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Bifurcating Nodoids in Hyperbolic Space

Mohamed Jleli
Department of Mathematics
King Saud University, Saudi Arabia
e-mail: jleli@ksu.edu.sa

Rafael López*
Departamento de Geometría y Topología
Universidad de Granada
18071 Granada, Spain
e-mail: rcamino@ugr.es

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Abstract

Consider in hyperbolic space $\mathbb{H}^3$ the one parameter family of immersed (non embedded) constant mean curvature surfaces of revolution $\mathcal{D}_\tau$ with constant mean curvature $H > 1$. The parameter $\tau \in (-\infty, 0)$ is the analogue of the "necksize" of the Delaunay surfaces in Euclidean space. It is proved that when $\tau \to -\infty$, there exists a branch of surfaces with constant mean curvature H which bifurcate from $\mathcal{D}_\tau$. Furthermore, we prove that these new surfaces have only a discrete group of symmetries. The proof consists in a detailed study of the behaviour of the eigenvalues of the Jacobi operator when τ tends to $-\infty$, together the bifurcation theorem of Crandall-Rabinowitz.

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1 Introduction and statement of the main result

Delaunay classified in [3] all rotational surfaces in Euclidean space $\mathbb{R}^3$ with constant mean curvature $H \neq 0$. Given a nonzero number H, the elements of this family parameterize as $\mathcal{D}_\tau$, where $\tau \in (-\infty, \tau_H]$. For $\tau \in (0, \tau_H)$, the surfaces $\mathcal{D}_\tau$ are called unduloids and are embedded and for negative values of τ, the surfaces $\mathcal{D}_\tau$ are called nodoids and are not embedded. The particular cases $\tau = \tau_H$ and $\tau = 0$ correspond, respectively, with the cylinder of radius $1/(2|H|)$ and a stack of spheres of radius $1/|H|$. Unduloids are stable in the sense that the only global constant mean curvature deformations of an unduloid are other elements of the same family, namely, unduloids. The same property holds for nodoids only when the parameter τ is sufficiently close to zero. On the other hand, Mazzeo and Pacard proved in [15] that as τ decreases to $-\infty$, infinitely many new complete, cylindrically bounded constant mean curvature surfaces bifurcate from a nodoid $\mathcal{D}_\tau$. The surfaces in these branches have only a discrete group of symmetries. Similar bifurcation results are obtained in [5] for constant mean curvature hypersurfaces in $\mathbb{R}^{n+1}$.

In hyperbolic space $\mathbb{H}^3$ and for $H > 1$, the family of rotational surfaces with constant mean curvature H are also called *Delaunay surfaces*. Fixing $H > 1$, Delaunay surfaces with constant mean curvature H can parameterize as $\{\mathcal{D}_\tau : \tau \in (-\infty, , \tau_H]\}$. For $\tau \in (0, \tau_H)$, $\mathcal{D}_\tau$ is embedded and it is called unduloid and if $\tau \in (-\infty, 0)$, the surface $\mathcal{D}_\tau$ is called a nodoid and it is not embedded. If $\tau = \tau_H$, the surface is a Killing cylinder and if $\tau = 0$, the surface is a hyperbolic sphere. Furthermore, as in the Euclidean space, we need to impose a lower bound $\tau^H < 0$ on the Delaunay parameter, where τ^H depends only on H, such that for $\tau \in [\tau^H, 0) \cup (0, \tau_H]$ the surface $\mathcal{D}_\tau$ is stable [6]. This fact was generalized for hypersurfaces of $\mathbb{H}^{n+1}$ by the first author in [7].

In this paper we extend the results of Mazzeo and Pacard for nodoids of hyperbolic space. We prove that for τ tending to $-\infty$, the nodoid $\mathcal{D}_\tau$ generates a family of a new constant mean curvature surfaces which are small normal perturbations of $\mathcal{D}_\tau$. The main result is:

Theorem 1.1. *Let $H > 1$. There exists $\ell_H \in \mathbb{N}$ depending only on H such that for each $\ell \in \mathbb{N}$, $\ell \geq \ell_H$, there exists $\tau_\ell < 0$ depending only ℓ and a smooth branch of surfaces in $\mathbb{H}^3$ with constant mean curvature equal to H which bifurcates from the nodoid $\mathcal{D}_{\tau_\ell}$. Moreover, if γ denotes the rotational axis of $\mathcal{D}_{\tau_\ell}$, any element of this branch is a non rotational surface invariant about the discrete group of symmetries generated by a rotation of angle $2\pi/\ell$ with respect to γ.*

As we have mentioned, there is an interest on problems of bifurcation of nodoids and, in general, of rotational surfaces with constant mean curvature in Euclidean space [4, 5, 9, 14, 15, 17, 19], as well as in other related contexts [1, 11, 12, 13, 18].

Theorem 1.1 is a consequence of the bifurcation theorem of Crandall-Rabinowitz for simple eigenvalues [2]. With this objective in mind, we need an analysis of the multiplicities of the eigenvalues of the Jacobi operator $\mathcal{L}_\tau$ of the nodoid $\mathcal{D}_\tau$ studying the behaviour of the number of negative eigenvalues of $\mathcal{L}_\tau$ when the parameter τ goes to $-\infty$. This will be carried out in section 3. Previously, in section 2, we calculate the expression of the mean curvature of a Delaunay surface in $\mathbb{H}^3$ in different parametrizations and we study the asymptotic behaviour close to $-\infty$ of the Delaunay period τ associated to these surfaces.

A formal proof of the existence of new constant mean curvature hypersurfaces in $\mathbb{R}^{n+1}, n > 2$ which bifurcates from the family of the immersed not embedded rotationally invariant constant mean curvature hypersurfaces is given in [8]. However, in this paper we give a complete details about the bifurcation phenomena for the case $n = 2$. In particular, a precise behavior concerning the period of the nodoids as the Delaunay parameter increases is given using some properties of well known special functions. Thus allows us to prove in the end of the paper that the bifurcation occurs for simple eigenvalue of the Jacobi operator which assure the smoothness of the bifurcated branches of surfaces. In [8] it is unable to study the regularity of the constructed hypersurfaces since no information about the period of the nodoids in $\mathbb{R}^{n+1}, n > 2$ is available.

