

Capillary surfaces in a cone

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Abstract

We show that a capillary surface in a solid cone, that is, a surface that has constant mean curvature and the boundary of surface meets the boundary of the cone with a constant angle, is radially graphical if the mean curvature is non-positive with respect to the Gauss map pointing toward the domain bounded by the surface and the boundary of the cone. In the particular case that the cone is circular, we prove that the surface is a spherical cap or a planar disc. The proofs are based on an extension of the Alexandrov reflection method by using inversions about spheres.

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1. Introduction and results

Consider a given amount of liquid deposited on a solid substrate with conical shape: see Figure 1. In absence of gravity, we analyse the equilibrium configurations when the drop reaches a state of critical interfacial area. To be precise, let (x, y, z) be the usual coordinates of \mathbb{R}^3 , where z indicates the vertical direction. Let \mathbb{S}^2 be the unit sphere centered at the origin O of \mathbb{R}^3 and let $D \subset \mathbb{S}^2$ be a simply-connected

domain of a hemisphere of \mathbb{S}^2 . Denote by C_D the cone defined by the union of all half-lines starting at O passing by all points of D and C_Γ the boundary of C_D , that is, the union of the rays that start at O through all points of $\Gamma = \partial D$. The point O is called the vertex of the cone. The cone C_D is called circular of opening angle $2\varphi \in (0, \pi)$ if $\Gamma \subset \mathbb{S}^2$ is a circle of radius $\sin(\varphi)$. A liquid drop in the cone is viewed as the closure of a domain $\Omega \subset \mathbb{R}^3$ which is confined in the cone C_D and whose boundary $\partial\Omega$ intersects C_Γ . The boundary $\partial\Omega$ of Ω is written by $\partial\Omega = T \cup S$, where $T \subset C_\Gamma$, $S = \partial\Omega \setminus T$ and $\partial\Omega$ is not smooth along its boundary ∂S . Physically, T is the wetted region by the liquid drop $\bar{\Omega}$ in C_Γ .

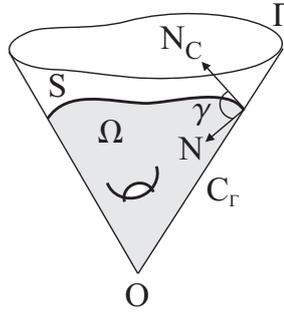


Figure 1: A drop Ω supported on a cone C_Γ . The interface S is a capillary surface, which means that S has constant mean curvature and the angle γ between the unit normal vectors N and N_C of S and C_Γ respectively, is constant along the boundary curve ∂S

In absence of gravity, and by the Young-Laplace equation, the shape of a liquid drop in equilibrium is characterized by the next two properties: 1) the surface S has constant mean curvature and 2) the angle γ which S meets C_Γ is constant. Here $\cos \gamma = \langle N, N_C \rangle$ along ∂S , where N and N_C denote the unit normal vector fields of S and C_Γ that point towards Ω and C_D respectively. With the above notation, we give the next definition:

Definition 1. *A capillary surface in a cone C_D is a compact embedded surface of constant mean curvature such that $\text{int}(S) \subset C_D$, $\partial S \subset C_\Gamma$ and which meets the cone C_Γ in a constant angle.*

In principle, a capillary surface in a cone can have higher topology, as well as, any number of boundary components. Physically capillary surfaces arise as the interface

of an incompressible liquid in a container (see [3] and references therein). In the context of liquid drops in a cone we refer to [7].

Examples of capillary surfaces in a circular cone C_D are some pieces of spheres and planar discs (see Figure 2). Indeed, take a round disc $D \subset \mathbb{S}^2$ centered at the north pole and radius $\sin(\varphi) \in (0, 1)$. Consider a sphere Σ centered at the positive z -axis of radius sufficiently large so that Σ intersects C_Γ . Then $S = \Sigma \cap C_D$ is a capillary surface in C_D . Depending on the values of γ and φ , we have: if $\gamma > \pi/2 + \varphi$, S is a concave interface (case (a)); if $\pi/2 - \varphi \leq \gamma < \pi/2 + \varphi$, S is a convex interface (case (b)); if $\gamma < \pi/2 - \varphi$, S is forced by two spherical caps (case (c)); and finally, if $\gamma = \pi/2 + \varphi$, we have a flat interface (case (d)).

