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Capillary surfaces of constant mean curvature in a right solid cylinder

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In this paper we investigate constant mean curvature surfaces with nonempty boundary in Euclidean space that meet a right cylinder at a constant angle along the boundary. If the surface lies inside of the solid cylinder, we obtain some results of symmetry by using the Alexandrov reflection method. When the mean curvature is zero, we give sufficient conditions to conclude that the surface is part of a plane or a catenoid.

1 Introduction

Capillary surfaces arise from the physical phenomena that occurs whenever two different materials are situated adjacent to each other and do not mix. If one (or both) of the materials is a fluid, which forms with another fluid (or gas) a free surface interface, then the interface is a capillary surface. A great deal of work has been devoted to nonparametric capillary surfaces since the initial works of Young and Laplace in the early nineteenth century (see [5]). In absence of gravity, a given amount of liquid placed on a solid substrate Σ in an equilibrium configuration meets Σ in a constant angle γ. Moreover, the mean curvature H of the air-liquid interface S of the drop is proportional to the pressure change across S. In the case that these pressures are constant, S is a surface of constant mean curvature.

Let $\phi : S \to \mathbb{R}^3$ be an immersion of a compact oriented surface S in Euclidean space $\mathbb{R}^3$. If $\phi : S \to \phi(S)$ is a homeomorphism, we say that S is an embedded surface in $\mathbb{R}^3$ and we identify S with $\phi(S)$. Consider a closed region $U \subset \mathbb{R}^3$ with $\Sigma = \partial U$. Assume that $\phi(int(S)) \subset int(U)$ and $\phi(\partial S) \subset \Sigma$ and separating a bounded domain $W \subset U$ with a prescribed volume. The domain W is bounded by S and by pieces of Σ. Let $\gamma \in [0, \pi]$. Consider the problem for seeking critical points of the energy functional $\mathcal{E} = \text{area}(S) - (\cos\gamma)\,\text{area}(\partial W \cap \Sigma)$ in the space of compact surfaces with boundary contained in Σ and interior contained in $int(U)$ and preserving the volume of W. Then S is a critical point of this functional if and only if its mean curvature H is constant and $\phi(S)$ meets Σ in a constant angle γ along $\phi(\partial S)$ [2]. In such a case, we say that S is a *constant mean curvature (in short, CMC) capillary surface in U*. In absence of gravity, or in micro-gravity where gravity is negligible, a liquid drop in U resting on Σ is then viewed as the closure of a domain $W \subset \mathbb{R}^3$ such that its boundary ∂W is written by $\partial W = T \cup S$, where $T \subset \Sigma$ is the portion of Σ which is wetted by $\overline{W}$, i.e., $T = \partial W \cap \Sigma$. If we seek local minima of the functional $\mathcal{E}$ and we require that $\overline{W}$ satisfies a volume constraint, then the free surface $S = \partial W \setminus T$ is an embedded CMC capillary surface. We refer the reader to the book of R. Finn [5] for the study of capillarity, mainly in the nonparametric case and in the more general situation of presence of gravity.

We fix the setting U which we study CMC capillary surfaces and the corresponding notation. Consider (x, y, z) the usual coordinates in $\mathbb{R}^3$, where the z-axis indicates the vertical direction. Let Π be the plane $z = 0$ and $C \subset \Pi$ a smooth simple closed curve. Denote by $\Omega \subset \Pi$ the bounded domain by C. Define the right cylinder on base C as the set $\Sigma = C \times \mathbb{R}$. If C is a circle, we say that Σ is a circular right cylinder and denote by L its axis. In this

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article we will consider that the region U is either the right solid cylinder $K = \overline{\Omega} \times \mathbb{R}$ or the outside of K, that is, $\mathbb{R}^3 \setminus (\Omega \times \mathbb{R})$. Under this context, the regularity of a CMC capillary surface in K or outside of K is guaranteed by a result of Taylor which asserts that the boundary of a CMC capillary surface is a smooth curve provided the supporting surface Σ is smooth [14]. In our case, the right cylinder Σ is smooth because of the smooth base curve C.