2 Delaunay surfaces in hyperbolic space

2.1 The mean curvature equation

Let $\mathbb{H}^3$ be the three-dimensional hyperbolic space and consider the upper half-space model of $\mathbb{H}^3$, that is,

$$\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$$

endowed with the metric

$$g_{hyp} = \frac{1}{z^2}(dx^2 + dy^2 + dz^2).$$

A surface of revolution in $\mathbb{H}^3$ is defined as a surface invariant by the hyperbolic rotations about a geodesic γ of $\mathbb{H}^3$. Without loss of generality, suppose that the geodesic is $\gamma(t) = (0, 0, e^t)$, $t > 0$. In such a case and from the Euclidean viewpoint, the surface is a rotational surface with respect to the z-axis. Assume that the generating curve is contained in the xz-plane. If this curve writes as a geodesic graph on γ of a function ψ, $\psi(t) \in (0, \pi/2)$, its parameterization is $t \longmapsto (e^t \sin(\psi(t)), 0, e^t \cos(\psi(t)))$, $t \in I \subset \mathbb{R}$. Then the corresponding rotational surface $X : \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R}^3_+$ is

$$X(t, \theta) = (e^t \sin(\psi(t)) \cos(\theta), e^t \sin(\psi(t)) \sin(\theta), e^t \cos(\psi(t))).$$

Let

$$f(t) = e^t \sin(\psi(t)), \quad g(t) = e^t \cos(\psi(t)).$$

Assume that the orientation of this surface is chosen by the unit normal vector field

$$N(t, \theta) = \frac{\cos(\psi(t))}{\sqrt{1 + \psi'(t)^2}} \left(-g'(t) \cos(\theta), -g'(t) \sin(\theta), f'(t)\right). \tag{1}$$

The mean curvature H of X with respect to the orientation (1) satisfies

$$\psi''(t)-\frac{1+\sin^2(\psi(t))}{\sin(\psi(t))\cos(\psi(t))}(1+\psi'(t)^2)+\frac{2H}{\cos(\psi(t))}(1+\psi'(t)^2)^{\frac{3}{2}}=0. \tag{2}$$

If H is constant, then (2) is equivalent that the Hamiltonian

$$\mathcal{H}(\psi,\psi')=\frac{\tan(\psi)}{\cos(\psi)\sqrt{1+\psi'^2}}-H\tan^2(\psi)$$

is constant. See [6]. From (2), we obtain two particular solutions. A first solution of (2) is when the function ψ is a constant $c_H\in(0,\pi/2)$. Then $\psi(t)=c_H$ is a solution of (2) if and only if

$$\sin(c_H)=H-\sqrt{H^2-1}. \tag{3}$$

The corresponding surface is a Killing cylinder. This surface is constructed as follows. Consider the Killing vector field associated to a geodesic γ. Let Γ be a circle of radius ρ contained in a geodesic plane $P\subset\mathbb{H}^3$ orthogonal to γ and centered at $\gamma\cap P$ and take the Killing vector field associated to γ. Then the Killing cylinder with base on Γ is the set of the orbits of all points of Γ by the Killing vector field. In our model of $\mathbb{H}^3$, if γ is the (positive) z-axis, the Killing vector field is given by the dilatations from the intersection of γ with the plane $z=0$. Moreover, P is a hemisphere orthogonal to γ and with boundary in the plane of equation $z=0$. Then the orbits are all the half straight-lines in $\mathbb{R}^3_+$ starting at the center of P through each point of Γ. Thus the Killing cylinder is the Euclidean cone

$$\{(x,y,z)\in\mathbb{R}^3_+ : x^2+y^2=\sinh(\rho)^2z^2\}$$

and the mean curvature is

$$H=\frac{\tanh(\rho)+\coth(\rho)}{2}=\frac{1+\sin(c_H)^2}{\sin(c_H)}.$$

The other explicit solution corresponds to the hyperbolic sphere. Here

$$\psi(t)=\arccos\left(\Big(\frac{\sqrt{H^2-1}}{H}\cosh(t)\right),\qquad e^t\in\left[\sqrt{\frac{H-1}{H+1}},\sqrt{\frac{H+1}{H-1}}\right].$$

2.2 The isothermal parameterization

In our analysis of the Jacobi operator of a Delaunay surface of $\mathbb{H}^3$, it will be more convenient to consider isothermal parameterizations. Let $X:\mathbb{R}\times\mathbb{S}^1\to\mathbb{R}^3_+$ be the parametrization given by

$$X(s,\theta)=\left(\varphi(s)\kappa(s)\cos(\theta),\varphi(s)\kappa(s)\sin(\theta),\kappa(s)\right), \tag{4}$$

where the functions $\varphi > 0$ and κ are determined by the conformality of X. Indeed, $X(s,\theta)$ is conformal if and only if

$$((\varphi\kappa)')^2 + (\kappa')^2 = \varphi^2\kappa^2, \tag{5}$$

where we drop the variable of a function if it is understood. See [6]. The first fundamental form of the surface $X(s,\theta)$ is

$$\varphi^2(ds^2 + d\theta^2). \tag{6}$$

The constant mean curvature equation (2) is now

$$\varphi'' - (1+2\varphi^2)\varphi + 2H\left(\varphi^2\varphi' + \frac{(1+\varphi^2)\varphi\kappa'}{\kappa}\right) = 0, \tag{7}$$

where the orientation (1) writes as

$$N(s,\theta) = \frac{1}{\varphi}\left(-\kappa'\cos(\theta), -\kappa'\sin(\theta), (\varphi\kappa)'\right). \tag{8}$$