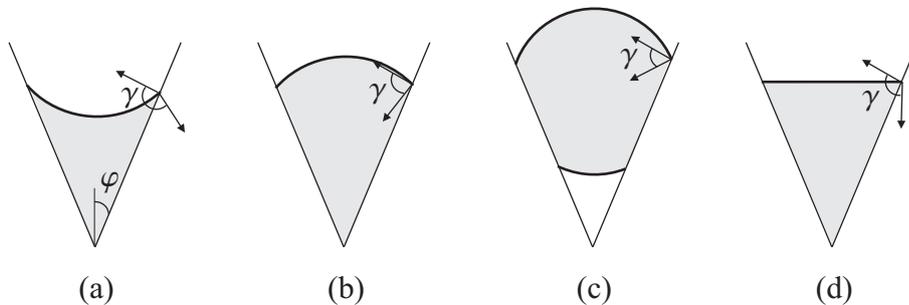


Figure 2: Different configurations of capillary spheres and discs on a circular cone

For a general cone, a first set of examples of capillary surfaces is considering non-parametric surfaces, that is, surfaces with a one-to-one central projection on D . We say then that the surface S is a radial graph, which implies that any half-line starting at O intersects S one point at most. Results of existence and uniqueness of radial graphs with constant mean curvature have been obtained in [8, 11, 14, 15, 17]. Radial graphs and capillary surfaces in a cone are the analogous examples of (vertical) graphs on a plane and capillary surfaces on a vertical cylinder if we move the vertex O of the cone at infinity. When the cone C_Γ is convex, Choe and Park have shown that if a parametric capillary surface S meets orthogonally C_Γ , then S is part of a sphere [2]. Ros and Vergasta proved that S has some topological restrictions if S is stable [12]. We also point out the results obtained by Vogel in [18]. He considered a drop Ω in a cone C_D such that its adherence $\bar{\Omega}$ contains the vertex O . Then he proved that the surface S is a radial graph if the mean curvature H is non-positive

with respect to the unit normal vector field pointing to Ω .

This is our starting point. We ask about the shape of a capillary surface in a cone and how the boundary of the surface imposes restrictions to the shape of the surface. For example, we pose the next questions:

1. Does exist a capillary surface in a cone C_D whose boundary is nullhomologous in $C_\Gamma \setminus \{O\}$? We do not know explicit examples and we hope that the answer is no at least if the surface is stable.
2. If the boundary ∂S is homologous to Γ in $C_\Gamma \setminus \{O\}$, is S a radial graph?
3. Under what conditions imply that a capillary surface is part of a sphere or a plane?
4. Under what conditions imply that a capillary surface is a bridge, that is, the surface has the topology of a portion of a cylinder bounded by two simple closed curves?

We extend the results of [18] assuming only the non-positivity of the mean curvature H . Recall that a compact embedded surface S contained in the cone C_D and $\partial S \subset C_\Gamma$ defines a closed 3-dimensional domain $\Omega \subset C_D$ such that its boundary $\partial\Omega$ is $\partial\Omega = T \cup S$ with $T \subset C_\Gamma$.

With the above notation, our result is the following:

Theorem 2. *Let S be a connected capillary surface supported in a cone C_D . Let fix the unit normal vector field N of S pointing towards the bounded domain Ω . If $H \leq 0$, then S is a radial graph and the boundary ∂S has only one connected component which is homologous to Γ in $C_\Gamma \setminus \{O\}$. In the particular case that the cone is circular, S is a planar disc or a spherical cap.*

In particular, Theorem 2 informs us that the non-positivity of the mean curvature of a capillary surface S in a cone implies that S is a topological disc.

2. The spherical reflection method

Alexandrov introduced the reflection method for surfaces with constant mean curvature proving that a sphere is the only closed embedded surface of constant mean curvature [1]. The idea is using the very surface as a barrier in a process of reflection

about a one parameter family of parallel planes and applying the maximum principle of the mean curvature equation. The Alexandrov method by planes has two steps. Firstly it is proved that for each spatial direction $v \in \mathbb{R}^3$, there exists an orthogonal plane Π_v to v such that the surface is invariant by reflections about Π_v . The second step proves that the only compact surface with this property is a sphere.

In this paper we use the *spherical reflection method* introduced by McCuan [10]. Studying capillary surfaces in a wedge, McCuan modifies successfully the Alexandrov reflection method by inversions about a one parameter family of spheres with the same center. Although an inversion does not preserve the mean curvature, one can have a certain control of the mean curvature of the inverted surface in order to use the maximum principle of elliptic PDEs theory. First, he proved that for each point of the axis of the wedge, there exists a sphere centered at this point such that the surface is invariant by the inversion about this sphere and next he showed that a sphere is the only surface with this property [9, 10, Th. 3.2]. Such as McCuan indicated in [9], the spherical reflection method can be of interest in other problems in global differential geometry of surfaces. Here we develop his technique on surfaces of constant mean curvature supported in a cone of Euclidean space.

