CMC graph in \mathbb{R}^3 with half-cylinders. In the present result we will strongly use that the ambient space is \mathbb{L}^3 because our comparison surfaces are connected components of hyperbolic cylinders, which are entire graphs over \mathbb{R}^2 . Finally, in Section 4

we present an example that shows that the hypothesis in Theorem 3 of asking the boundary of Σ to be a pseudo-elliptic curve cannot be removed.

2. Preliminaries

Let \mathbb{L}^n be the n -dimensional Lorentz-Minkowski space, that is \mathbb{R}^n endowed with the metric

$$\langle, \rangle = dx_1^2 + \dots + dx_{n-1}^2 - dx_n^2,$$

where (x_1, \dots, x_n) are the canonical coordinates in \mathbb{R}^n .

A (connected) hypersurface Σ^{n-1} in \mathbb{L}^n is said to be a spacelike hypersurface if \mathbb{L}^n induces a Riemannian metric on Σ , which is also denoted by \langle, \rangle . Given a spacelike hypersurface Σ we can choose a unique future-directed unit normal vector field N on Σ . We will refer to N as the future-pointing Gauss map of Σ . Let $\bar{\nabla}$ and ∇ denote the Levi-Civita connections in \mathbb{L}^n and Σ , respectively. Then the Gauss and Weingarten formulae for the spacelike hypersurface Σ are

$$\bar{\nabla}_X Y = \nabla_X Y - \langle AX, Y \rangle N \quad (1)$$

and

$$AX = -\bar{\nabla}_X N, \quad (2)$$

respectively, for any tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$, where $A : \mathfrak{X}(\Sigma) \rightarrow \mathfrak{X}(\Sigma)$ stands for the shape operator of Σ with respect to N . The mean curvature function of Σ related to N is defined by

$$H = -\frac{1}{n-1} \operatorname{tr} A = -\frac{1}{n-1} (k_1 + \dots + k_{n-1})$$

where k_i , $i = 1, \dots, n-1$, stand for the principal curvatures of Σ .

It is well known that there exists no closed spacelike hypersurface in \mathbb{L}^n [18, 19]. Therefore, every compact spacelike hypersurface Σ in the Lorentz-Minkowski space necessarily has non-empty smooth boundary.

As a first result, let us see that every compact spacelike non-planar hypersurface Σ in \mathbb{L}^n with planar boundary $\partial\Sigma$ necessarily has an elliptic point, that is, a point at which all the principal curvatures have the same sign.

Lemma 1. *Let Σ be a compact spacelike hypersurface in \mathbb{L}^n such that $\partial\Sigma \subset \Pi$, where Π is a hyperplane. If Σ is not contained in Π , it necessarily has an elliptic point.*

Proof. Since $\partial\Sigma$ is closed and spacelike, Π is a spacelike hyperplane. Up to an isometry of \mathbb{L}^n , we assume that Π is the hyperplane of equation $x_n = 0$ and that there exist points of Σ under Π .

It is easy to prove the existence of a point q such that the function $f : \Sigma \rightarrow \mathbb{R}$ defined by $f(p) = \langle p - q, p - q \rangle$ is negative and attains its maximum at an interior point p^* of Σ . Indeed, if we take a disk in Π centered at the origin and containing $\partial\Sigma$, we can consider the family of hyperboloids whose upper connected component contains the boundary of the disk and whose center is located under Σ . It is enough to choose one of them intersecting Σ , and denote its center by q . At the intersection, f takes values bigger than at $\partial\Sigma$. And so, it attains its maximum at an interior point.

Therefore,

$$\nabla f(p^*) = 0 \tag{3}$$

and

$$\text{Hess}f_{p^*}(v, v) \leq 0 \quad \forall v \in T_{p^*}\Sigma, \tag{4}$$

where ∇ and Hess denote the gradient and Hessian operators of Σ .