Let us introduce a parameter τ by

$$\varphi_\tau(s) = |\tau| e^{\sigma(s)}$$

for some scalar function $\sigma(s)$. Equation (5) becomes

$$(1+\tau^2 e^{2\sigma})\left(\frac{\kappa'}{\kappa}\right)^2 + 2\tau^2 e^{2\sigma}\sigma'\left(\frac{\kappa'}{\kappa}\right) + \tau^2 e^{2\sigma}\left(\sigma'^2 - 1\right) = 0, \tag{9}$$

and the mean curvature equation (7) is

$$\sigma'' - \left(1 + 2\tau^2 e^{2\sigma} - \sigma'^2\right) + 2H\left(\tau^2 e^{2\sigma}\sigma' + \frac{(1+\tau^2 e^{2\sigma})\kappa'}{\kappa}\right) = 0. \tag{10}$$

Hence, in order to find constant mean curvature surfaces of revolution, we have to solve (10) together the conformality condition (9). Define the positive real number τ_H as

$$\tau_H^2 = \frac{1}{2}(H - \sqrt{H^2-1}).$$

For all $\tau \in (-\infty, 0) \cup (0, \tau_H]$, let σ_τ be the unique non constant solution of

$$\sigma'^2 + \tau^2\Big(\left(He^{\sigma} + \iota e^{-\sigma}\right)^2 - e^{2\sigma}\Big) = 1, \tag{11}$$

with the initial condition $\sigma'(0) = 0$. Here ι is the sign of τ. The function $\kappa = \kappa_\tau$ will be a solution of

$$\frac{\kappa'}{\kappa} = \left(H - \sigma' + \iota e^{-2\sigma}\right)\frac{\tau^2 e^{2\sigma}}{1+\tau^2 e^{2\sigma}}.$$

It is not difficult to check that σ_τ and κ_τ solve (9) and (10). Moreover, the function κ_τ is monotone increasing for $\tau \in (0, \tau_H]$ since $\kappa'_\tau(s) > 0$. However, if $\tau < 0$, the function κ_τ is not monotone anymore since $\kappa_\tau'(s)$ changes sign. Hence, the surface parameterized by

$$X_\tau(s,\theta) = \left(|\tau| e^{\sigma_\tau(s)}\kappa_\tau(s)\cos(\theta), |\tau| e^{\sigma_\tau(s)}\kappa_\tau(s)\sin(\theta), \kappa_\tau(s)\right),$$

for $(s,\theta) \in \mathbb{R}\times\mathbb{S}^1$, is a constant mean curvature surface of revolution which is embedded for $\tau \in (0,\tau_H]$ and non-embedded if $\tau < 0$. See Fig. 1 for a nodoid of $\mathbb{H}^3$. When τ tends to 0, the family of unduloids converges to a sequence of hyperbolic spheres of $\mathbb{H}^3$.

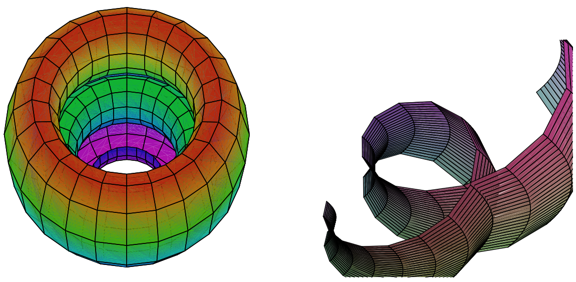

Figure 1: The surface $\mathcal{D}_{-1}$ with $H = 2$.

As in the Euclidean space, the constant τ_H corresponds to the Delaunay parameter of the embedding limit surface which is a Killing cylinder. Exactly, for $\tau = \tau_H$, the function σ is constant and the function φ is equal to $\tan(c_H)$, where c_H was defined in (3).

In order to know the behaviour of the Delaunay surfaces as τ tends to 0, define the function

$$\eta_\tau(s) = |\tau| e^{-\sigma_\tau(s)}.$$

Using (11), one checks that the functions φ_τ and η_τ are non constant solutions of

$$\varphi_\tau'^2 = \varphi_\tau^2(1+\varphi_\tau^2 - (H\varphi_\tau + \iota\,\eta_\tau)^2) \tag{12}$$

and

$$\eta_\tau'^2 = \eta_\tau^2(1+\varphi_\tau^2 - (H\varphi_\tau + \iota\,\eta_\tau)^2). \tag{13}$$

In addition, we have

$$\varphi_\tau\eta_\tau = \tau^2. \tag{14}$$

A first lemma states that the functions φ_τ and η_τ and their derivatives, are uniformly bounded with respect to τ in the C^k-topology on compact sets provided that $|\tau|$ remains bounded.

Lemma 2.1. *Assume that $\tau_0 < 0$ is fixed. Then for all $k \in \mathbb{N}$, there exists a constant $c_k = c_k(\tau_0, k) > 0$ depending only on τ_0 and k, such that*

$$\|\varphi_\tau\|_{C^k} + \|\eta_\tau\|_{C^k} \leq c_k$$

for all $\tau \in [\tau_0, 0) \cup (0, \tau_H]$.

Proof. When $\tau > 0$, observe that (12) already implies that the functions φ_τ and η_τ are uniformly bounded by 1. When $\tau < 0$ is bounded from below by τ_0, Equations (12) and (13) imply that the functions φ_τ and η_τ are uniformly bounded by a constant only depending on τ_0. By combining (12) and (13) inductively, we prove that the same property holds for the derivatives of φ_τ and η_τ. □

Assume that $\{s_n\}$ and $\{\tau_n\}$ are two sequences of real numbers with $\tau_n \to 0$ as n tends to ∞. For each $n \in \mathbb{N}$, define

$$\varphi_n(s) = \varphi_{\tau_n}(s - s_n) \qquad \text{and} \qquad \eta_n(s) = \eta_{\tau_n}(s - s_n).$$

The previous result together with Ascoli's theorem allows to extract from $\{(\varphi_n, \eta_n)\}_n$, a subsequence which converges to $(\varphi_\infty, \eta_\infty)$ in the C^k-topology on compact sets. The following result classifies the possible limits $(\varphi_\infty, \eta_\infty)$.

Lemma 2.2. *Under the above hypothesis, the following holds:*

1. *Either $\varphi_\infty = \eta_\infty = 0$,*
2. *or $\varphi_\infty = 0$ and there exists s_∞ such that*
$$\eta_\infty(s) = \frac{1}{\cosh(s - s_\infty)},$$
3. *or $\eta_\infty = 0$ and there exists s_∞ such that*
$$\varphi_\infty(s) = \frac{1}{\sqrt{H^2 - 1}\cosh(s - s_\infty)}$$
and the surface corresponds with a sphere.