We describe briefly the main ingredients of the spherical reflection method and we refer to the reader to [9, 10]. Let $\mathbb{S}_r \subset \mathbb{R}^3$ be the sphere of radius r centered at O . The spherical reflection about \mathbb{S}_r is the inversion mapping defined by

$$\phi_r : \mathbb{R}^3 \setminus \{O\} \rightarrow \mathbb{R}^3 \setminus \{O\}, \quad \phi_r(p) = \frac{r^2}{|p|^2}p.$$

Here the center of inversion is taken as the origin O and r is the radius of the inversion. If the radius r is known in the context, we only write ϕ instead ϕ_r . We introduce the next notation. If $p \in \mathbb{R}^3$, we denote $\hat{p} = \phi(p)$. If $A \subset \mathbb{R}^3$, denote $A(r)^- = \{p \in A : |p| > r\}$ and $A(r)^+ = \{p \in A : |p| < r\}$. Let $\hat{A}(r) = \phi(A(r)^-)$ be the image of $A(r)^-$ by the spherical reflection about \mathbb{S}_r . We write $A \geq B$ if for every $p \in A$, $q \in B$ such that p and q lie in the same ray from the origin, we have $|p| \geq |q|$.

Let S be an oriented surface of \mathbb{R}^3 , $O \notin S$ and let N be the Gauss map of S . By a spherical reflection ϕ , $\phi(S)$ is a surface where the Gauss map \hat{N} is

$$\hat{N}(\hat{p}) = N(p) - \frac{2\langle N(p), p \rangle}{|p|^2}p. \tag{1}$$

An easy calculation yields

$$\hat{\lambda}_i(\hat{p}) = \frac{\lambda_i(p)|p|^2 + 2\langle N(p), p \rangle}{r^2}, \quad i = 1, 2,$$

where $\hat{\lambda}_i$ and λ_i are the principal curvature of the surface $\phi(S)$ and S respectively. Hence the mean curvature \hat{H} of $\phi(S)$ at \hat{p} is

$$\hat{H}(\hat{p}) = \frac{H|p|^2 + 2\langle N(p), p \rangle}{r^2}, \quad (2)$$

For the second part of the spherical reflection method, we need the following result and whose proof done below simplifies the given one in [10].

Proposition 3. *Let S be a surface of constant mean curvature. If as a set, S is invariant by an inversion about a sphere \mathbb{S}^2 , then S is either part of a sphere or part of a plane passing through the center of the sphere.*

Proof. After a rigid motion, we assume that S is invariant by the inversion about the sphere \mathbb{S}_r . Since $\hat{H}(\hat{p})$ is given by (2) and $\phi(S(r)^-) = S(r)^+$, then

$$|p|^2 H + 2\langle N(p), p \rangle = Hr^2 \quad (3)$$

on S . At each point $p \in S$, let $\{e_1, e_2\}$ be a basis of principal directions, that is, $(dN)_p(e_i) = -\lambda_i(p)e_i$, $i = 1, 2$. Then the derivative of (3) along e_i gives

$$2(H - \lambda_i(p))\langle p, e_i \rangle = 0, \quad i = 1, 2. \quad (4)$$

Assume that at a point $p \in S$, $N(p)$ is not parallel to p . Then this occurs in an open set $V \subset S$ around p . We conclude by (4) that for some $i \in \{1, 2\}$, $\lambda_i = H$ on V . Since $2H = \lambda_1 + \lambda_2$, we derive that $\lambda_1 = \lambda_2 = H$ on V and all points of V are umbilics. Because the umbilics in a non-umbilical surface of constant mean curvature are isolated [5], we conclude that S is part of a sphere or part of a plane passing through the origin. If $N(p)$ is parallel to p on S , that is, $N(p) = \pm p/|p|$ on S , then one easily concludes that S is a sphere centered at the origin O , obtaining the result again. \square

We also need the tangency principle based on the maximum principle of mean curvature equation.

Proposition 4 (Tangency principle). *Let S_1 and S_2 be two surfaces that are expressed locally as graphs of functions u_1 and u_2 over the closure of a planar domain $\Omega \subset \mathbb{R}^2$. Assume $p_0 \in S_1 \cap S_2$ is a common tangent point and denote $p_0 = u_i(x_0, y_0)$, $(x_0, y_0) \in \overline{\Omega}$. Suppose p_0 satisfies one of the next conditions:*

1. $p_0 \in \text{int}(S_1) \cap \text{int}(S_2)$, or equivalently, $(x_0, y_0) \in \Omega$ (interior point).
2. $p_0 \in \partial S_1 \cap \partial S_2$ and ∂S_1 and ∂S_2 are tangent at p_0 . This is equivalent to $(x_0, y_0) \in \partial\Omega$ and $\partial\Omega$ is smooth at (x_0, y_0) (boundary point).
3. $(x_0, y_0) \in \partial\Omega$ is a point where Ω is a quadrant of \mathbb{R}^2 and (x_0, y_0) is the corner of this quadrant (corner point).