Observe that for every $X \in \mathfrak{X}(\Sigma)$ it holds

$$X(f) = 2\langle p - q, X \rangle = \langle 2(p - q)^\top, X \rangle,$$

which implies that

$$\nabla f(p) = 2(p - q)^\top = 2(p - q + \langle p - q, N(p) \rangle N(p)), \quad p \in \Sigma.$$

Thus, by (3) we get

$$\langle p^* - q, N(p^*) \rangle = -R, \tag{5}$$

where $R > 0$ is such that $f(p^*) = -R^2 < 0$.

On the other hand, by the Gauss (1) and Weingarten (2) equations we obtain

$$\nabla_X \nabla f = 2(X - \langle p - q, N \rangle AX)$$

for every tangent vector field $X \in \mathfrak{X}(\Sigma)$. Therefore,

$$\text{Hess}f_p(v, v) = 2(\langle v, v \rangle - \langle p - q, N(p) \rangle \langle A_p(v), v \rangle), \quad p \in \Sigma, \quad v \in T_p \Sigma. \quad (6)$$

Let $\{e_1, \dots, e_{n-1}\}$ be an orthonormal basis of $T_{p^*} \Sigma$ which diagonalizes A_{p^*} , that is $A_{p^*}(e_i) = k_i(p^*)e_i$, then (4) and (6) yield

$$\text{Hess}f_{p^*}(e_i, e_i) = 2(1 - \langle p^* - q, N(p^*) \rangle k_i(p^*)) \leq 0, \quad i = 1, \dots, n-1,$$

which jointly with (5) implies

$$k_i(p^*) \leq -\frac{1}{R} < 0, \quad i = 1, \dots, n-1.$$

□

Remark 1. Let us observe that a geometric proof of the above result can be given. It consists in comparing Σ with an $(n-1)$ -dimensional hyperbolic space, $\mathbb{H}^{n-1}(R)$, of big enough radius, tangent to Σ at an interior point p^* , and such that Σ is contained in the closure of the region determined by Π and $\mathbb{H}^{n-1}(R)$ (see Figure 1). By comparison of the normal curvatures of both surfaces at p^* we conclude that p^* is an elliptic point.

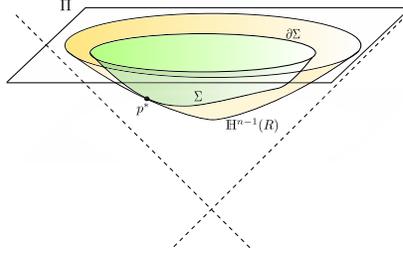


Figure 1: Existence of an elliptic point in Σ by comparison with a hyperbolic space $\mathbb{H}^{n-1}(R)$.

Remark 2. In the three dimensional case, a point $p \in \Sigma$ is elliptic if and only if the Gaussian curvature of the surface Σ , $K = -\det(A) = -\kappa_1 \kappa_2$, is negative at p . We will use this characterization in the proof of Theorem 3.

It is known that every compact spacelike hypersurface Σ in \mathbb{L}^n with simple boundary contained in a hyperplane (which can be supposed to be $x_n = 0$)

can be regarded as a spacelike graph over the region Ω bounded by $\partial\Sigma$ (see [5, Corollary 12.1.8] for the three dimensional case and notice that the proof works for the general case). That is, there exists a function $u \in C^\infty(\Omega)$ such that

$$\Sigma = \Sigma_u = \{(x_1, \dots, x_{n-1}, u(x_1, \dots, x_{n-1})) : (x_1, \dots, x_{n-1}) \in \Omega\} \subset \mathbb{L}^n.$$