Proof. Passing the limit in (14) we get $\varphi_\infty \eta_\infty = 0$. This implies that, at least one of the functions φ_∞ and η_∞ has to be identically equal to 0. It only remains to identify the possible non trivial limits.

If $\varphi_\infty = 0$, then (5) implies that κ is constant and the limit surface is singular by (6). Moreover, (13) implies $\eta'^2 = \eta^2(1 - \eta^2)$, whose solution is $\eta_\infty(s) = 1/\cosh(s - s_\infty)$, for a constant $s_\infty \in \mathbb{R}$.

Now, if $\eta_\infty = 0$, we can pass to the limit in (12) to get

$$\varphi'^2 = \varphi^2(1 + (1 - H^2)\varphi^2).$$

The non trivial solution of this equation is of the form

$$\varphi_\infty(s) = \frac{1}{\sqrt{H^2-1}\cosh(s-s_\infty)},$$

for some $s_\infty \in \mathbb{R}$. With this value of φ, we solve (7) obtaining that, up constants,

$$\kappa(s) = \frac{1}{H-\tanh(s-s_\infty)}.$$

Then the limit surface is a sphere with contant mean curvature H. □

We point out that the last part of lemma says that we can extend the one parameter family of surfaces $\mathcal{D}_\tau$ for the value $\tau = 0$ and the surface is a sphere, showing that a stack of spheres are the transition surfaces between the family of unduloids and the one of nodoids. This extends to the hyperbolic ambient, the similar result in Euclidean space.

2.3 The Delaunay period

From equation (11), we deduce that the curve $(\sigma_\tau(s), \sigma'_\tau(s))$ is closed which implies that the function $s \mapsto \sigma_\tau(s)$ is periodic. Denote by s_τ its period, that is, $\sigma_\tau(s+s_\tau) = \sigma_\tau(s)$. The behaviour of s_τ when the parameter τ tends to $-\infty$ will be crucial in our analysis.

Before starting this study, we need to recall some elementary results concerning to the elliptic integrals and elliptic functions [20]. First, recall that for all $\xi \in (0,1)$, the incomplete elliptic integrals of first kind and second kind are defined by

$$F(\phi,\xi) = \int_0^\phi \frac{d\psi}{\sqrt{1-\xi\sin^2\psi}},$$

and

$$E(\phi,\xi) = \int_0^\phi \sqrt{1-\xi\sin^2\psi}\,d\psi,$$

respectively, where $\phi \in [0,\pi/2]$. Here ξ is called the modulus and ϕ is the amplitude angle, which will be denoted by $\mathbf{amp}(z,\xi)$. In particular, the functions

$$K(\xi) = F(\frac{\pi}{2},\xi), \quad E(\xi) = E(\frac{\pi}{2},\xi)$$

are the complete elliptic integrals of the first and second kind, respectively. By letting $z = F(\phi,\xi)$, the inversion of the elliptic integral F gives

$$\phi = F^{-1}(z,\xi) = \mathbf{amp}(z,\xi).$$

Define the Jacobi elliptic functions as

$$\sin\phi = \sin(\mathbf{amp}(z,\xi)) = sn(z,\xi), \quad \cos\phi = cn(z,\xi)$$

$$dn(z,\xi) = \sqrt{1 - \xi sn^2(z,\xi)}.$$

Furthermore, the function $dn(z,\xi)$ is $2K(\xi)$-periodic on the variable z and it satisfies the system of equations

$$\begin{aligned} (\partial_z dn)^2 &= \left(dn^2 - 1\right)\left(1 - \xi - dn^2\right) && (15) \\ 2(1-\xi)\partial_\xi dn &= sn\ cn\left((1-\xi)z + E(\mathbf{amp}, .)\right) - sn^2\ dn && (16) \\ \sqrt{1-\xi} &= dn(z,\xi)\ dn(z - K(\xi), \xi). && (17) \end{aligned}$$

Now we are able to prove:

Lemma 2.3. *The following expansions hold as τ tends to $-\infty$*

$$s_\tau = -2\frac{K(\frac{2}{H+1})}{\tau\sqrt{H+1}}\left(1 + O(\tau^{-2})\right) \tag{18}$$

$$\partial_\tau s_\tau = -\frac{s_\tau}{\tau}\left(1 + O(\tau^{-2})\right). \tag{19}$$

Proof. We know from (11) that the function σ is periodic and

$$\frac{\sigma'(s)}{\sqrt{1 - \tau^2 g(\sigma(s))}} = 1, \qquad s \in (\sigma_-, \sigma_+).$$

Here the function g is defined by

$$g(s) = (H^2 - 1)e^{2\sigma} + e^{-2\sigma} - 2H,$$

the period s_τ is

$$s_\tau = 2\int_{\sigma_-}^{\sigma_+} \frac{d\sigma}{\sqrt{1 - \tau^2 g(\sigma)}}$$

and $\sigma_- < \sigma_+$ are the solutions of

$$g(\sigma_\pm) = \frac{1}{\tau^2}.$$

Let us perform the change of variables $y = e^\sigma$ and this yields

$$s_\tau = -\frac{2}{\tau\sqrt{H^2-1}}\int_{\alpha_\tau}^{\beta_\tau} \frac{dy}{\sqrt{(\beta_\tau^2 - y^2)(y^2 - \alpha_\tau^2)}},$$

with

$$\alpha_\tau^2 = \frac{1 + 2H\tau^2 - \sqrt{1 + 4H\tau^2 + 4\tau^4}}{2(H^2-1)\tau^2}$$

$$\beta_\tau^2 = \frac{1 + 2H\tau^2 + \sqrt{1 + 4H\tau^2 + 4\tau^4}}{2(H^2-1)\tau^2}.$$