An example of a CMC capillary surface in a circular right solid cylinder K determined by Σ is an appropriate piece of a Delaunay surface whose axis agrees with the one of Σ. A Delaunay surface is one of catenoids, unduloids, nodoids, cylinders and spheres which are obtained rotating the roulettes of the conics [3]. Another CMC capillary surface in the same circular right solid cylinder K is the round disc $\overline{\Omega} \times \{t\}$, which is a minimal surface ($H = 0$) intersecting orthogonally Σ. In all these examples, the boundaries of the surfaces are curves which are homotopic to C on Σ. Our interest is also addressed on CMC capillary surface S where the boundary ∂S is non-homotopic to C on Σ as it occurs when we deposit a small liquid drop in the wall Σ of the tube K in absence of gravity.

In Section 2, we study compact CMC capillary surfaces in a right solid cylinder. We prove that certain compact embedded CMC capillary surfaces have some symmetric planes by the Alexandrov reflection method (Theorems 2.2 and 2.4). In the case that the mean curvature of the capillary surface is zero and the boundary is graph on C, we prove that the minimal capillary surface is a planar surface $\overline{\Omega} \times \{t\}$ (Theorem 2.7).

In the last Section 3, we consider complete capillary surfaces of zero mean curvature in the exterior of Σ. Besides $\Pi \setminus \Omega$ meeting Σ with a right angle, a first example is obtained when we take a circle C in a catenoid. This circle separates the catenoid into two pieces and one of them is a graph on the exterior of the round disk Ω bounded by C. Moreover, this surface is a minimal surface included outside of K and meeting the right circular cylinder $\Sigma = C \times \mathbb{R}$ with a constant angle. We prove that these surfaces are the only minimal capillary surfaces lying outside of a right cylinder under reasonable assumptions on the behavior of the end of the surface (Theorem 3.2).

2 Compact CMC capillary surfaces in a right solid cylinder

In this section we consider a compact embedded CMC capillary surface S in a right solid cylinder K, where $\partial K = \Sigma = C \times \mathbb{R}$ and C is a simple closed smooth planar curve. Denote by $\partial S = \Gamma_1 \cup \cdots \cup \Gamma_n$, the decomposition of ∂S into its connected components. Our techniques use the Hopf maximum principle for constant mean curvature surfaces in its interior or boundary versions ([1], [12]) to compare the mean curvatures of two surfaces at a tangent point, which in our context it is the so-called *tangency principle*.

Proposition 2.1 (Tangency principle) *Let S_1 and S_2 be two orientable surfaces in $\mathbb{R}^3$ that are tangent at a common point p. Assume that p lies in the interiors of both S_1 and S_2 or $p \in \partial S_1 \cap \partial S_2$ and the tangent lines of ∂S_1 and ∂S_2 coincide at p. Let us orient S_1 and S_2 such that both orientations $N(p)$ agree. Assume that with respect to the reference system determined by $N(p)$, the surface S_1 lies above S_2 around p with respect to $N(p)$, which it will denoted by $S_1 \geq S_2$. If $H_1 \leq H_2$ at p, then S_1 and S_2 coincide in a neighborhood of p.*

The first result informs us about the symmetries of a liquid drop inside of a right cylinder without gravity assuming that the boundary is nullhomotopic in the right cylinder.

Theorem 2.2 *Let Σ be a right cylinder and let S be a compact connected embedded CMC capillary surface in K. Assume that all the curves Γ_i, $i = 1, \ldots, n$, are nullhomotopic on Σ. Then S has a plane of symmetry parallel to Π. In addition, if Σ is a circular right cylinder with axis L and ∂S is strictly contained in a half cylinder determined by a plane containing L, then S has a plane of symmetry containing L and S is topologically a disc.*

P r o o f . Since the curves Γ_i, $i = 1, \ldots, n$, are nullhomotopic on Σ, the surface S together a bounded set $\Lambda \subset \Sigma$ defines a bounded 3-domain $W \subset K$. Let us orient S by the Gauss map N pointing to W and let H be the mean curvature of S. Consider the one parameter family of horizontal planes $\{\Pi(t) : t \in \mathbb{R}\}$ where $\Pi(t)$ is the plane of equation $z = t$ and we apply the Alexandrov reflection method [1]. See Figure 1, left. We describe briefly the main ingredients of the technique. If $p \in \mathbb{R}^3$, we denote $p = (p_1, p_2, p_3)$. Let us introduce the next notation. If $A \subset \mathbb{R}^3$, let $A(t)^+ = \{p \in A : p_3 \geq t\}$ and $A(t)^- = \{p \in A : p_3 \leq t\}$. Denote by $\hat{A}(t)$ the reflection