Conversely, given a graph Σ_u in \mathbb{L}^n defined over a connected domain $\Omega \subset \mathbb{R}^{n-1}$, it is easy to check that it is a spacelike hypersurface if and only if $|Du| < 1$, where D and $|\cdot|$ stand for the gradient operator and the norm in the Euclidean space \mathbb{R}^{n-1} , respectively. In this case, it is well known that the mean curvature of Σ_u is given by

$$\operatorname{div} \left(\frac{Du}{\sqrt{1 - |Du|^2}} \right) = (n - 1)H,$$

where div is the divergence operator of the Euclidean space \mathbb{R}^{n-1} . Therefore, if Σ_u is a compact spacelike graph in \mathbb{L}^n with planar boundary and constant mean curvature H , the function u satisfies the Dirichlet problem

$$\begin{aligned} \operatorname{div} \left(\frac{Du}{\sqrt{1 - |Du|^2}} \right) &= (n - 1)H, & |Du| < 1 & \quad \text{in} \quad \Omega, \\ u &= 0 & & \quad \text{along} \quad \partial\Omega. \end{aligned} \quad (7)$$

Notice that the previous equation is quasilinear and locally uniformly elliptic in Ω [20]. This equation is known as the CMC spacelike hypersurface equation in \mathbb{L}^n .

3. A result on the convexity of CMC spacelike compact graphs

Before stating our main result let us define a special family of closed curves.

Definition 1. A closed curve in \mathbb{R}^2 (and up to an isometry, in any spacelike plane in \mathbb{L}^3) is said to be *pseudo-elliptic* if it intersects any branch of any hyperbola at most at five points.

Observe that ellipses are particular cases of pseudo-elliptic curves.

Proposition 2. *Pseudo-elliptic curves are convex.*

Proof. We take a pseudo-elliptic curve in \mathbb{R}^2 , γ , and we proceed by the method of reductio ad absurdum. Let p and q be two points in γ such that the open segment pq is outside the closure of the interior domain of γ . We consider the line determined by pq , r , and we work in the half-plane, Π^+ , containing a connected piece of γ between p and q . Since our curve is closed, there has to be another connected piece of γ contained in Π^+ with boundary in r . Take a line r' parallel to r , contained in Π^+ and close enough to r in order to get at least four connected pieces of γ between r and r' . Hence, we can choose two half-lines with the same initial point, not parallel to each other, and intersecting γ at least at eight points between r and r' . We can take a branch of hyperbola as close to these two half-lines as needed, so that it intersects γ in eight points, which is a contradiction.

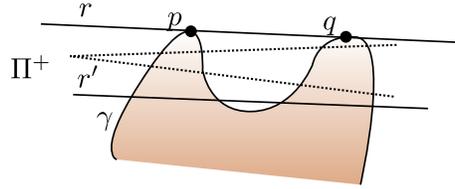


Figure 2: Technique to construct a hyperbola cutting γ at least at eight points.

□

In this section we prove the following result:

Theorem 3. *Let Σ be a spacelike compact surface in \mathbb{L}^3 with constant mean curvature $H \neq 0$, such that its boundary is a planar curve which is pseudo-elliptic. Then Σ has negative Gaussian curvature in all its interior points. In particular, Σ is a convex surface.*

Proof. Under the assumptions of Theorem 3, and up to an isometry, the surface Σ can be regarded as a constant mean curvature spacelike compact graph with $H \neq 0$ defined over a bounded domain $\Omega \subset \mathbb{R}^2$, where $\partial\Omega$ is a pseudo-elliptic curve which coincides with $\partial\Sigma$. Notice that we are identifying \mathbb{R}^2 and the plane

$z = 0$ in \mathbb{L}^3 . Thus there exists a function $u \in \mathcal{C}^\infty(\Omega)$ such that $\Sigma = \Sigma_u$, u being a solution of the Dirichlet problem (7). Without loss of generality, we can assume that $H > 0$. Observe that, by the comparison principle for quasilinear elliptic equations we get that $u < 0$ on Ω [20].

By Lemma 1, Σ_u has an interior elliptic point, that is, there exists a point $p^* \in \Sigma_u$ such that $K(p^*) < 0$.

