Invariant surfaces in the homogeneous space Sol with constant curvature

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A surface in homogeneous space Sol is said to be an invariant surface if it is invariant under some of the two 1-parameter groups of isometries of the ambient space whose fix point sets are totally geodesic surfaces. In this work we study invariant surfaces that satisfy a certain condition on their curvatures. We classify invariant surfaces with constant mean curvature and constant Gaussian curvature. Also, we characterize invariant surfaces that satisfy a linear Weingarten relation.

1 Introduction

In 1982, W.P. Thurston formulated a geometric conjecture for three dimensional manifolds, namely every compact orientable 3-manifold admits a canonical decomposition into pieces, each of them having a canonical geometric structure from the following eight maximal and simply connected homogeneous Riemannian spaces: $\mathbb{E}^3$, $\mathbb{S}^3$, $\mathbb{H}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{PSL}(2, \mathbb{R})$, Nil and Sol. See e.g. [18]. A good comprehension of this topic requires the use of modern differential geometry of Ricci flows, and the attention of mathematicians has turned, during last years, to homogeneous 3-spaces. Although Sol is a homogeneous space and the action of the isometry group is transitive, the number of isometries is low, its isometry group has dimension 3. Therefore, the knowledge of the geometry of its submanifolds is far to be complete. For example, the geodesics of Sol are known since 1998 (see [20]), while the geometry of the totally umbilical surfaces (cf. [16]) and some properties of surfaces with constant mean curvature (cf. [4, 8, 10]) are found much more recently. See also [6].

The space Sol can be constructed as the semi-direct product $\mathbb{R}^2 \rtimes \mathbb{R}$, where $\mathbb{R}$ acts on $\mathbb{R}^2$ by $z \mapsto diag(e^{-z}, e^z)$. Hence Sol is identified with $\mathbb{R}^3(x, y, z)$ on which the group operation

$$(x, y, z) * (x', y', z') = (x + e^{-z}x', y + e^z y', z + z')$$

confers to Sol the structure of an unimodular Lie group, which it is solvable but not nilpotent. The Lie algebra $\mathfrak{sol}$ of Sol is spanned by the following left invariant vector fields

$$E_1 = e^{-z}\frac{\partial}{\partial x},\ \ E_2 = e^z\frac{\partial}{\partial y},\ \ E_3 = \frac{\partial}{\partial z}.$$

Among left invariant metrics on Sol (constructed starting from a positively defined symmetric bilinear form in $\mathfrak{sol}$) let us choose one such that the basis $\{E_1, E_2, E_3\}$ becomes orthonormal, namely $g = \theta_1^2 + \theta_2^2 + \theta_3^2$, where $\{\theta_1 = e^z dx,\ \theta_2 = e^{-z}dy,\ \theta_3 = dz\}$ is the dual basis. Thus, as Riemannian manifold, the space Sol can be represented by $\mathbb{R}^3$ equipped with the left invariant metric

$$\langle\ ,\ \rangle = ds^2 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2.$$

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In [17], Takahashi proved that a simply connected Riemannian homogeneous 3-space can be isometrically immersed in the hyperbolic space $\mathbb{H}^4(-1)$ with type number 2, if and only if it is isometric to Sol and he called it B-manifold. Indeed, in Japanese literature, Sol is known as *Takahashi's B-manifold*. Furthermore, Kowalski explains in [9], the geometry of Sol where it is realized as the Lie group $E(1,1)$ of rigid motions of Minkowski plane $\mathbb{E}^2_1 = (\mathbb{R}^2, dxdy)$, endowed with the metric $\langle\ ,\ \rangle$ described above. Moreover, staying in Riemannian context, in [11, Th. 3.3], the space Sol is described as the 3-dimensional simply connected generalized pointwise symmetric space which is not symmetric.

The space Sol belongs to a large family of simply connected homogeneous Riemannian 3-manifolds, depending on two parameters μ_1 and μ_2, realized as solvable matrix Lie groups $G(\mu_1, \mu_2)$ (e.g. [8]). This family includes Sol as $G(1,-1)$ and also other spaces from the Thurston's list: the Euclidean 3-space $\mathbb{E}^3 = G(0,0)$, the hyperbolic 3-space $\mathbb{H}^3 = G(1,1)$ and $\mathbb{H}^2 \times \mathbb{R} = G(0,1)$. The geometry of Sol is often called *solve-geometry* (see [2], [8]).