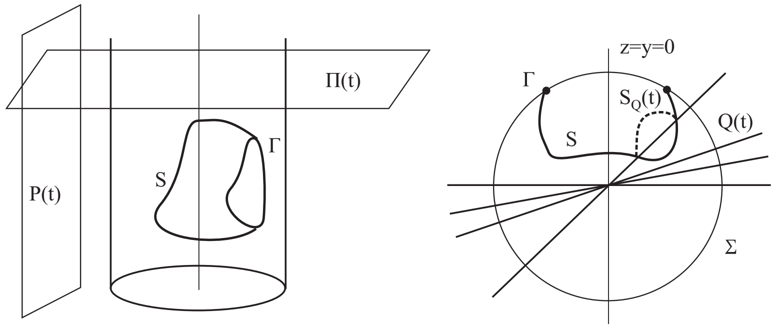

Fig. 1 Left: the planes $\Pi(t)$ and $P(t)$ in the Alexandrov reflection method; right: the capillary surface S and the planes $Q(t)$ viewed from the top of the right cylinder Σ.

of $A(t)^-$ about $\Pi(t)$. Since $\overline{W}$ is compact, consider t close to $-\infty$ such that $S(t)^- = \emptyset$. We increase the value t until the first value t_0 such that $\Pi(t_0)$ touches $\overline{W}$. Since S is embedded, for a small $\epsilon > 0$, we have

$$\hat{S}(t) \subset \overline{W} \text{ and } \hat{S}(t) \text{ is a graph on } \Pi(t), \tag{2.1}$$

for every $t \in (t_0, t_0 + \epsilon)$. Next move up $\Pi(t)$ and reflect $S(t)^-$ about $\Pi(t)$ until the first touching point p between $\hat{S}(t)$ and $S(t)^+$ occurs. Exactly, define

$$t_1 = \sup\left\{t > t_0 : \hat{S}(t) \subset \overline{W}, \hat{S}(t) \text{ is a graph on } \Pi(t)\right\}.$$

Since S is a compact surface, $t_1 < \infty$. Then for any $t > t_1$, some of the two conditions in (2.1) fails. We analyze the different possibilities.

1) There exists a common interior point p of $\hat{S}(t_1)$ and $S(t_1)^+$. Then $\hat{S}(t_1)$ and $S(t_1)^+$ are tangent at p. The induced orientation on $\hat{S}(t_1)$ given by the reflection about $\Pi(t_1)$ points towards W and the mean curvature is H. Thus $\hat{S}(t_1) \geq S(t_1)^+$ around p. Denote by T^+ the connected component of $S(t_1)^+$ which contains p, and T' the reflection about $\Pi(t_1)$ of a certain connected component T of $S(t_1)^-$ and containing p as an interior point. Then tangency principle implies that T^+ and T' agree in a neighborhood of p. By the analytic property of CMC surfaces, $T^+ = T' = \hat{T}(t_1)$. Also $T^+ \cup T$ forms a smooth connected component of S. Since S is connected, $S = T^+ \cup T$ and hence $\Pi(t_1)$ is a plane of symmetry of S, proving the result.
2) There exists $p \in (\partial\hat{S}(t_1) \cap \partial S(t_1)^+) \setminus \partial S$ where $\hat{S}(t_1)$ is not a graph on $\Pi(t_1)$ around p. In particular, the tangent plane T_pS is orthogonal to $\Pi(t_1)$. Then $\partial\hat{S}(t_1)$ and $\partial S(t_1)^+$ are tangent at p and $\hat{S}(t_1) \geq S(t_1)^+$ around p. By the boundary version of the tangency principle, $\hat{S}(t_1)$ coincides with $S(t_1)^+$ in a neighborhood of p. A similar argument as above proves that $\Pi(t_1)$ is a plane of symmetry of S.
3) There exists $p \in \partial\hat{S}(t_1) \cap \partial S(t_1)^+$ such that $p \in \partial S$. The hypothesis that S meets Σ with constant angle implies that $\hat{S}(t_1)$ and $S(t_1)^+$ are tangent at p. Moreover, both surfaces may be expressed locally as a graph on a quadrant in the common tangent plane T_pS, where p is the corner of the quadrant. Then we apply a tangency principle of the Hopf boundary point version so-called the Serrin corner lemma [13]. Again, the plane $\Pi(t_1)$ is a plane of symmetry of S.