Let us continue with the proof by assuming the existence of interior points on Σ_u with non-negative Gaussian curvature. In particular, by a continuity argument, there exists an interior point $p_0 = (x_0, y_0, u(x_0, y_0)) \in \Sigma_u$ such that $K(p_0) = 0$. Consider the upper connected component of a hyperbolic cylinder in \mathbb{L}^3 , S , with radius $r = 1/(2H)$, tangent to Σ_u at p_0 and such that its generators lines are parallel to a zero curvature direction of Σ_u at p_0 (see Figure 3). Let us recall that a hyperbolic cylinder is, up to an isometry, the spacelike surface of equation $x^2 - z^2 = -r^2$, for some positive constant r . Each connected component of these surfaces is an entire graph over \mathbb{R}^2 with constant mean curvature $|H| = 1/(2r)$ and zero Gaussian curvature.

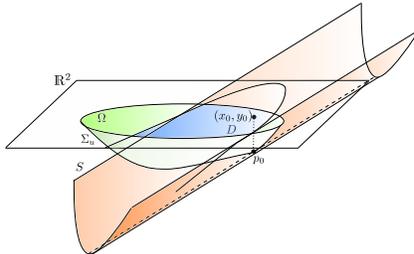


Figure 3: Comparison of Σ_u with the hyperbolic cylinder S .

The intersection of S and \mathbb{R}^2 is either a branch of a hyperbola or two parallel lines. Hence, $S \cap \mathbb{R}^2$ divides \mathbb{R}^2 into two or three domains. The piece of S with negative height is the graph of a function $v \in \mathcal{C}^\infty(\Omega')$, $v < 0$, defined in one of the domains Ω' determined by $S \cap \mathbb{R}^2$.

Define $D = \Omega \cap \Omega'$ and consider the difference function $w = u - v$ on D . Let us observe that $w(x_0, y_0) = 0$. Our next objective is to control the behavior of

the zero set of w on D ,

$$\mathcal{Z} = \{(x, y) \in D : w(x, y) = 0\}.$$

By construction of S , both Σ_u and S have the same mean curvature and zero Gaussian curvature at the point p_0 . Then, both surfaces have a contact of (at least) second order at that point. On the other hand, u and v are both solutions of the equation in (7). Using these two facts, a classical argument by Alexandrov [17] can be extended to prove that there exists a neighborhood \mathcal{U} of (x_0, y_0) such that $\mathcal{Z} \cap \mathcal{U}$ consists of at least three smooth arcs intersecting at (x_0, y_0) and dividing \mathcal{U} into $2n$ sectors with $n \geq 3$, on which the sign of w alternates. The proof of the result is identical to the one in [11], but for two details. Firstly, Lemma 1 in [11] needs to be formulated for the CMC spacelike surface equation in \mathbb{L}^3 (the proof can be easily adapted). Secondly, to apply Lemma 2 in [11] we need w to be analytic, which holds since the solutions of (7) are analytic, see [21].

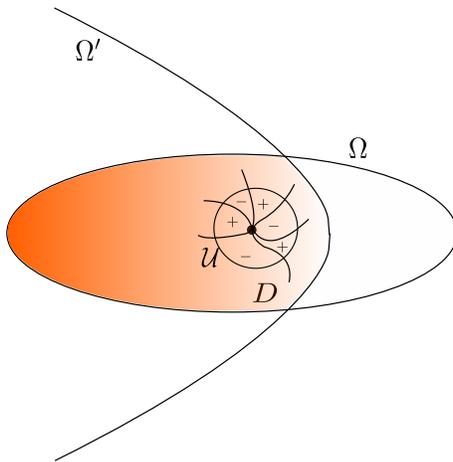


Figure 4: The zero set of $w = u - v$ around (x_0, y_0) .