We orient S_i by the unit normal vector fields such that $N_1(p_0) = N_2(p_0)$ and consider the reference system of \mathbb{R}^3 where the vector $N_i(p_0)$ indicates the positive vertical line. Denote by H_i the mean curvature of S_i according to the orientation given by N_i , $i = 1, 2$. If in a neighborhood of p_0 , S_1 lies above S_2 and $H_1 \leq H_2$, then S_1 coincides with S_2 in an open set around p_0 .

The first two cases are a direct consequence of the Hopf maximum principle [4], usually called, the interior point version and the boundary point version of the tangency principle. See also [13]. The third case was shown by Serrin in [16] and is called the Serrin corner version of the tangency principle.

3. Proof of Theorem 2

Denote $\mathcal{S} = \partial\Omega$ and we assume that the vertex O is the origin of coordinates. Recall that $\mathcal{S} = T \cup S$, where T is the part of $\partial\Omega$ in the cone C_Γ . Consider the family of spheres $\{\mathbb{S}_r : r > 0\}$ centered at O . Because $\overline{\Omega}$ is compact, consider r so large such that $S(r)^- = \emptyset$. We decrease the radius of the sphere \mathbb{S}_r until the first value $r_0 > 0$ such that \mathbb{S}_{r_0} touches $\overline{\Omega}$. Because S is embedded, for a small $\epsilon > 0$, we have

$$\hat{\mathcal{S}}(r) \geq \mathcal{S}(r)^+ \quad \text{and} \quad \hat{\mathcal{S}}(r) \subset \Omega,$$

for all $r \in (r_0 - \epsilon, r_0)$. For these values of r , the support function $f(p) = \langle N(p), p \rangle$ is negative for any $p \in S(r)^-$ because N points to Ω . Next, we decrease the radius of \mathbb{S}_r and reflecting $\mathcal{S}(r)^-$ about \mathbb{S}_r . Let

$$r_1 = \inf\{r : \hat{\mathcal{S}}(r) \geq \mathcal{S}(r)^+, \hat{\mathcal{S}}(r) \subset \Omega\}.$$

We have two possibilities:

Case 1. $r_1 = 0$. In such a case, for all $r \in (0, r_0)$, $\hat{\mathcal{S}}(r) \subset \Omega$. This shows that S is a radial graph on D , proving the first part of Theorem.

Case 2. $r_1 > 0$. We are going to prove that this case is not possible. For this value r_1 , there exists a touching point p_0 between $\mathcal{S}(r_1)^+$ and $\hat{\mathcal{S}}(r_1)$. Denote $q_0 \in S(r_1)^-$ such that $p_0 = \hat{q}_0 = \phi_{r_1}(q_0)$. The possibilities of touching point p_0 are the following (see Theorem 4 in [9]):

[T1] The point p_0 belongs to $\text{int}(S)$ with $p_0 \in \text{int}(S(r_1)^+) \cap \text{int}(\hat{S}(r_1))$. Then $|p_0| < r_1$ and $f < 0$ on $S(r)^-$ for $r \in [r_1, r_0]$. In particular, p_0 is common tangent point of $S(r_1)^+$ and $\hat{S}(r_1)$.

[T2] The point p_0 belongs to $\text{int}(S)$ with $p_0 \in \partial S(r_1)^+ \cap \partial \hat{S}(r_1)$. In such case $|p_0| = r_1$, $f(p_0) = 0$. Moreover, $f < 0$ on $S(r)^-$ for $r \in (r_1, r_0]$. Again p_0 is common tangent point of $S(r_1)^+$ and $\hat{S}(r_1)$ and $\partial S(r_1)^+$ and $\partial \hat{S}(r_1)$ are tangent at p_0 .