Assume now the second part of theorem, that is, Σ is a circular right cylinder. Suppose that L is the z-axis and that ∂S lies in the half cylinder $\Sigma \cap \left\{(x, y, z) \in \mathbb{R}^3 : x > 0\right\}$. The proof consists of applying the Alexandrov reflection method with two families of planes. First, denote by $P(t)$ the vertical plane of equation $x = t$ and consider the one parameter family $\{P(t) : t \in (-\infty, 0]\}$. See Figure 1, left. Consider a similar notation as above indicating the subscript $-_P$ that the reflection is done with respect to the planes of type $P(t)$. Then we work with the Alexandrov method with reflections across the planes $P(t)$. For values t close to $-\infty$, $S_P(t)^- = \emptyset$. Next, let $t \to +\infty$ until $P(t)$ touches S the first time at $t = t_0$. Next we reflect $S_P(t)^-$ about $P(t)$ for $t > t_0$. Observe that by the symmetry of Σ we can go reflecting until at most $t = 0$. Suppose that one of the two properties in (2.1) fails for some value $t \leq 0$. Then define t_1 as above, where $t_1 \leq 0$. We prove that this situation yields a contradiction. Since the cylinder is circular and $\partial S \subset \left\{(x, y, z) \in \mathbb{R}^3 : x > 0\right\}$, the only possible cases of contact points are 1) and 2). The tangency principle would say that $P(t_1)$ is a plane of symmetry of S, which it is a contradiction with the fact that $\partial S \subset \{(x, y, z) \in \mathbb{R}^3 : x > 0\}$. Therefore, in the reflection process we can arrive until $t = 0$ having $\hat{S}_P(0) \subset \overline{W}$ and $\hat{S}_P(0)$ is a graph on $P(0)$. Notice that it is possible that $t_0 > 0$, that is, no plane $P(t)$, $t \leq 0$, intersects S, which it occurs if S is included in the half cylinder $K \cap \{(x, y, z) \in \mathbb{R}^3 : x > 0\}$.

Once arrived at the position $P(0)$, we change the reflection technique by other family of planes. Denote by $Q(t)$ the plane containing L where $t \in [-\pi/2, \pi/2]$ is the angle between the unit normal vector $(n_1(t), n_2(t), n_3(t))$, $n_1(t) \geq 0$ of $Q(t)$ and the vector $(1, 0, 0)$. Consider the one parameter family $\{Q(t) : t \in [-\pi/2, \pi/2]\}$. The Alexandrov reflection technique with changing directions appeared in [11]. We start from $t = 0$, reflecting the surface and letting $t \to \pi/2$ first. Here the subscript $-_Q$ denotes the reflection about the planes $Q(t)$. Remark that $P(0) = Q(0)$ in such way that the Alexandrov method works. The planes $Q(t)$ varying t in the interval $[0, \pi/2]$ sweep out the half of the cylinder $\Sigma \cap \{(x, y, z) \in \mathbb{R}^3 : x > 0\}$. See Figure 1, right. Assume there is a first time $t_1 \in (0, \pi/2)$ such that one of the two properties in (2.1) fails.Then we have the three possibilities of touching point as in the first part of the proof. The tangency principle assures that $Q(t_1)$ is a plane of symmetry of S. On the contrary, that is, when such time $t = t_1$ does not occur, we rotate $Q(t)$ in the opposite direction, that is, $t \searrow -\pi/2$. Then there is necessarily the first time $t_2 \in (-\pi/2, 0)$ such that $\hat{S}_Q(t_2)$ and $S_Q(t_2)^+$ have a common (interior, boundary or corner) touching point. By the tangency principle, $Q(t_2)$ is a plane of symmetry of S. So, S has a symmetric plane containing L. The statement on the topology of S is a consequence that the surface S is invariant by the symmetries about the planes $\Pi(t_1)$ and $Q(t_1)\big(\text{ or } Q(t_2)\big)$, which are orthogonal. Moreover, as a consequence of the Alexandrov method, each one of the parts that both planes separates S is a graph on these planes. □

