Capillary drops with free boundary in a wedge

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Abstract

We study constant mean curvature compact surfaces with free boundary in a wedge. Under the hypothesis of stability or embeddedness, we prove that the surface is part of a sphere centered at the vertex.

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1. Introduction and the results

A wedge W in Euclidean space $\mathbb{R}^3$ is the region of the space between two intersecting planes and its vertex is the intersection of both planes. The free boundary Plateau problem in the wedge consists into find a compact orientable immersed surface M of stationary area among surfaces in W whose boundary lies on ∂W and preserving the volume enclosed by the surface. In such case, the surface M is characterized by the fact it has constant mean curvature and it meets orthogonally ∂W along its boundary. We say that M is a capillary surface with free boundary in W. In a similar context, constant mean curvature surfaces orthogonally meeting the boundary of a bounded 3-domain B appear as solutions for the partitioning problem to separate

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B into two bodies of prescribed volume [12]. In general, a capillary surface included in the wedge W is a compact surface immersed in $\mathbb{R}^3$ with constant mean curvature meeting ∂W at a constant angle γ along its boundary. Capillary surfaces are models of an incompressible liquid drops supported in a given substrate in the absence of gravity and the angle γ depends on the materials of the support. See the book of R. Finn [5] for an account of the subject.

A first example of a capillary surface with free boundary in a wedge is the part of a sphere centered at the vertex of W. If we displace this sphere with its center equidistant from the two planes that define W, then the part of the sphere included in W is an example of a capillary surface with contact angle $\gamma \neq \pi/2$. In [6], it is proved that if M is a capillary immersed disc with piecewise smooth boundary and the number of vertices is less than or equal to 3, then it is a piece of a sphere. Motivated by some previous results obtained by McCuan [9, 10], Park has shown that if a capillary surface is embedded, topologically an annulus and does not touch the vertex of the wedge, then it has to be part of a round sphere again [13].

In this paper, we study capillary surfaces with free boundary in a wedge with no restriction about its topology and assuming that the boundary ∂M can have many components. We address in two problems. First on stability, where a capillary surface is stable if the second variation of area is non negative for all volume preserving variations.

Theorem 1.1. *Let $\phi : M \to \mathbb{R}^3$ be a stable capillary compact surface with free boundary in a wedge W. Then $\phi(M)$ describes part of a sphere centered at the vertex.*

The result can experimentally be observed when one deposits a liquid drop on a wedge. Then one see that the fluid tends to the vertex to be physically more stable until that the drop converts spherical. Related with Theorem 1.1, if we change a wedge by two parallel planes, the only stable capillary surfaces which lie between two parallel planes are spherical caps and sufficiently short cylinders [2, 18]

The second problem consists into assume that the surface is embedded in $\mathbb{R}^3$, that is, with no self intersections. Our two results are:

Theorem 1.2. *Let $\phi : M \to \mathbb{R}^3$ be an embedded capillary compact surface with free boundary in a wedge W. Then $\phi(M)$ describes part of a sphere centered at the vertex.*

We point out that all these results establish a certain analogy between capillary surfaces with free boundary in a wedge with the three classical theorems that characterize the sphere in the family of closed constant mean curvature surfaces in $\mathbb{R}^3$. Indeed, the Hopf theorem [7] assumes that the closed surface has genus 0, which corresponds here with the Finn-McCuan's theorem; Theorem 1.1 corresponds with the Barbosa-do Carmo theorem on stability [3] and Theorem 1.2 with the Alexandrov theorem [1] assuming that the surface is embedded.