The global behavior of the zero set \mathcal{Z} in D is also controlled. In fact, the component of $\overline{D} \setminus \mathcal{Z}$ containing a given component of $\mathcal{U} \cap (\overline{D} \setminus \mathcal{Z})$ must intersect ∂D in a non-empty portion consisting of open arcs. Otherwise, the classical comparison principle would imply that $u = v$ in an open set, and since both

are analytic, we would obtain that $u = v$ on D , which is a contradiction. It is also clear that two different components of $\mathcal{U} \cap (\overline{D} \setminus \mathcal{Z})$ can never be included in the same component of $\overline{D} \setminus \mathcal{Z}$. Therefore, there exist at least six regions of ∂D where the function w alternates sign, and so, ∂D is divided into at least six arcs with alternate signs of w .

However, the intersection of $\partial\Omega$, a pseudo-elliptic curve, and $\partial\Omega'$, a branch of a hyperbola or two parallel lines, consists of at most five points, a line segment and at most two points or two line segments. In the first case, ∂D consists of at most five arcs contained in either $\partial\Omega$ or $\partial\Omega'$. In the arcs belonging to $\partial\Omega$ w is positive, and negative in those of $\partial\Omega'$. In the last two cases, w vanishes on the line segments, and has constant sign on the other arcs, which are at most three. In any case, we arrive to a contradiction regarding the number of arcs on which w alternates sign. This finishes the proof of Theorem 3. \square

It is interesting to observe that, as a direct consequence of the above result, under the assumptions of Theorem 3, the level curves of Σ , but the boundary, are strictly convex.

4. An example of a non-strictly convex CMC spacelike graph with planar strictly convex boundary

In this section we prove that the hypothesis that $\partial\Sigma$ is a pseudo-elliptic curve cannot be removed in Theorem 3. For this purpose, we show an example of a CMC spacelike graph in \mathbb{L}^3 with planar strictly convex boundary, which is not a convex surface. The motivation of our example comes from the Euclidean setting, where there exists subsets of unduloids obtained by cutting off with a parallel plane to the axis of rotation in such a way that the compact portions contain points with negative Gaussian curvature [22].

In [23], the authors consider the surface obtained when rotating around the x -axis in \mathbb{L}^3 the spacelike piece of an elliptic catenary contained in the xz -plane. They prove that this surface, which is a graph over \mathbb{R}^2 , has non-zero constant mean curvature.

Let us remember that an elliptic catenary is the curve described by a focus of an ellipse as the ellipse rolls without slipping along a line. The following parametrization of this curve can be found in [24]

$$\alpha(s; H, B) = \left(\int_0^s \frac{1 + B \sin(2Ht)}{\sqrt{1 + B^2 + 2B \sin(2Ht)}} dt, \frac{\sqrt{1 + B^2 + 2B \sin(2Hs)}}{2|H|} \right),$$

where $H \in \mathbb{R} - \{0\}$ is the mean curvature of the surface of revolution and $B \in (0, 1)$ is a real parameter.

We will show that B can be chosen in order to prove that a piece of the surface generated by α is a CMC spacelike graph whose Gaussian curvature can not be signed.

Firstly, we compute $\langle \alpha'(s), \alpha'(s) \rangle$ so that we can take a spacelike piece of α . It is easy to check that

$$\langle \alpha'(s), \alpha'(s) \rangle = 1 - \frac{2B^2 \cos^2(2Hs)}{1 + B^2 + 2B \sin(2Hs)}.$$

Therefore, if we take $B < \sqrt{2} - 1$, the curve α restricted to $[\pi/(2H), \pi/H]$ is spacelike. From now on, when we mention α , we are considering its restriction to $[\pi/(2H), \pi/H]$.

Secondly, we focus on the sign of the curvature $\kappa(s)$ of α . An easy computation yields

$$|\alpha'(s)|^3 \kappa(s) [1 + B^2 + 2B \sin(2Hs)] = 2|H|B (B + \sin(2Hs)).$$

From this expression we get that the curvature takes both positive and negative values at $\pi/(2H)$ and $3\pi/(4H)$, respectively. At any point of α , the curve is a normal section of the surface we are considering. Also, the sign of the curvature of the orbits through any of those points does not vary. Therefore, we have proved that the Gaussian curvature of the surface of revolution generated by α takes both positive and negative values.