Since the reflections about the planes $P(t)$ and $Q(t)$ do not change the third spatial coordinate, we deduce from the proof of Theorem 2.2:

Corollary 2.3 *Let Σ be a circular right cylinder and let S be a compact connected embedded surface included in K such that $\partial S \subset \Sigma$ and S meets Σ with a constant contact angle. Assume that the mean curvature H depends only on the z-coordinate. If ∂S is strictly contained in a half cylinder determined by a plane containing L, then there exists a plane P containing L such that S is invariant by the symmetry with respect to P.*

We note that the model of a liquid drop inside of a right solid cylinder with a vertical gravitational field is a surface whose mean curvature H is a linear function on the z-coordinate.

Theorem 2.4 *Let K be a right solid cylinder and let S be a compact connected embedded CMC capillary surface in K.*

1. *If ∂S is a connected curve homotopic to C in Σ, then S is a graph on Ω.*
2. *If ∂S has two connected components $\Gamma_1 \cup \Gamma_2$ and each curve Γ_i, $(i = 1, 2)$, is homotopic to C in Σ, then S has a plane of symmetry parallel to Π.*

Proof.

1. The boundary curve ∂S separates Σ into two components. Denote by Σ^+ the component such that for some $m > 0$, $\Sigma^+ \cap \Pi(t) \neq \emptyset$ for all $t \geq m$. Then S together Σ^+ separates $int(K)$ in two unbounded components and let us denote by W the component such that $W \cap \Pi(t) \neq \emptyset$ for all $t > m'$ and for some $m' > m$. We apply the Alexandrov method by the planes $\Pi(t)$ again as in Theorem 2.2 starting with planes $\Pi(t)$ for values t close to $-\infty$. Since S is compact, we obtain the value t_0 as in Theorem 2.2. Next, we increase t, reflecting $S(t)^-$ about $\Pi(t)$ until the value $t = t_1$. Since W is not compact, it can occur that $t_1 = +\infty$. In such a case, $\hat{S}(t)$ is a graph on $\Pi(t)$ for all t. Hence S is a graph on Ω, proving the result.
   The other possibility is that $t_1 < \infty$. We have three different possibilities about the point p similarly as it occurs in the proof of Theorem 2.2. So, the point p could be either an interior point between $\hat{S}(t_1)$ and $S(t_1)^+$, or a boundary point of both $\hat{S}(t_1)$ and $S(t_1)^+$ and $p \in \Pi(t_1)$, or a corner point of $\hat{S}(t_1)$ and $S(t_1)^+$. Using the hypothesis of constant contact angle between S and Σ, in the three cases we have $\hat{S}(t_1) \geq S(t_1)^+$ around p. Then we use the tangency principle in its interior, boundary and corner version to conclude that $\Pi(t_1)$ is a horizontal plane of symmetry of S. This is impossible because ∂S is connected and homotopic to C. This contradiction proves that the case $t_1 < \infty$ does not occur and hence S is a graph on Ω.

2. The curves Γ_1 and Γ_2 define a bounded set $\Lambda \subset \Sigma$ such that together S defines a bounded 3-domain $W \subset K$. We apply the reflection method as in the first part of Theorem 2.2 with the planes $\Pi(t)$. There are three types of touching points, namely, an interior point, a boundary point and a corner point. In all these cases, $\Pi(t_1)$ is a plane of symmetry of S. □

Remark 2.5 By anonymous referee(s), it is reveal that the first of Theorem 2.4 is a special case of Theorem 9 in [15].

In the case that the surface has zero mean curvature, we have a strong result without embeddedness condition. Previously, we recall the flux formula for surfaces of constant mean curvature, which was used firstly for embedded surfaces ([6]–[8]) but it is also valid for immersed surfaces.

Lemma 2.6 *Let S be a compact surface with boundary ∂S and let $\phi : S \to \mathbb{R}^3$ be an immersion of constant mean curvature H. If $\alpha = \phi_{|\partial S} : \partial S \to \mathbb{R}^3$, then*

$$H \int_{\partial S} \langle \alpha(s) \times \alpha'(s), a \rangle \, ds = - \int_{\partial S} \langle N(s) \times \alpha'(s), a \rangle \, ds, \tag{2.2}$$

where $a \in \mathbb{R}^3$, $\times$ is the vectorial product in $\mathbb{R}^3$ and N is the Gauss map.