The integral in the expression of s_τ is elliptic. Using the change of variable $y = \beta_\tau\cos(\omega)$, we deduce

$$s_\tau = -\frac{2}{\tau\beta_\tau\sqrt{H^2-1}\sin(\omega_\tau)}\int_0^{\omega_\tau}\frac{dw}{\sqrt{1-\frac{\sin^2(\omega)}{\sin^2(\omega_\tau)}}}$$

with

$$\omega_\tau = \arccos(\frac{\alpha_\tau}{\beta_\tau}).$$

A new change of variables by $\sin\phi = \sin^2(\omega)/\sin^2(\omega_\tau)$ gives

$$s_\tau = -\frac{2}{\tau\beta_\tau\sqrt{H^2-1}}\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-\sin^2(\omega_\tau)\sin^2\phi}} = -2\frac{K(1-\frac{\alpha_\tau^2}{\beta_\tau^2})}{\tau\beta_\tau\sqrt{H^2-1}}. \tag{20}$$

Using that

$$\alpha_\tau^2 = \frac{1}{H+1}\left(1+O(\tau^{-2})\right) \quad \text{and} \quad \beta_\tau^2 = \frac{1}{H-1}\left(1+O(\tau^{-2})\right),$$

we obtain (18).

In order to give the asymptotic expansion of the derivative of s_τ with respect to τ, we differentiate (20) with respect to τ obtaining

$$\partial_\tau s_\tau = -\left(\frac{1}{\tau}+\frac{\partial_\tau\beta_\tau}{\beta_\tau}+\frac{K'(1-\frac{\alpha_\tau^2}{\beta_\tau^2})}{K(1-\frac{\alpha_\tau^2}{\beta_\tau^2})}\partial_\tau\left(\frac{\alpha_\tau}{\beta_\tau}\right)^2\right)s_\tau.$$

As above, we have

$$\begin{aligned}\frac{\partial_\tau\beta_\tau}{\beta_\tau} &= O(|\tau|^{-3})\\ \left(\frac{\alpha_\tau}{\beta_\tau}\right)^2 &= \frac{H-1}{H+1}\left(1+O(\tau^{-2})\right)\\ \partial_\tau\left(\frac{\alpha_\tau}{\beta_\tau}\right)^2 &= O(|\tau|^{-3}).\end{aligned}$$

Since $K'(\xi) = K(1-\xi)$ for all $\xi\in(0,1)$, we use the above expansions to deduce that

$$\frac{K'(1-\frac{\alpha_\tau^2}{\beta_\tau^2})}{K(1-\frac{\alpha_\tau^2}{\beta_\tau^2})} = O(1),$$

and this proves (19). □

We end this section giving an expansion of the function $\varphi(s)$ using the elliptic function $z \longmapsto dn(z,\xi)$. Let

$$\varphi_m = \varphi(0),\ \varphi_M = \varphi(\frac{s_\tau}{2})$$

be the minimal and the maximal values of the function φ. Define the function $v = v(s)$ by the identity

$$\varphi(s) = \varphi_M\ dn(v(s),\xi), \qquad s \in [0, \frac{s_\tau}{2}].$$

In order to ensure uniqueness of v, we require that

$$v(\frac{s_\tau}{2}) = 0 \qquad \text{and} \qquad v'(\frac{s_\tau}{2}) < 0.$$

The function σ_τ satisfies (11), hence

$$\varphi'^2 = (H^2-1)(\varphi^2-\varphi_M^2)(\varphi_m^2-\varphi^2),$$

that together with (15) yields

$$v(s) = \sqrt{H^2-1}\varphi_M(\frac{s_\tau}{2}-s), \qquad s \in [0, \frac{s_\tau}{2}].$$

Therefore the function φ is written as

$$\varphi(s) = \varphi_M\ dn\left(\sqrt{H^2-1}\varphi_M(s-\frac{s_\tau}{2}),\xi\right), \tag{21}$$

with $\xi = 1-\varphi_m^2/\varphi_M^2$.

Remark 2.1. *Using the same argument as above, the last expansion of φ holds for all $s \in \mathbb{R}$. Then the relation between s_τ, the period of φ and $K(\xi)$, the half-period of $dn(.,\xi)$, is*

$$2K(\xi) = \sqrt{H^2-1}s_\tau\varphi_M, \tag{22}$$

with $\xi = 1-\varphi_m^2/\varphi_M^2 = 1-\alpha_\tau^2/\beta_\tau^2$.

2.4 The Jacobi operator of $\mathcal{D}_\tau$

Recall that a surface close enough to $\mathcal{D}_\tau$ can be locally parameterized as the graph in geodesic cylindrical coordinates of a smooth function ω defined on $\mathcal{D}_\tau$, namely,

$$X_\omega = X_\tau + \omega N_\tau,$$

and with exponential decay rate for any C^k norm ([10, Th. 6.9]). Denote by $\mathcal{D}_\tau(\omega)$ the surface obtained by X_ω and by $H(\omega)$ its mean curvature. We have

$$H(\omega) = H + \mathcal{L}_\tau(\omega) + \mathcal{Q}_\tau(\omega), \tag{23}$$

where $\mathcal{L}_\tau$ is the linearized mean curvature operator on $\mathcal{D}_\tau$, which is the Jacobi operator, and $\mathcal{Q}_\tau$ collects all the nonlinear terms. It is well known that

$$\mathcal{L}_\tau = \Delta_\tau + |A_\tau|^2 + Ric_{\mathbb{H}^3}(N_\tau, N_\tau),$$

where Δ_τ is the Laplace-Beltrami operator on $\mathcal{D}_\tau$, $|A_\tau|^2$ is the square of the norm of the second fundamental form on $\mathcal{D}_\tau$ and $Ric_{\mathbb{H}^3}$ is the Ricci tensor on $\mathbb{H}^3$. A computation of the Jacobi operator in coordinates (s, θ) leads to

$$\tau^2 e^{2\sigma} \mathcal{L}_\tau = \partial_s^2 + \partial_\theta^2 + 2\tau^2 \left((H^2 - 1)e^{2\sigma} + e^{-2\sigma} \right). \tag{24}$$

By letting

$$L_\tau = \tau^2 e^{2\sigma} \mathcal{L}_\tau, \tag{25}$$

the mapping properties of $\mathcal{L}_\tau$ are easily translated for L_τ. With a slight abuse of terminology, we shall refer to any of them as the Jacobi operator on $\mathcal{D}_\tau$.