Next, we will restrict to the piece of the surface of revolution generated by α with height smaller or equal to $\sqrt{1 + B^2}/(2|H|)$, which is the height of the boundary of α . We prove that the boundary of this surface is a strictly convex

curve. This curve is parametrized by

$$s \mapsto \left(\int_0^s \frac{1 + B \sin(2Ht)}{\sqrt{1 + B^2 + 2B \sin(2Ht)}} dt, \pm \frac{\sqrt{-2B \sin(2Hs)}}{2|H|}, \frac{1 + B^2}{2|H|} \right).$$

It suffices to prove that the curve

$$\gamma(s; H, B) = \left(\int_0^s \frac{1 + B \sin(2Ht)}{\sqrt{1 + B^2 + 2B \sin(2Ht)}} dt, \frac{\sqrt{-2B \sin(2Hs)}}{2|H|} \right)$$

defined in the interval $[\pi/(2H), \pi/H]$, and with $B < \sqrt{2} - 1 < 1/2$, has non-zero curvature. Notice that γ is contained in a spacelike plane. After some computations we arrive to

$$\begin{aligned} |\gamma'(s)|^3 \kappa_\gamma(s) &= \frac{-2|H|B}{\sqrt{1 + B^2 + 2B \sin(2Hs)} \sqrt{-2B \sin(2Hs)}} \left\{ -\sin(2Hs) - B \right. \\ &\quad \left. + \frac{(1 + B \sin(2Hs))B \cos^2(2Hs)(1 + B^2)}{-2B \sin(2Hs)(1 + B^2 + 2B \sin(2Hs))} \right\} \end{aligned} \quad (8)$$

for $s \in (\pi/(2H), \pi/H)$.

At this point we need to distinguish cases. If $-\sin(2Hs) \geq B$, then $\kappa_\gamma < 0$. Otherwise $\sin^2(2Hs) < B^2$, and so

$$\cos^2(2Hs) > 1 - B^2 \quad \text{and} \quad -2B \sin(2Hs)(1 + B^2 + 2B \sin(2Hs)) < 2B^2(1 + B^2).$$

Since $B < \sqrt{2} - 1 < 1/2$, the expression between the brackets in (8) is positive.

We have proved that κ_γ is positive for $s \in (\pi/(2H), \pi/H)$. To extend this inequality to $[\pi/(2H), \pi/H]$, we study the limit of $\kappa_\gamma(s)$ in $\pi/(2H)$ and π/H .

After computing $\langle \gamma'(s), \gamma'(s) \rangle$, and using (8), we conclude that the previous limits are both equal to $-2|H|/(B\sqrt{1 + B^2}) \neq 0$.

Summing up, we have proved that when we consider the surface of revolution generated by α , being $B < \sqrt{2} - 1$, and we take the piece of height less or equal to $\sqrt{1 + B^2}/(2|H|)$, we get a CMC spacelike graph with strictly convex boundary contained in a plane, but the graph is not a convex surface.