Proof. See [9, Lemma]. The 1-form

$$\omega_p(v) = \langle (H\phi(p) + N(p)) \times (d\phi_p)(v), a \rangle,$$

for $v \in T_pS$, is closed because H is constant. Then we apply the Stokes' theorem. □

Theorem 2.7 *Let Σ be a right cylinder and let $\phi : S \to \mathbb{R}^3$ be a capillary minimal immersion of a compact connected surface S in K. If $\phi(\partial S)$ is a graph on C, then $\phi(S)$ is a (horizontal) planar domain.*

Proof. Denote by $\mathbf{n}$ the unit normal vector to Σ pointing outside K and let N be a Gauss map on S. Assume first that $\langle N, \mathbf{n} \rangle = 0$ along ∂S, that is, the contact angle is $\gamma = \pi/2$. Take a horizontal plane $\Pi(t)$ of equation $z = t$ for t sufficiently large such that $\Pi(t) \cap S = \emptyset$. We move $\Pi(t)$ vertically down until the first time that $\Pi(t)$ intersects S at $t = t_1$ occurs. If $\Pi(t_1)$ touches S at an interior point, then both surfaces are tangent and $S \geq \Pi(t_1)$ around this point. Since $\Pi(t_1)$ and S are both minimal, the tangency principle implies that $S \subset \Pi(t_1)$, proving the result. The other possibility is that $\Pi(t_1)$ and S touch at a boundary point $p \in \partial S$. In fact, this occurs at the highest point of ∂S with respect to Π. At such a point p, the tangent vector to ∂S is horizontal. Since $\mathbf{n}$ is also horizontal, then necessarily N is vertical and so (up an orientation on S which does not change the value of the mean curvature) the vector N agrees with the normal to $\Pi(t_1)$. Then $S \geq \Pi(t_1)$ around p and the tangency principle in its boundary version concludes that S is a subset of the plane $\Pi(t_1)$.

Assume now that $\gamma \neq \pi/2$ and we arrive to a contradiction. We write $\alpha(s) = \beta(s) + \langle \alpha(s), a \rangle a$, where β is the orthogonal projection of α on Π and $a = (0, 0, 1)$. Then $\alpha'(s) = \beta'(s) + \langle \alpha'(s), a \rangle a$ and $\beta'(s) \neq 0$ because ∂S is a graph on C. For each $s \in \mathbb{R}$, we consider $\mathbf{t}(s)$ the unit tangent vector to C such that $\mathbf{n} \times \mathbf{t} = a$. Then $\beta'(s) = \varphi(s)\mathbf{t}(s)$ for some non-zero function φ. On the other hand, by the flux formula (2.2), when $H = 0$ we have:

$$\begin{aligned} 0 &= \int_{\partial S} \langle N(s), \alpha'(s) \times a \rangle \, ds = \int_{\partial S} \langle N(s), \beta'(s) \times a \rangle \, ds \\ &= \int_{\partial S} \varphi(s) \langle N(s), \mathbf{n}(s) \rangle \, ds = \cos\gamma \int_{\partial S} \varphi(s) \, ds, \end{aligned} \tag{2.3}$$

which it is a contradiction because $\varphi(s) \neq 0$ for all s. □

We give a result of non-existence of CMC capillary surfaces in a right solid cylinder orthogonally intersecting a right cylinder.