Some Jacobi functions, that is, solutions ω of the homogeneous problem $\mathcal{L}_\tau \omega = 0$, are obtained from the action of a one parameter family of rigid motions of $\mathbb{H}^3$. The corresponding Jacobi function is obtained by projecting onto the normal bundle of $\mathcal{D}_\tau$ the Killing vector field associated to the family of rigid motions under consideration. We describe two examples of such Jacobi functions. Previously, and in order to know the rigid motions of $\mathbb{H}^3$, recall that by the Liouville theorem the isometries of $\mathbb{H}^3$ in the upper half-space model are the restrictions to $\mathbb{R}^3_+$ of the conformal transformations of $\mathbb{R}^3$ which carry $\mathbb{R}^3_+$ onto itself. Let $H > 1$, $\tau \in (-\infty, 0) \cup (0, \tau_H]$ and denote by $\mathcal{D}_\tau$ the rotational surface of $\mathbb{H}^3$ with constant mean curvature H for the parameter τ.

1. Consider the parabolic translations orthogonal to the axis of revolution of $\mathcal{D}_\tau$. In the upper half-space model, and assuming that the axis of revolution is the z-axis, these isometries are Euclidean translations along a horizontal direction $(a, b) \in \mathbb{R}^2$:

   $$(x, y, z) \in \mathbb{R}^3_+ \longmapsto (x + a, y + b, z) \in \mathbb{R}^3_+.$$

   Let

   $$\chi_1 = (a, b, 0) \in \mathbb{R}^3_+$$

   denote the Killing field corresponding to this isometry. Taking into account (8), the projection on the normal bundle of $\mathcal{D}_\tau$ is

   $$g_{hyp}(N_\tau, \chi_1) = -\varphi^{-1} \kappa^{-2} \kappa' \left(a \cos(\theta) + b \sin(\theta) \right).$$

   Taking (a, b) to be the canonical basis of $\mathbb{R}^2$, we obtain the Jacobi functions

   $$\Phi_\tau^{\pm 1}(s, \theta) = \varphi^{-1}(s) \kappa^{-2}(s) \kappa'(s) e^{\pm i\theta}, \tag{26}$$

   Let us observe that $\Phi_\tau^{\pm 1}$ are periodic with respect to the variable s.

2. Consider the dilations

$$(x,y,z)\in\mathbb{R}^3_+\longmapsto(\lambda x,\lambda y,\lambda z)\in\mathbb{R}^3_+$$

for $\lambda>0$, whose Killing field is

$$\chi_2(x,y,z)=(x,y,z).$$

Then the projection of χ_2 onto the normal bundle fo $\mathcal{D}_\tau$ is the Jacobi function

$$\Phi^0_\tau(s,\theta)=g_{hyp}(N_\tau,\chi_2)=\varphi^{-1}(s)\varphi'(s) \tag{27}$$

Again, Φ^0_τ only depends on s and it is periodic in this variable.

3 Bifurcation results

3.1 The spectrum of the Jacobi operator

We study the spectrum of the operator L_τ in terms of the parameter τ, specially, the behavior of the eigenvalues of L_τ when τ varies from 0 to $-\infty$. Since the function σ_τ, and hence the potential of L_τ defined in (25), is periodic with a period which depends on τ, it will be more convenient to work in a function space where the period is fixed. To this aim, define the variable t by

$$t=\frac{2\pi}{s_\tau}s, \tag{28}$$

where s_τ is the period of σ_τ. By (24), the operator L_τ reads in the new variables (t,θ) as

$$L_\tau=\left(\frac{2\pi}{s_\tau}\right)^2\partial_t^2+\partial_\theta^2+2\tau^2\left((H^2-1)e^{2\sigma}+e^{-2\sigma}\right),$$

where σ_τ is now a 2π-periodic function of t. Recall that the eigenvalues of ∂_θ^2 on $\mathbb{S}^1$ are $\{j^2 : j\in\mathbb{N}\}$. Decompose L_τ into a direct sum of ordinary differential operators $L_{\tau,j}$ with

$$L_{\tau,j}=\left(\frac{2\pi}{s_\tau}\right)^2\partial_t^2+2\tau^2\left((H^2-1)e^{2\sigma}+e^{-2\sigma}\right)-j^2=L_{\tau,0}-j^2, \tag{29}$$

with

$$L_{\tau,0}=\left(\frac{2\pi}{s_\tau}\right)^2\partial_t^2+2\tau^2\left((H^2-1)e^{2\sigma}+e^{-2\sigma}\right).$$

Then

$$\mathrm{Spec}(L_\tau)=\bigcup_{j\in\mathbb{N}}\mathrm{Spec}(L_{\tau,j}).$$

Therefore, by (29), the spectrum of the operator L_τ is known by the spectrum of $L_{\tau,0}$. Define L^2_{per} (resp. H^2_{per}) the space of L^2_{loc}-functions (resp. H^2_{loc}-functions) which are 2π-periodic and consider the operator

$$\begin{array}{rccc} l_\tau: & H^2_{per} & \longrightarrow & L^2_{per} \\ & u & \longmapsto & L_{\tau,0}(u) \end{array}$$

It is easy to check that l_τ is a self adjoint operator and hence it has discrete spectrum

$$\mathrm{Spec}(l_\tau) = \{\lambda_0(\tau) < \lambda_1(\tau) \leq \ldots\}.$$

Observe that the multiplicity of each eigenvalue of the operator l_τ is at most equal to 2 because l_τ is a second order ordinary differential operator.