2. Proof of Theorem 1.1

We introduce the setting of our results: see [3, 17] for more details. Let W be a 3-domain of $\mathbb{R}^3$. Let M be an orientable (connected) compact surface with boundary ∂M and $\phi : M \to \mathbb{R}^3$ an immersion such that $\phi(M) \subset W$ and $\phi(\partial M) \subset \partial W$. A variation of ϕ is a a differentiable map $\Phi : M \times (-\epsilon, \epsilon) \to \mathbb{R}^3$, with $\Phi(p, 0) = \phi(p)$ and Φ_t is an immersion for any t. We say que Φ is admissible if $\Phi_t(\mathrm{int}M) \subset \mathrm{int}\ W$ and $\Phi_t(\partial M) \subset \partial W$ and the variation is called normal if $\frac{\partial \Phi}{\partial t}\big|_{t=0}$ is a normal vector field on M. Define the area functional

$$A(t) = \int_M dA_t,$$

where dA_t is the area element of M computed with the induced metric by Φ_t, and the volume is

$$V(t) = -\frac{1}{3} \int_M \langle N, \phi \rangle \ dA_t.$$

Here $V(t)$ represents the oriented volume of a cone over $\Phi_t(M)$ with respect to the origin on $\mathbb{R}^3$. The variation Φ is called volume-preserving if $V(t) = V(0)$ for all t.

We say that the immersion ϕ is stationary if $A'(0) = 0$ for any volume-preserving admissible variation of ϕ. Then an immersion ϕ is stationary if and only if ϕ has constant mean curvature and $\phi(M)$ intersects ∂W orthogonally.

A stationary immersion ϕ is said to be stable if $A''(0) \geq 0$ for all volume-preserving admissible normal variations of ϕ. The formula of the second variation of the area is

$$A''(0) = -\int_M (f\Delta f + |\sigma|^2 f^2) \ dM - \int_{\partial M} \Big(f \frac{f}{\partial \nu} + \Pi(N, N) f^2 \Big) ds, \qquad (1)$$

for all smooth functions f on M with $\int_M f\, dM = 0$. Here N is the Gauss map of ϕ, σ is the second fundamental form of ϕ, Δ is the Laplacian of the induced metric on M by ϕ, Π is the second fundamental form of ∂W in W, ν is the inward unit conormal along ∂M and $\partial f/\partial\nu$ is the partial derivative of f with respect to ν.

Lemma 2.1. *Let M be a surface and let $\phi : M \to \mathbb{R}^3$ be an immersion with constant mean curvature H with respect to the Gauss map N. Then the Laplacian of the support function $\langle N, \phi\rangle$ is*

$$\Delta\langle N, \phi\rangle = -2H - |\sigma|^2\langle N, \phi\rangle, \tag{2}$$

where $|\sigma|^2$ is the norm of the second fundamental form.

Proof. Let $a \in \mathbb{R}^3$ and we consider the tangent vector field $a^T = \nabla\langle\phi(p), a\rangle$, the tangent part of a. Then its divergence is

$$\Delta\langle\phi(p), a\rangle = \mathrm{trace}(v \longmapsto \nabla_v a^T) = \langle N(p), a\rangle \mathrm{trace}(\sigma) = 2H\langle N(p), a\rangle. \tag{3}$$

This formula holds for any surface of $\mathbb{R}^3$ without assuming that H is constant. On the other hand, let Y be a vector field of $\mathbb{R}^3$ and we write $Y = Z + uN$, where Z is a tangent vector field to M and $u = \langle N, Y\rangle$ is the normal component of Y. We consider a smooth variation M_t, $t \in (-\epsilon, \epsilon)$, of M whose variation vector field is uN. Then the variation of the mean curvature H_t of M_t changes according to

$$\partial_t(H_t)_{t=0} = \frac{1}{2}(\Delta u + |\sigma|^2 u) + \langle\nabla H, Z\rangle. \tag{4}$$

Assume that H is constant. Let us take the vector field $Y = a$, whose associated one-parameter subgroup generates translations. Then H_t is constant, obtaining from (4)

$$\Delta\langle N(p), a\rangle + |\sigma|^2\langle N(p), a\rangle = 0. \tag{5}$$

Taking into account that $f(p) = \sum_{i=1}^{3}\langle N(p), a_i\rangle\langle\phi(p), a_i\rangle$ where $\{a_i\}$ is an orthonormal basis of $\mathbb{R}^3$, we get (2) combining (3) and (5). □