Theorem 2.8 *Let Σ be a right cylinder and let $\phi : S \to \mathbb{R}^3$ be a CMC capillary immersion of a compact surface S in K. If $\phi(\partial S)$ is homotopic to C in Σ and $\phi(S)$ meets Σ orthogonally along ∂S, then S is a minimal surface.*

Proof. Let $a = (0, 0, 1)$. Since $\phi(\partial S)$ is homotopic to C, the integral $\int_{\partial S} \langle \alpha(s) \times \alpha'(s), a \rangle \, ds$ in the left-hand side of (2.2) represents the algebraic area of C, that is, it coincides, up a sign, with twice the area of Ω. In particular, $\int_{\partial S} \langle \alpha(s) \times \alpha'(s), a\rangle \, ds \neq 0$. Assume that $\gamma = \pi/2$. We use the notation of the proof of Theorem 2.7 writing $\beta'(s) = \varphi(s)\mathbf{t}(s)$ where now φ is a smooth function on ∂S. Then the right-hand side of the flux formula (2.2) is

$$\int_{\partial S} \langle N(s) \times \alpha'(s), a\rangle \, ds = \cos\left(\frac{\pi}{2}\right) \int_{\partial S} \varphi(s) \, ds = 0.$$

It follows from (2.2) that $H = 0$. □

3 Complete minimal capillary surfaces outside of a right solid cylinder

Consider a right solid cylinder K where $\partial K = \Sigma = C \times \mathbb{R}$ and C is a simple closed smooth planar curve. Let $\phi : S \to \mathbb{R}^3$ be a CMC capillary immersion of a surface S in $\mathbb{R}^3 \setminus int(K)$. We say that S is a *CMC capillary surface outside of* K. This section is motivated by the next two examples of minimal capillary surfaces outside K. First, a piece of a catenoid S bounded by a circle such that S is a graph on the exterior of a round disc $\Omega \subset \Pi$, where Ω is the domain bounded by C. Then S intersects the cylinder $\Sigma = \partial\Omega \times \mathbb{R}$ with constant angle $\gamma \neq \pi/2$. The second example is the planar domain $\Pi \setminus \Omega$, where here the contact angle is $\gamma = \pi/2$.

Consider now S a surface of a single puncture and $\phi : S \to \mathbb{R}^3$ a complete minimal immersion. The immersion of a small neighborhood of the puncture of S, topologically a punctured open disk, is the *end* of S. Geometrically, it is a connected component of $S \setminus B$, where $B \subset \mathbb{R}^3$ is a sufficiently large compact domain. We say that the end of S is *regular at infinity* if the end is expressed by a graph of the following function

$$u(x, y) = c \log\left(\sqrt{x^2 + y^2}\right) + b + \frac{mx + ny}{x^2 + y^2} + O\left(\frac{1}{x^2 + y^2}\right), \tag{3.1}$$

on the exterior of a bounded domain in a plane $\Pi_0 \subset \mathbb{R}^3$ with constants c, b, m and n. When $c = 0$ (resp. $c \neq 0$) the end is called *planar* (resp. *catenoidal*) [12]. We say that the end of S is *parallel to* Π if Π_0 is parallel to Π.

We need a version of the maximum principle for properly immersed minimal surfaces with boundary.

Lemma 3.1(Maximum principle at infinity [10]) *Let $S_1, S_2 \subset \mathbb{R}^3$ be two disjoint, connected, properly immersed minimal surfaces with boundary. If $\partial S_1 \neq \emptyset$ or $\partial S_2 \neq \emptyset$, then after possibly reindexing, the distance between S_1 and S_2 is equal to* $\inf\{dist(p, q) : p \in \partial S_1, q \in S_2\}$.

Theorem 3.2 *Let K be a right solid cylinder and S be a connected surface with a puncture and let $\phi : S \to \mathbb{R}^3$ be a complete embedded minimal capillary immersion outside of K, where the end of S is parallel to Π. Assume that ∂S is a graph on C.*

1. *If the end of S is planar, then S is part of a parallel plane to Π.*
2. *If the end of S is catenoidal, then $\gamma \neq \pi/2$.*
3. *Assume that K is a circular right solid cylinder with axis L. If the end of S is catenoidal, then S is part of a catenoid where L is its rotational axis.*

Proof.