Proposition 3.1. *When the Delaunay parameter τ tends to $-\infty$, we have*

$$-2(1+2H\tau^2) \leq \lambda_0(\tau) \leq -4\tau^2\sqrt{H^2-1} \tag{30}$$

$$\partial_\tau \lambda_0(\tau) - \frac{2}{\tau}\lambda_0(\tau) = O(|\tau|^{-1}). \tag{31}$$

In addition, for all $\tau < 0$ we have

$$\lambda_1(\tau) = -1, \;\; \lambda_2(\tau) = 0. \tag{32}$$

Proof. We use the explicit expression of the Jacobi vector functions Φ^0_τ and $\Phi^{\pm 1}_\tau$. In particular, from (26) we know that

$$L_\tau(\varphi^{-1}\kappa^{-2}\kappa' e^{\pm i\theta}) = 0.$$

Hence

$$l_\tau\left(\varphi^{-1}\kappa^{-2}\kappa'\right) = -\left(\varphi^{-1}\kappa^{-2}\kappa'\right).$$

We know from subsection 2.2 that if $\tau < 0$, κ' changes of sign. Then the function $\varphi^{-1}\kappa^{-2}\kappa'$ has two nodal domains and so $\lambda = -1$ must be equal either to the second or the third eigenvalue of l_τ. We also know from (27) that

$$L_\tau(\varphi^{-1}\varphi') = 0.$$

From this equation and since the function $\varphi^{-1}\varphi'$ has two nodal domains, the eigenvalue $\lambda = 0$ must also be equal either to the second or the third eigenvalue of l_τ. As a conclusion of this discussion, we have that -1 is the second eigenvalue and 0 is the third eigenvalue of l_τ, proving (32).

From (21) we have

$$\left(\frac{s_\tau}{2\pi}\right)^2 l_\tau = \partial_t^2 + \left(\frac{s_\tau}{2\pi}\right)^2 P_\tau(t),$$

where the potential P_τ is

$$P_\tau(t) = 2\left((H^2-1)\varphi^2 + \tau^4\varphi^{-2}\right)(\frac{s_\tau}{2\pi}t).$$

Using (17), (22) and (21) we get

$$\varphi(\frac{s_\tau}{2\pi}t) = \varphi_M \; dn \left(\frac{K(\xi)}{\pi} t - K(\xi), \xi \right) = \varphi_m \; dn^{-1} \left(\frac{K(\xi)}{\pi} t, \xi \right), \tag{33}$$

where $\xi = 1 - \varphi_m^2 / \varphi_M^2$.

Recall that (21) yields

$$\varphi_m \varphi_M = \tau^2 \sqrt{H^2 - 1}.$$

Combining this together (22) and (33), we obtain

$$\left(\frac{s_\tau}{2\pi} \right)^2 P_\tau(t) = 2 \left(\frac{K(\xi)}{\pi} \right)^2 \left(\frac{1-\xi}{dn^2} + dn^2 \right) \left(\frac{K(\xi)}{\pi} t, \xi \right). \tag{34}$$

A simple study of the function

$$z \longmapsto (1-\xi)\, z^{-2} + z^2,$$

with $z \in [\sqrt{1-\xi}, 1]$, together with (34) shows that for all $t \in \mathbb{R}$, we have

$$4 \left(\frac{K(\xi)}{\pi} \right)^2 \sqrt{1-\xi} \leq \left(\frac{s_\tau}{2\pi} \right)^2 P_\tau(t) \leq 2(2-\xi) \left(\frac{K(\xi)}{\pi} \right)^2. \tag{35}$$

Using (22) again, we get (30).

In order to prove (31), the expression (34) yields

$$\partial_\tau \left(\left(\frac{s_\tau}{2\pi} \right)^2 P_\tau \right) = 2 \partial_\tau \xi \frac{K'}{K} \left(\frac{s_\tau}{2\pi} \right)^2 P_\tau - 2 \left(\frac{K}{\pi} \right)^2 \left(\frac{\partial_\tau \xi}{dn^2} + \partial_\tau dn \left(\frac{2(1-\xi)}{dn^3} - dn \right) \right).$$

By using (35), together with the properties of the Jacobi functions given in (15)-(16)-(17) and that $\partial_\tau \xi = O(|\tau|^{-3})$, we deduce that the derivative of the potential (34) with respect to τ is bounded by a constant times $|\tau|^{-3}$.

Denote by $\psi_0(\tau)$ the eigenfunction of $L_{\tau,0}$ associated to $\lambda_0(\tau)$, which will be normalized to have norm equal to 1 with respect to the L^2-inner product $\langle , \rangle$. Owing to these estimates and to the standard formula from the eigenvalue perturbation theory [16], we have

$$\begin{aligned} \partial_\tau \left(\left(\frac{s_\tau}{2\pi} \right)^2 \lambda_0 \right) &= -\langle \partial_\tau \left(\left(\frac{s_\tau}{2\pi} \right)^2 L_{\tau,0} \right) (\psi_0), \psi_0 \rangle_{L^2} \\ &= -\langle \partial_\tau \left(\left(\frac{s_\tau}{2\pi} \right)^2 P_\tau \right) \psi_0, \psi_0 \rangle_{L^2} \\ &= O(|\tau|^{-3}). \end{aligned}$$

Then

$$2\lambda_0 s_\tau \partial_\tau s_\tau + (s_\tau)^2 \partial_\tau \lambda_0 = O(|\tau|^{-3}).$$

With this identity, we use the expansions of s_τ and $\partial_\tau s_\tau$ given in (18) to get the estimate (31). □

3.2 Proof of Theorem 1.1

Let us now apply the previous spectral analysis of the Jacobi operator for nodoids to prove Theorem 1.1. We start with the next definition.