[T3] The point p_0 belongs to ∂S with $|p_0| < r_1$. In particular, $p_0 \in \partial \hat{S}(r_1) \cap \partial S(r_1)^+$. We claim that the tangent planes of $\hat{S}(r_1)$ and $S(r_1)^+$ coincide. Denote by N_C the Gauss map of C_Γ pointing to Ω . Let $\alpha = \alpha(s)$ be an arc-length parametrization of ∂S . At the boundary points ∂S , we take the orthonormal frame

$$\left\{ N_C, \frac{\alpha}{|\alpha|}, \mathbf{b} := N_C \times \frac{\alpha}{|\alpha|} \right\},$$

where \times denotes the cross product of \mathbb{R}^3 . Then the expression of N is

$$N = (\cos \gamma)N_C + \lambda(s)\frac{\alpha}{|\alpha|} + \mu(s)\mathbf{b},$$

where γ is the contact angle and λ and μ satisfy $\lambda^2 + \mu^2 = (\sin \gamma)^2$. By (1),

$$\hat{N}(s) = (\cos \gamma)N_C(s) - \lambda(s)\frac{\alpha(s)}{|\alpha(s)|} + \mu(s)\mathbf{b}(s). \quad (5)$$

Let $p_0 = \hat{\alpha}(s_0) = \phi(\alpha(s_0))$. Since $q_0 = \alpha(s_0)$ and $\hat{\alpha}(s_0)$ lie on the same ray starting from O ,

$$N(p_0) = N(\hat{\alpha}(s_0)) = (\cos \gamma)N_C(s_0) + \lambda_1\frac{\alpha}{|\alpha|}(s_0) + \mu_1\mathbf{b}(s_0), \quad (6)$$

with $\lambda_1, \mu_1 \in \mathbb{R}$. Recall

$$\phi(p) = \frac{r_1^2}{|p|^2} p.$$

Then

$$d\phi(\alpha'(s)) = \frac{r_1^2}{|\alpha(s)|^2} \alpha'(s) - 2r_1^2 \frac{\langle \alpha(s), \alpha'(s) \rangle}{|\alpha(s)|^4} \alpha(s), \quad (7)$$

and from (5),

$$0 = \langle \hat{N}(s), d\phi(\alpha'(s)) \rangle = \lambda(s) \frac{r_1^2}{|\alpha(s)|^3} \langle \alpha(s), \alpha'(s) \rangle + \mu(s) \frac{r_1^2}{|\alpha(s)|^2} \langle \mathbf{b}(s), \alpha'(s) \rangle. \quad (8)$$

Since $|p_0| < r_1$, the contact between $\partial \hat{S}(r_1)$ and $\partial S(r_1)^+$ at p_0 is tangential. Then $d\phi(\alpha'(s_0))$ is parallel to $\hat{\alpha}'(s_0)$ and (7) implies

$$\begin{aligned} 0 &= \langle N(\hat{\alpha}(s_0)), d\phi(\alpha'(s_0)) \rangle \\ &= -\lambda_1 \frac{r_1^2}{|\alpha(s_0)|^3} \langle \alpha(s_0), \alpha'(s_0) \rangle + \mu_1 \frac{r_1^2}{|\alpha(s_0)|^2} \langle \mathbf{b}(s_0), \alpha'(s_0) \rangle. \end{aligned} \quad (9)$$

As $\langle \alpha(s_0), \alpha'(s_0) \rangle$ and $\langle \mathbf{b}(s_0), \alpha'(s_0) \rangle$ can not simultaneously vanish because it would imply $\alpha'(s_0) = 0$, we conclude from (8) and (9) that $\mu_1 \lambda(s_0) + \lambda_1 \mu(s_0) = 0$. As $\lambda(s_0)^2 + \mu(s_0)^2 = (\sin \gamma)^2 = \lambda_1^2 + \mu_1^2$, then $\lambda(s_0) = -\lambda_1$ and $\mu(s_0) = \mu_1$. Then (5) and (6) imply $N(\hat{\alpha}(s_0)) = \hat{N}(s_0)$. Thus the tangent planes of $\hat{S}(r_1)$ and $S(r_1)^+$ coincide at $p_0 = \hat{\alpha}(s_0)$. This completes the proof of the claim.

[T4] The point p_0 belongs to ∂S with $|p_0| = r_1$. Then $p_0 = q_0$. The same reasoning as above proves both $\hat{S}(r_1)$ and $S(r_1)^+$ as well as $\partial \hat{S}(r_1)$ and $\partial S(r_1)^+$ are tangent at p_0 .

In order to apply the tangency principle (Proposition 4), we compare the mean curvatures of $\hat{S}(r_1)$ and $S(r_1)^+$ in a neighborhood of the contact point p_0 . Denote by \hat{H} the mean curvature of $\hat{S}(r_1)$ with respect to the unit normal vector \hat{N} . Using (1), we have that both N and \hat{N} point towards Ω because $f < 0$ for any $r \in (r_1, r_0)$. On the other hand, remark that for all $r \in [r_1, r_0]$ and $q \in S(r)^-$, we have

$$\hat{H}(\hat{q}) = \frac{|q|^2 H + \langle N(q), q \rangle}{r_1^2} \leq \frac{|q|^2}{r_1^2} H \leq H, \quad (10)$$

because $H \leq 0$ on S and $|q| \geq r_1$. In the McCuan's notion, the inequality (10) is expressed by saying that S satisfies the boundedness property [9].