The component of the identity in Iso(Sol) is generated by the following families of isometries:

$$T_{1,c}(x,y,z) := (x+c, y, z)\ , \quad T_{2,c}(x,y,z) := (x, y+c, z)\ , \quad T_{3,c}(x,y,z) := (e^{-c}x, e^{c}y, z+c)$$

where $c \in \mathbb{R}$ is a real parameter. These isometries are left multiplications by elements of Sol and so, they are left-translations with respect to the structure of Lie group. The set of fixed points of $T_{1,c}$ and $T_{2,c}$ are totally geodesic surfaces in Sol. In this work we consider surfaces invariant under the 1-parametric group of isometries $T_{i,c}$, with $i = 1, 2$.

Definition 1.1 A surface S in Sol is said to be an invariant surface if it is invariant under the action of one of the 1-parameter groups of isometries $\{T_{i,c}; c \in \mathbb{R}\}$, with $i = 1, 2$.

After an isometry of the ambient space, an invariant surface under the group $\{T_{2,c}\}_{c\in\mathbb{R}}$ converts into an invariant surface under the group $\{T_{1,c}\}_{c\in\mathbb{R}}$: this can done by taking the isometry of Sol given by $\phi(x,y,z) = (y, x, -z)$. Thus, throughout this work, we consider invariant surfaces under the first group $\{T_{1,c}\}_{c\in\mathbb{R}}$, and sometimes we abbreviate by saying invariant surfaces.

In this paper we study invariant surfaces in Sol with some condition on their curvatures, for example, that the mean curvature of the Gaussian curvature is constant. The study of surfaces with constant curvature, specially with constant mean curvature, in homogeneous 3-spaces and invariant under the action of a one-parameter group of isometries of the ambient space has been recently of interest for many geometers. Several results were obtained when the ambient space is the Heisenberg group ([3, 5, 7, 13, 19]) and the product space $\mathbb{H}^2 \times \mathbb{R}$ ([12, 14, 15]).

The paper is divided according the type of curvature that is considered. So, in Section 3 we classify all invariant surfaces of Sol with constant mean curvature H, including minimal surfaces (some pictures of surfaces with $H \neq 0$ appeared in [4]). In Section 4 we construct and classify all invariant surfaces with constant (intrinsic or extrinsic) Gaussian curvature (K_{int} and K_{ext}). The fact that the Sol geometry has no constant sectional curvature makes that the constancy of K_{int} and of K_{ext} do not imply each other. Finally in Section 5 we initiate the study of linear Weingarten invariant surfaces by considering a relation of type $\kappa_1 = m\kappa_2$, where κ_i, $i = 1, 2$ are the principal curvatures and $m \in \mathbb{R}$.

Remark 1.2 One can also consider the study of surfaces in Sol invariant by the group of isometries $\{T_{3,c}\}_{c\in\mathbb{R}}$ with some restrictions on its curvature. However, the authors have not been able to obtain reliable results due to the fact that the computations are quite complicated and difficult to manage for this type of surfaces.

2 Local computations of curvatures

In this section we will recall some basic geometric properties of the space Sol and we will compute the curvatures of an invariant surface. See also [4, 20]. The following transformations

$$(x,y,z) \longmapsto (y, -x, -z) \quad \text{and} \quad (x,y,z) \longmapsto (-x, y, z)$$

span a group of isometries of (Sol, g) having the origin as fixed point. This group is isomorphic to the dihedral group (with 8 elements) D_4. It is, in fact, the complete group of isotropy [20]. The other elements of the group are $(x,y,z) \longmapsto (-x,-y,z)$, $(x,y,z) \longmapsto (-y,x,-z)$, $(x,y,z) \longmapsto (y,x,-z)$, $(x,y,z) \longmapsto (y,x,z)$ and $(x,y,z) \longmapsto (x,-y,z)$. They can be unified as follows (cf. [15]):

$$(x,y,x) \longmapsto (\pm e^{-c}x + a, \pm e^{c}y + b, z + c)\ , \quad (x,y,z) \longmapsto (\pm e^{-c}y + a, \pm e^{c}x + b, z + c).$$

The key point to understanding the geometry of Sol is to consider the following three foliations:

$$\mathcal{F}_1 : \left\{P_t = \{(t, y, z); y, z \in \mathbb{R}\}\right\}_{t\in\mathbb{R}}$$
$$\mathcal{F}_2 : \left\{Q_t = \{(x, t, z); x, z \in \mathbb{R}\}\right\}_{t\in\mathbb{R}}$$
$$\mathcal{F}_3 : \left\{R_t = \{(x, y, t); x, y \in \mathbb{R}\}\right\}_{t\in\mathbb{R}}.$$

The first two foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are determined by the isometry groups $\{T_{1,c}\}_{c\in\mathbb{R}}$ and $\{T_{2,c}\}_{c\in\mathbb{R}}$ respectively, and they describe (the only) totally geodesic surfaces of Sol, each leaf being isometric to a hyperbolic plane. The third foliation $\mathcal{F}_3$ realizes by minimal surfaces, and all them are isometric to the Euclidean plane.