Definition 3.1. *Let γ be a geodesic of $\mathbb{H}^3$. Given $\ell \in \mathbb{N}$, denote by $\Re_\ell$ the rotation about γ by angle $2\pi/\ell$ about the γ. A surface of $\mathbb{H}^3$ is $\Re_\ell$-symmetric if it is invariant by $\Re_\ell$, but not by any $\Re_{\ell'}$ for $\ell' \in \mathbb{N}$, $\ell' > \ell$.*

Recall that for all $\tau < 0$, any surface close enough to $\mathcal{D}_\tau$ corresponds to a smooth function ω over $\mathcal{D}_\tau$, namely, by $X_\omega = X_\tau + \omega N_\tau$ for some small smooth function ω. We know from (23) that this new surface has mean curvature equal to H if and only if ω is a solution of

$$L_\tau(\omega) + Q_\tau(\omega) = 0,$$

where Q_τ is a second order nonlinear differential operator. As it was already mentioned above, we shall use the variables (t, θ) defined in (28) instead of (s, θ) and thus, all functions involved in the parameterization of $\mathcal{D}_\tau$ are 2π-periodic on t. Define the following functional space:

Definition 3.2. *Given $\ell \in \mathbb{N}$ and $\alpha \in (0, 1)$, consider*

$$C^{2,\alpha}_\ell(\mathbb{R} \times \mathbb{S}^1) = \{u \in C^{2,\alpha}(\mathbb{R} \times \mathbb{S}^1) : u(t + 2\pi, \theta) = u(t, \theta)$$
$$u(-t, \theta) = u(t, \theta), u(t, \frac{2\pi}{\ell} + \theta) = u(t, \theta)\}.$$

Then the operators

$$L_\tau, Q_\tau : C^{2,\alpha}_\ell(\mathbb{R} \times \mathbb{S}^1) \longrightarrow C^{0,\alpha}_\ell(\mathbb{R} \times \mathbb{S}^1)$$

are well defined. This follows from the fact that if $\omega \in C^{2,\alpha}_\ell(\mathbb{R} \times \mathbb{S}^1)$, the surface X_ω is $\Re_\ell$-symmetric. Then its mean curvature belongs to $C^{0,\alpha}_\ell(\mathbb{R} \times \mathbb{S}^1)$ and hence, $Q_\tau(\omega) \in C^{0,\alpha}_\ell(\mathbb{R} \times \mathbb{S}^1)$.

Consider the Fourier decomposition of a function $u \in C^{2,\alpha}_\ell(\mathbb{R} \times \mathbb{S}^1)$ given by

$$u(t, \theta) = \sum_{j \in \mathbb{Z}} u_j(t) e^{ij\ell\theta},$$

where

$$u_j(t + 2\pi) = u_j(t), \qquad u_{-j}(t) = u_j(-t).$$

Then the spectrum of L_τ acting in $C^{2,\alpha}_\ell(\mathbb{R} \times \mathbb{S}^1)$ is

$$\mathrm{Spec}(L_\tau) = \bigcup_{k \in \mathbb{N}} \bigcup_{j \in \mathbb{N}} \left(\lambda_k(\tau) + j^2\ell^2\right),$$

where $\lambda_k(\tau)$ is the k-th eigenvalue of $L_{\tau,0}$. We are in conditions to prove Theorem 1.1. Fix $j, \ell \in \mathbb{N}$. By Proposition 3.1, the function

$$\tau \longmapsto \lambda_0(\tau) + j^2\ell^2$$

changes the sign as the parameter τ decreases from 0 to $-\infty$. Let

$$\tau'_{j,\ell} = \inf\{\tau \in (-\infty, 0) : \lambda_0(\tau) + j^2\ell^2 = 0\}$$

$$\tau_{j,\ell} = \sup\{\tau \in (\infty, 0) : \lambda_0(\tau) + j^2\ell^2 = 0\}.$$

By (30) and (31), the function $\tau \longmapsto \lambda_0(\tau)$ is increasing for τ close enough to $-\infty$. We conclude there exist $j_H, \ell_H \in \mathbb{N}$ such that, for all $j \geq j_H, \ell \geq \ell_H$, it holds

$$\tau'_{j,\ell} = \tau_{j,\ell}.$$

Denote

$$\tau_\ell = \tau'_{j,\ell} = \tau_{j,\ell}.$$

We check that the assumptions of Crandall-Rabinowitz theorem are fulfilled [2]. Let $\tau \leq \tau_\ell$.

1. First we have to prove that $\mathrm{Ker}(L_\tau)$ is a one dimensional subspace. This is because the properties of the functional space $C^{2,\alpha}_\ell(\mathbb{R} \times \mathbb{S}^1)$ together with that L_τ is a second order elliptic operator. Assume that $\mathrm{Ker}(L_\tau)$ spanned by $\psi_0(\tau)e^{ij\ell\theta}$.

2. Second, we need to check that $\mathrm{Im}(L_\tau)$ has codimension 1. This follows because L_τ is a self adjoint elliptic operator with one dimensional kernel.

3. Finally we will prove that the derivative with respect to τ of L_τ applied to $\psi_0(\tau)e^{ij\ell\theta}$ does not belongs to $\mathrm{Im}(L_\tau)$. We will argue by absurd. Assume that there exists $\psi \in C^{2,\alpha}_\ell(\mathbb{R} \times \mathbb{S}^1)$ such that

   $$L_\tau(\psi) = \partial_\tau L_\tau(\psi_0(\tau)e^{ij\ell\theta}).$$

   Multiply this equality by $\psi_0(\tau)e^{ij\ell\theta}$ with the L^2-inner product $\langle , \rangle$ and using that L_τ is self-adjoint, we conclude that

$$\begin{aligned}
\partial_\tau \lambda_0(\tau) &= \partial_\tau \left(\lambda_0(\tau) + j^2\ell^2\right) = -\partial_\tau \langle L_\tau(\psi_0(\tau)e^{ij\ell\theta}), \psi_0(\tau)e^{ij\ell\theta}\rangle \\
&= -\langle \partial_\tau(L_\tau)(\psi_0(\tau)e^{ij\ell\theta}), \psi_0(\tau)e^{ij\ell\theta}\rangle - 2\langle L_\tau \partial_\tau(\psi_0(\tau)e^{ij\ell\theta}), \psi_0(\tau)e^{ij\ell\theta}\rangle \\
&= -\langle L_\tau(\psi), \psi_0(\tau)e^{ij\ell\theta}\rangle - 2\langle \partial_\tau(\psi_0(\tau)e^{ij\ell\theta}), L_\tau(\psi_0(\tau)e^{ij\ell\theta})\rangle \\
&= -\langle \psi - 2\partial_\tau(\psi_0(\tau)e^{ij\ell\theta}), L_\tau(\psi_0(\tau)e^{ij\ell\theta})\rangle \\
&= 0
\end{aligned}$$

   which is a contradiction by Proposition 3.1.

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