We analyse the different cases of touching points that have appeared in the previous discussion.

- [T1] Because $\hat{S}(r_1) \geq S(r_1)^+$ around p_0 and $N(p_0) = \hat{N}(p_0)$ point to Ω , the surface $\hat{S}(r_1)$ lies above $S(r_1)^+$ in a neighborhood of the point p_0 . In this neighborhood, $\hat{H} \leq H$ and the tangency principle implies that $\hat{S}(r_1) = S(r_1)^+$ in an open set around p_0 . Repeating the reasoning, it is proved that the set of points where $\hat{S}(r_1)$ coincides with $S(r_1)^+$ is an open set in S , that is, S is invariant by the inversion about \mathbb{S}_{r_1} .
- [T2] This case is similar to the above one with the difference that p_0 is a boundary point of $\hat{S}(r_1)$ and $S(r_1)^+$. The tangency principle in its boundary point version concludes that S is invariant by the inversion about \mathbb{S}_{r_1} .
- [T3] This case is the same the above one because $|p| < r_1$ and thus, it is a common boundary point of $\hat{S}(r_1)$ and $S(r_1)^+$.
- [T4] Now the surfaces $\hat{S}(r_1)$ and $S(r_1)^+$ are tangent at a common boundary point p_0 where both surfaces may be expressed locally as a graph over a quadrant in the common tangent plane at p_0 . We apply the Serrin corner point version of the tangency principle.

In all cases, an argument using connectedness proves finally that S is invariant by the inversion about \mathbb{S}_{r_1} . Clearly, any plane passing through the origin can not be a capillary surface in the cone C_D and, finally, Proposition 3 gives that S is part of a sphere. Since S is invariant by the inversion about \mathbb{S}_{r_1} , S is either part of \mathbb{S}_{r_1} or part of sphere which is not containing the origin inside of S . With the unit normal vector pointing toward bounded domain, the mean curvature is strictly positive. This contradicts the condition on the mean curvature. An analysis of the different configurations of a capillary sphere in a cone C_D with non-positive mean curvature with respect to the unit normal vector field pointing Ω gives that C_Γ is a circular cone and S is a concave sphere (Figure 2, case (a)). But this sphere is not invariant by inversions about a sphere of type \mathbb{S}_r . This proves that the case 2) does not occur.

We prove the last statement of Theorem 2. Assume now that C_Γ is a circular cone. By the previous argument, we know that S is a radial graph on D . We claim that ∂S is a circle that lies a plane which is perpendicular to the axis L of the circular

cone C_Γ .

Suppose on the contrary. Denote by p_M and p_m the points of ∂S which lie at the maximum and minimum distance from O . Let l_M and l_m the half-lines from O passing through p_M and p_m respectively. Since ∂S is not a circle and it is a radial graph on Γ , $l_M \neq l_m$. Rotate S about L until l_M coincides with l_m and we call S^* the new position of S . If $A \subset \mathbb{R}^3$ is a subset of \mathbb{R}^3 , denote by ρA the image of A by the homothety of ratio $\rho > 0$. Let $\rho < 1$ be small enough such that ρS^* is disjoint to S . By homothetically expanding ρS^* , that is, increasing the value of ρ , we arrive until the first value $\rho = \rho_1$ such that $\rho_1 S^*$ touches S . Because $|p_m| < |p_M|$, the value ρ_1 satisfies $\rho_1 < 1$. We analyze the different types of contact points between $\rho_1 S^*$ and S .

If the contact point p_0 occurs at some interior point between $\rho_1 S^*$ and S , then the unit normal vectors of $\rho_1 S^*$ and S agree at p_0 because both vectors point in the opposite direction of p_0 . Around p_0 , consider the reference system whose vertical direction is given by the unit normal vector at p_0 . With respect to this reference system, the surface $\rho_1 S^*$ lies above S in a neighborhood of p_0 . However, the mean curvature of $\rho_1 S^*$ is H/ρ_1 and because $H \leq 0$, we have $H/\rho_1 < H$ if $H < 0$ or $H/\rho_1 = H$ if $H = 0$. This is a contradiction with the tangency principle.