1. We claim that $\gamma = \pi/2$. Since the end of S is parallel to Π, we may assume after a vertical translation, that $\Pi_0 = \Pi$. Let $C_R \subset \Pi$ be the circle of radius R and centered at the origin of Π. Since the end of S is planar, for sufficiently large R, $\Gamma_R = S \cap (C_R \times \mathbb{R})$ is asymptotic to a circle which is a vertical translation of C_R and S asymptotically meets $C_R \times \mathbb{R}$ at a right angle along Γ_R. Let S_R be the compact subset of S contained in $\Omega_R \times \mathbb{R}$, where $\Omega_R \subset \Pi$ is the round disk bounded by C_R. For sufficiently large R, $\partial S_R = \partial S \cup \Gamma_R$. As in the proof of Theorem 2.7, we write a parametrization of ∂S given by $\alpha(s) = \beta(s) + \langle \alpha(s), a\rangle a$, with $a = (0, 0, 1)$. By the flux formula (2.2) and since $H = 0$, we have
$$\cos\gamma \int_{\partial S} \varphi(s) \, ds = \int_{\Gamma_R} \langle \nu, a\rangle \, ds, \tag{3.2}$$
where ν is outward unit conormal vector field of S_R along Γ_R. By (3.1) and a direct computation, as the radius R goes to infinity, $\int_{\Gamma_R} \nu \, ds$ converges to a vector $2\pi c N_0$, where N_0 is the unit normal vector of Π_0.

Since the end of S is planar and parallel to Π, the integral $\int_{\Gamma_R} \langle \nu, a\rangle\, ds$ converges to zero as $R \nearrow \infty$. By using that ∂S is a graph on C, the function $\varphi(s)$ does not vanish, obtaining by (3.2) that $\gamma = \pi/2$.

Once we have proved that $\gamma = \pi/2$, we now show that S is part of a parallel plane to Π. Denote by $E(t)$ the vertical translation of $\Pi \setminus \Omega$ at the height $z = t$. Since the end of S is regular, the total curvature of S is finite and by [4, Proposition 11.5], S is proper. As S is a planar end asymptotic to a horizontal plane, for t sufficiently large, $E(t)$ is disjoint from S. Next, we descend $E(t)$ by letting $t \searrow 0$ until that $E(t)$ touches S the first time at $t = t_0$. Since S and $E(t_0)$ are proper and included in the outside of K with $\partial S, \partial E(t_0) \subset \Sigma$, by the maximum principle at infinity, the contact between S and $E(t_0)$ occurs at boundary points. Taking into account that S and $E(t_0)$ meet Σ with the same contact angle $\gamma = \pi/2$, the tangency principle and argument on connectedness imply that S and $E(t_0)$ coincide, that is, S is a vertical displacement of $\Pi \setminus \Omega$.

2. We use the same notations as in the previous item. Since the end of S is catenoidal and parallel to Π, for sufficiently large R, $\Gamma_R = S \cap (C_R \times \mathbb{R})$ is asymptotic to a circle which is a vertical translation of C_R. Moreover, S asymptotically meets $C_R \times \mathbb{R}$ at a constant angle along Γ_R different from $\pi/2$. Similarly as above, $\int_{\Gamma_R} \langle \nu, a\rangle\, ds$ converges to a nonzero constant $2\pi c$ as $R \nearrow \infty$, where c is the number given in (3.1). So, by Equations (2.3) and (3.2), we have that the integral

$$\int_{\partial S} \langle N(s) \times \alpha'(s), a\rangle\, ds = \cos\gamma \int_{\partial S} \varphi(s)\, ds$$

does not vanish. Hence $\gamma \neq \pi/2$.

3. Denote by S_θ the rotation of S about L and $S_\theta(t) = S_\theta + (0, 0, t)$ the vertical translation of S_θ of height t. Since the right cylinder Σ is circular, the boundary of S_θ remains to be included in Σ. Similarly, S and $S_\theta(t)$ are proper because they are minimal surfaces with finite total curvature. Fix θ. Let $t \searrow -\infty$ so ∂S_θ is disjoint of $\partial S_\theta(t)$, which it is possible because ∂S is compact and the end of S is catenoidal and parallel to Π . Next, we move up $S_\theta(t)$ vertically. By the maximum principle at infinity, the first contact point occurs at a boundary point $p \in \partial S_\theta(t_0) \cap \partial S$ for some $t_0 \in \mathbb{R}$. As the (constant) contact angle of $S_\theta(t_0)$ with Σ agrees with the one of S, the surfaces $S_\theta(t_0)$ and S are tangent at p with $S_\theta(t_0) \geq S$ around p. The tangency principle implies that both surfaces coincide. Since this property holds for any angle θ, S is a surface of revolution about the axis L. Hence S is a part of a catenoid and this completes the proof of theorem. □

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