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References

- [1] J. L. M. Barbosa, V. Oliker, Stable spacelike hypersurfaces with constant mean curvature in Lorentz space, in: *Geometry and global analysis (Sendai, 1993)*, Tohoku Univ., Sendai, 1993, pp. 161–164.
- [2] A. Brasil, Jr., A. G. Colares, On constant mean curvature spacelike hypersurfaces in Lorentz manifolds, *Mat. Contemp.* 17 (1999) 99–136, 10th School on Differential Geometry (Portuguese) (Belo Horizonte, 1998).
- [3] Y. Choquet-Bruhat, The problem of constraints in general relativity: solution of the Lichnerowicz equation, in: *Differential geometry and relativity*, Reidel, Dordrecht, 1976, pp. 225–235. *Mathematical Phys. and Appl. Math.*, Vol. 3.
- [4] J. E. Marsden, F. J. Tipler, Maximal hypersurfaces and foliations of constant mean curvature in general relativity, *Phys. Rep.* 66 (3) (1980) 109–139.
- [5] R. López, *Constant mean curvature surfaces with boundary*, Springer Monographs in Mathematics, Springer, Heidelberg, 2013.
- [6] R. Bartnik, L. Simon, Spacelike hypersurfaces with prescribed boundary values and mean curvature, *Comm. Math. Phys.* 87 (1) (1982/83) 131–152.
- [7] L. J. Alías, R. López, J. A. Pastor, Compact spacelike surfaces with constant mean curvature in the Lorentz-Minkowski 3-space, *Tohoku Math. J.* (2) 50 (4) (1998) 491–501.
- [8] J. Serrin, The Dirichlet problem for surfaces of constant mean curvature, *Proc. London Math. Soc.* (3) 21 (1970) 361–384.

- [9] J. Serrin, On surfaces of constant mean curvature which span a given space curve., *Math. Z.* 112 (1969) 77–88.
- [10] R. López, Area monotonicity for spacelike surfaces with constant mean curvature, *J. Geom. Phys.* 52 (3) (2004) 353–363.
- [11] J. T. Chen, W. H. Huang, Convexity of capillary surfaces in the outer space, *Invent. Math.* 67 (2) (1982) 253–259.
- [12] W.-H. Huang, C.-C. Lin, Negatively curved sets on surfaces of constant mean curvature in \mathbf{R}^3 are large, *Arch. Rational Mech. Anal.* 141 (2) (1998) 105–116.
- [13] B. Kawohl, Rearrangements and convexity of level sets in PDE, Vol. 1150 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin, 1985.
- [14] N. Korevaar, Capillary surface convexity above convex domains, *Indiana Univ. Math. J.* 32 (1) (1983) 73–81.
- [15] N. J. Korevaar, Convex solutions to nonlinear elliptic and parabolic boundary value problems, *Indiana Univ. Math. J.* 32 (4) (1983) 603–614.
- [16] J. McCuan, Concavity, quasiconcavity, and quasilinear elliptic equations, *Taiwanese J. Math.* 6 (2) (2002) 157–174.
- [17] A. D. Aleksandrov, Uniqueness theorems for surfaces in the large. I, *Amer. Math. Soc. Transl.* (2) 21 (1962) 341–354.
- [18] J. A. Aledo, L. J. Alías, On the curvatures of bounded complete spacelike hypersurfaces in the Lorentz-Minkowski space, *Manuscripta Math.* 101 (3) (2000) 401–413.
- [19] L. J. Alías, A. Romero, M. Sánchez, Uniqueness of complete spacelike hypersurfaces of constant mean curvature in generalized Robertson-Walker spacetimes, *Gen. Relativity Gravitation* 27 (1) (1995) 71–84.

- [20] D. Gilbarg, N. S. Trudinger, Elliptic partial differential equations of second order, *Classics in Mathematics*, Springer-Verlag, Berlin, 2001, reprint of the 1998 edition.
- [21] S. Bernstein, Sur la nature analytique des solutions des équations aux dérivées partielles du second ordre, *Math. Ann.* 59 (1-2) (1904) 20–76.
- [22] R. Finn, Comparison principles in capillarity, in: *Partial differential equations and calculus of variations*, Vol. 1357 of *Lecture Notes in Math.*, Springer, Berlin, 1988, pp. 156–197.
- [23] J.-i. Hano, K. Nomizu, Surfaces of revolution with constant mean curvature in Lorentz-Minkowski space, *Tohoku Math. J. (2)* 36 (3) (1984) 427–437.
- [24] S. C. García-Martínez, Aplicaciones del principio del máximo generalizado de omori-yau al estudio de la geometría global de hipersuperficies en espacios de curvatura constante, Ph.D. thesis, University of Murcia (2012).