If the touching point p_0 is a boundary point, then $\rho_1 S^*$ and S are tangent at p_0 because the constancy of the contact angle γ (homotheties and rotations about L do not change the value of γ). We apply again the tangency principle, obtaining a contradiction again. As a conclusion, $|p_M| = |p_m|$ and every point of ∂S has the same distance from O . Because Γ is a circle, then ∂S is a circle too. This completes the proof of the claim.

Once proved that ∂S is a circle, the proof of Theorem 2 ends as follows. If $H = 0$, S is a planar disc by the convex hull property that satisfy minimal surfaces. Suppose now $H \neq 0$. Let Π be the plane containing the circle ∂S . Since S is a radial graph, $S \cap (\Pi \setminus \overline{W}) = \emptyset$ where $W \subset \Pi$ is the disc bounded by ∂S . A result due to Koiso [6] shows that S lies in one side of Π and finally, the Alexandrov reflection method by using planes orthogonal to Π proves that S is a spherical cap. This completes the proof of Theorem 2. \square

As a consequence of this theorem, we give some partial answers to the questions posed in the Introduction. Using that Theorem 2 asserts that the boundary curve

∂S is homologous to Γ in $C_\Gamma \setminus \{O\}$, we obtain:

Corollary 5. *There are no capillary surfaces on a cone C_D with non-positive mean curvature whose boundary is nullhomologous in $C_\Gamma \setminus \{O\}$.*

Corollary 6. *There are no capillary bridges on a cone with non-positive mean curvature.*

We point out that for some contact angles, there exist capillary spherical domains in non-circular cones. For instance, this occurs when the contact angle $\gamma = \pi/2$. For this, take a domain D in a hemisphere of \mathbb{S}^2 and consider $D_\rho = \rho D$ the image of D by the homothety of ratio $\rho > 0$. Then D_ρ is a capillary surface in the cone C_D intersecting orthogonally C_Γ . In this example, the mean curvature of D_ρ is positive with respect to the unit normal vector pointing to Ω .

The next result has its origin in Propositions 2.1 and 2.2 of [18], but we do not assume that $\bar{\Omega}$ wets the origin.

Corollary 7. *Let C_D be a circular cone of opening angle 2φ . Let S be a connected capillary surface S in C_D whose unit normal vector field N points to Ω . Let γ be the contact angle.*

1. *If $\gamma < (\pi + \varphi)/2$, then the mean curvature H is positive.*
2. *If $\gamma > (\pi + \varphi)/2$, then S is a concave spherical cap.*
3. *If $\gamma = (\pi + \varphi)/2$, then S is a planar domain.*

Proof. For each γ , there exists an umbilical surface U_γ in C_D spanning Γ and meeting C_Γ with angle γ . Consider on U_γ the unit normal vector pointing to the domain Ω_γ bounded by U_γ and C_Γ . Because $U_\gamma \subset C_D$, then $0 \leq \gamma \leq \pi$. In fact, the surface U_γ is a planar disc if $\gamma = (\pi + \varphi)/2$ and a spherical cap if $\gamma \neq (\pi + \varphi)/2$. See Figure 2. Moreover,

- (i) If $\gamma < (\pi + \varphi)/2$, U_γ lies over the plane Π containing Γ and the mean curvature is positive. If $\gamma < \pi/2 - \varphi$, then U_γ is a large spherical cap and if $\gamma > \pi/2 - \varphi$, U_γ is a small spherical cap. In the case $\gamma = \pi/2 - \varphi$, U_γ is a hemisphere.
- (ii) If $\gamma > (\pi + \varphi)/2$, U_γ lies below the plane Π containing Γ and the mean curvature is negative.

Assume that $H \leq 0$. Then Theorem 2 implies that S is a planar disc or a spherical cap. By the above description of the configurations of all U_γ depending on the contact angle γ , the only possibilities are the cases (2) and (3). As a conclusion, if $\gamma < (\pi + \varphi)/2$, then H must be positive. \square

Finally, we obtain a result of symmetry using the Alexandrov reflection method by planes.

Proposition 8. *Consider a circular cone C_Γ whose axis L is the vertical z -axis of \mathbb{R}^3 and let S be a capillary surface in C_D . Assume that the boundary ∂S lies in a half-cone, that is, $\partial S \subset \{p \in C_\Gamma : \langle p, a \rangle > 0\}$ for some unit horizontal vector $a \in \mathbb{R}^3$. Then S is symmetric with respect to a vertical plane.*

Proof. Without loss of generality, we assume that $a = (1, 0, 0)$. As usual, let $\Omega \subset C_D$ be the bounded domain by S and C_Γ and let choose on S the Gauss map N that points towards Ω . Consider the planes $P(t)$ of equation $x = t$ for $t \in (-\infty, 0]$. We view the x -axis as an oriented line where the right direction points towards $x \rightarrow +\infty$.