Other ingredient in our proof is the so-called first Minkowski formula, more familiar in the boundaryless closed case (see for example [11]) and which is an immediate consequence of

$$\Delta|\phi|^2 = 4 + 4H\langle N, \phi\rangle.$$

Integrating and using the divergence theorem, we obtain

$$\int_M (1 + H\langle N, \phi\rangle)\ dM = -\frac{1}{2}\int_{\partial M} \langle \nu, \phi\rangle\ ds. \tag{6}$$

We are in position to prove Theorem 1.1. Let W be a wedge which, after a translation if necessary, we assume that its vertex contains the origin of $\mathbb{R}^3$. Let ϕ be an immersion under the hypothesis of theorem. Because $\phi(M)$ is orthogonal to the planes ∂W, the inward unit conormal vector ν is orthogonal to ∂W. Thus $\langle \nu, \phi\rangle = 0$ along ∂M because the origin of $\mathbb{R}^3$ lies in the vertex of W. From (6), we have

$$\int_M (1 + H\langle N, \phi\rangle)\ dM = 0. \tag{7}$$

Then the function $f = 1 + H\langle N, \phi\rangle$ is a candidate to be a test-function in (1).

On the other hand, according to a result of Joachimsthal [8], if two surfaces meet in a constant angle and if the intersection curve is a curvature line on one of the surfaces, then it is a curvature line on the another one. In our situation, $\phi(\partial M)$ is a curvature line in the support planes ∂W. Thus, $\sigma(\alpha'(s), \nu(s)) = 0$ along ∂M, where $\alpha(s)$ is a parametrization of ∂M. With this information, we compute $\partial f/\partial \nu$:

$$\frac{\partial f}{\partial \nu} = H\langle dN(\nu), \alpha\rangle = -H\Big(\sigma(\alpha', \nu)\langle \alpha', \phi\rangle + \sigma(\nu, \nu)\langle \nu, \alpha\rangle\Big) = 0. \tag{8}$$

Using (8) and the fact that $\Pi = 0$ on ∂W, the expression of $A''(0)$ writes now as

$$A''(0) = -\int_M (f\Delta f + |\sigma|^2 f^2) dM. \tag{9}$$

Finally, and because $\langle dN(\nu), \alpha\rangle = 0$ along ∂M, the divergence theorem in (2) yields

$$\int_M |\sigma|^2 \langle N, \phi\rangle\ dM = -\int_M 2H\ dM. \tag{10}$$

From (2), the integrand in (9) is

$$f\Delta f + |\sigma|^2 f^2 = |\sigma|^2 - 2H^2 - H(2H^2\langle N, \phi\rangle - |\sigma|^2\langle N, \phi\rangle).$$

Then using (6), (10) and the stability of M, we have

$$\begin{aligned} 0 \ \leq \ A''(0) &= \int_M (2H^2 - |\sigma|^2)\ dM + 2H^3 \int_M \langle N, \phi\rangle\ dM - \int_M 2H^2\ dM \\ &= \int_M (2H^2 - |\sigma|^2)\ dM - 2H^2 \int_M f\ dM = \int_M (2H^2 - |\sigma|^2)\ dM \leq 0, \end{aligned}$$

where the last inequality is due to $|\sigma|^2 \geq 2H^2$, which it holds on any surface. As a conclusion, $|\sigma|^2 = 2H^2$ on M, that is, ϕ is an umbilical immersion and so, $\phi(M)$ is an open of a plane or a sphere. The only possibility to be a capillary compact surface with free boundary in a wedge is that $\phi(M)$ is a part of a sphere with center at the vertex.

3. Proof of Theorem 1.2

In this part of the article we use the ideas from [4, Sect. 2]. For completeness, we briefly the method. We extend the Ros formula [15] in our setting of embedded capillary surfaces with free boundary in a wedge.

Lemma 3.1. *Let M be an embedded compact surface in a wedge such that M meets orthogonally ∂W. Let V denote the volume of the domain Ω enclosed by M and W. Assume that the mean curvature H of M is positive. Then*

$$\int_M \frac{1}{H}\ dM \geq 3V, \tag{11}$$

and the equality holds if and only if M is part of a sphere.