We start with the Alexandrov method with reflections about the planes $P(t)$ from $t = -\infty$. We move $P(t)$ to the right by $t \nearrow 0$ until the first contact with S . A possible case is that the contact occurs at $t = t_0 < 0$. We move $P(t)$ slightly to the right and we reflect the left side of S about $P(t)$. For values $t \in (t_0, t_0 + \epsilon)$, the reflected surface lies in Ω . We move $P(t)$ on the right and reflecting the part of S on the left with respect to $P(t)$. If there is a contact point at $t = t_1$ with $t_1 \leq 0$ and $t_0 < t_1$, the touching point must be interior by the geometry of the circular cone C_Γ and because the boundary ∂S lies in the half-space $\mathbb{R}_a^3 = \{(x, y, z) \in \mathbb{R}^3 : x > 0\}$. Then the tangency principle implies that S has a symmetry plane $P(\bar{t})$ for some $\bar{t} \leq 0$, which it is impossible since ∂S lies in the half-space \mathbb{R}_a^3 . Moreover, by the geometry of C_Γ there is not contact point with some boundary point. This means that we arrive until $t = 0$ and the reflected surface lies in Ω . We call this last possibility case (i).

Case (ii). The other possibility is that when we move $P(t)$ from $t = -\infty$, there is no contact point between $P(t)$ and S for all $t \leq 0$.

In both cases (i) and (ii), we arrive with the planes $P(t)$ until the position $t = 0$ and we stop here.

Now we change the planes $P(t)$ in the reflection method by the one parameter family of planes $Q(t)$ that contain the axis L and parametrized by the angle $t \in [0, \pi]$ that makes $Q(t)$ with the vector $(1, 0, 0)$. Exactly,

$$Q(t) = \{(x, y, z) \in \mathbb{R}^3 : \cos(t)x + \sin(t)y = 0\}.$$

Introduce the next notation:

$$S_Q(t)^- = \{(x, y, z) \in S : \cos(t)x + \sin(t)y < 0\},$$

$$S_Q(t)^+ = \{(x, y, z) \in S : \cos(t)x + \sin(t)y > 0\}$$

and $\tilde{S}_Q(t)$ the reflection of $S_Q(t)^-$ about $Q(t)$. For $t = 0, \pi$, $Q(0) = P(0) = Q(\pi)$. By the process followed until here, we have two possibilities according to the above discussion of cases (i) and (ii):

1. *Case (i)*. We know that $P(t)$ touches the first time S at $t = t_0 < 0$ and by the argument of moving to the right and reflecting, the reflected surface about the plane $P(0)$ lies in the domain Ω . At this position, indeed, when we stop at $P(0)$, we change by the reflection process about $Q(t)$, that is, we begin reflecting $S_Q(t)^-$ about $Q(t)$. Remark that the reflection about $Q(t)$ leaves invariant C_Γ because $Q(t)$ contains the axis L of C_Γ . For values t in some interval $[0, \epsilon)$ around $t = 0$, the reflected surface $\tilde{S}_Q(t)$ lies in the domain Ω . Letting $t \nearrow \pi$, we arrive the first time $t = t_1 < \pi$ such that $\tilde{S}_Q(t_1)$ has a first contact point p_0 with $S_Q(t_1)^+$. This occurs because $\partial S \subset \mathbb{R}_a^3$ and the planes $Q(t)$ sweep the half-cone $C_\Gamma \cap \mathbb{R}_a^3$. If p_0 is an interior point, then the interior tangency principle implies that $Q(t_1)$ is a plane of symmetry of S , proving the result. If $p_0 \in \partial S$, then the tangent planes of S and the reflected surface agree at p_0 by the constancy of the contact angle and because reflections do not change this angle. The tangency principle of the boundary point version or corner point version concludes that $Q(t_1)$ is a plane of symmetry of S again.
2. *Case (ii)*. By moving $P(t)$ from $t = -\infty$, we arrive until $t = 0$ and we have no contact with S . Then we do a similar argument of reflection with the planes $Q(t)$ as above. The difference now is that firstly we arrive until a first position that the planes $Q(t)$ touches S at some time $t = t_o \in (0, \pi)$. Next, we follow increasing $t \nearrow \pi$ and reflecting as in the above case, until that we find a time $t = t_1 \in (0, \pi)$, $t_1 > t_o$, such that $\tilde{S}_Q(t_1)$ touches the first time with $S_Q(t_1)^+$. The same argument for the case (i) proves that $Q(t_1)$ is a plane of symmetry of S .

□

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