Proof. We recall the Reilly formula [14] for a bounded domain $D \subset \mathbb{R}^3$:

$$\int_D \left((\bar{\Delta} u)^2 - |\bar{\nabla}^2 u|^2 \right) = \int_{\partial D} \left((-2\Delta u + 2H \frac{\partial u}{\partial \eta}) \frac{\partial u}{\partial \eta} + \Pi(\nabla u, \nabla u) \right),$$

where the symbol $^-$ indicates that the computation is given in $\mathbb{R}^3$, Π is the second fundamental form of ∂D and η is the inward unit normal of ∂D. Take D a domain with smooth boundary obtained from Ω rounding smoothly ∂M a small distance $\epsilon > 0$. Let u be a smooth solution to the mixed problem

$$\begin{array}{ll} \bar{\Delta} u = 1 & \text{in } D \\ u = 0 & \text{on } \partial D \setminus \partial W \\ \dfrac{\partial u}{\partial \eta} = 0 & \text{on } \partial D \cap \partial W \end{array}$$

Using that $\Pi = 0$ on ∂W, the Reilly formula writes

$$\mathrm{Vol}(D) = 2\int_{\partial D \setminus \partial W} H\Big(\frac{\partial u}{\partial \eta}\Big)^2 + \int_D |\bar{\nabla}^2 u|^2 \geq 2\int_{\partial D \setminus \partial W} H\Big(\frac{\partial u}{\partial \eta}\Big)^2 + \frac{1}{3}\mathrm{Vol}(D),$$

where in the inequality we have used $(\bar{\Delta} u)^2 \leq 3|\bar{\nabla}^2 u|^2$ by the Cauchy-Schwarz inequality. Thus

$$\mathrm{Vol}(D) \geq 3\int_{\partial D \setminus \partial W} H\Big(\frac{\partial u}{\partial \eta}\Big)^2. \tag{12}$$

Moreover the divergence theorem gives

$$\mathrm{Vol}(D)^2 = \Big(\int_D \bar{\Delta} u\Big)^2 = \int_{\partial D \setminus \partial W} \Big(\frac{\partial u}{\partial \eta}\Big)^2 \leq \int_{\partial D \setminus \partial W} H\Big(\frac{\partial u}{\partial \eta}\Big)^2 \int_{\partial D \setminus \partial W} \frac{1}{H}$$

This inequality together (12) yields $3\mathrm{Vol}(D) \geq \int_{\partial D \setminus \partial W} 1/H$. Letting $\epsilon \to 0$, we obtain (11). If the equality holds, the Cauchy-Schwarz inequality implies that $\bar{\nabla}^2 u$ is proportional to the identity and because $\bar{\Delta} u = 1$, then $u(p) = (|p|^2 - a^2)/6$ for some constant a. Since $u = 0$ on M, then M is part of a sphere. $\square$

Once established the inequality (11), the proof of Theorem 1.2 is as follows. Consider M a surface under the hypothesis of theorem and let Ω be the bounded domain by M and ∂W, where the vertex of W contains the origin of $\mathbb{R}^3$. The volume $\mathrm{Vol}(\Omega)$ of Ω agrees with the one obtained by the cone with vertex at the origin, that is,

$$V = -\frac{1}{3}\int_M \langle N, \phi\rangle \, dM,$$

where N is the Gauss map in M pointing towards Ω because the unit normal vector to $\partial\Omega \cap \partial W$ is orthogonal to the position vector.

We claim that the mean curvature H of M is positive. For this, take $p_0 \in \partial M$ the point of maximum distance of ∂M from the vertex. Take P the parallel plane to the vertex of W which is orthogonal at the point p_0 to the plane of ∂W where p_0 lies. We parallel displace P sufficiently far from the vertex and intersecting W until a position that P does not touch M. New we come back P to its original position until the first time that P touches M again. By the definition of p_0 and P, this occurs at some interior point of M or just at the point p_0, where P and M are tangent by the orthogonality of M to ∂W. In both cases, we use the tangency principle to compare the mean curvatures of M and P concluding $H > 0$. This proves the claim.

The Minkowski formula (7) writes now as $A - 3HV = 0$, which yields equality in the Reilly formula (11). This proves that M is a part of a sphere and since must orthogonally meet ∂W, then the center of M lies in the vertex of W.

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