The Riemannian connection ∇ of Sol with respect to $\{E_1, E_2, E_3\}$ is

$$\begin{array}{lll} \nabla_{E_1}E_1 = -E_3 & \nabla_{E_1}E_2 = 0 & \nabla_{E_1}E_3 = E_1 \\ \nabla_{E_2}E_1 = 0 & \nabla_{E_2}E_2 = E_3 & \nabla_{E_2}E_3 = -E_2 \\ \nabla_{E_3}E_1 = 0 & \nabla_{E_3}E_2 = 0 & \nabla_{E_3}E_3 = 0. \end{array}$$

A T_1-invariant surface S is determined by the intersection curve α with each of the leaves of the corresponding foliation together with the group of isometries. Any curve α is called *a generating curve* of the surface, and by our choice of group, we will consider that α is the intersection curve of S with the plane $x = 0$. Thus when we impose some condition on the curvature of S, this equivalently translates into properties of the generating curve. Let us take a parametrization of α given by $\alpha(s) = (0, y(s), z(s))$, $s \in I$, where s is the arc-length parameter. Thus

$$e^{-z(s)}y'(s) = \cos\theta(s), \qquad z'(s) = \sin\theta(s)$$

where $\theta = \theta(s)$ is a certain smooth function. We parametrize S by

$$X(s, t) = (t, y(s), z(s)), \ \ s \in I \subset \mathbb{R}, \ t \in \mathbb{R}.$$

From now on, we shall often not write the parameter s explicitly in our formulas. We have an orthogonal pair of vector fields on S, namely,

$$e_1 := X_s = (0, y', z') = \cos\theta E_2 + \sin\theta E_3\ , \quad e_2 := X_t = (1, 0, 0) = e^z E_1.$$

We choose as Gauss map

$$N = -\sin\theta E_2 + \cos\theta E_3$$

and herewith we choose the orientation of S throughout the paper. Now we are going to compute the curvatures of an invariant surface. Let H and K_{ext} be the mean curvature and the extrinsic Gaussian curvature of S, respectively. Using classical notation, we have

$$H = \frac{1}{2}\frac{En - 2Fm + Gl}{EG - F^2}, \qquad K_{ext} = \frac{ln - m^2}{EG - F^2}$$

with $E = \langle e_1, e_1\rangle$, $F = \langle e_1, e_2\rangle$, $G = \langle e_2, e_2\rangle$, $l = \langle N, \nabla_{e_1}e_1\rangle$, $m = \langle N, \nabla_{e_1}e_2\rangle$, $n = \langle N, \nabla_{e_2}e_2\rangle$.

In our case, the first fundamental form (i.e. the restriction of the metric $\langle\ ,\ \rangle$ to S) is

$$g = ds^2 + e^{2z(s)}dt^2$$

where $z(s)$ is a primitive of $\sin\theta$. The values of $\nabla_{e_i}e_j$ (for $i, j = 1, 2$) are

$$\begin{array}{rcl} \nabla_{e_1}e_1 &=& (\theta' + \cos\theta)(-\sin\theta E_2 + \cos\theta E_3) \\ \nabla_{e_1}e_2 &=& \nabla_{e_2}e_1 = \sin\theta e^z E_1 \\ \nabla_{e_2}e_2 &=& -e^{2z}E_3. \end{array}$$

Then we obtain

$$l = \theta' + \cos\theta, \quad m = 0, \quad n = -e^{2z}\cos\theta.$$

Consequently, the two curvatures can be expressed as

$$H = \frac{1}{2}\,\theta', \qquad K_{ext} = -\cos\theta(\theta' + \cos\theta). \tag{2.1}$$

Hence the principal curvatures are

$$\kappa_1 = \theta' + \cos\theta \quad \text{and} \quad \kappa_2 = -\cos\theta.$$

In order to obtain the intrinsic Gauss curvature K_{int}, recall that $K_{int} = K_{ext} + K(e_1 \wedge e_2)$, where $K(e_1 \wedge e_2)$ is the sectional curvature of each tangent plane and

$$K(e_1 \wedge e_2) = \frac{\langle \nabla_{e_1}\nabla_{e_2}e_2 - \nabla_{e_2}\nabla_{e_1}e_2 - \nabla_{[e_1,e_2]}e_2, e_1\rangle}{EG - F^2}\ .$$

Now we easily compute

$$\begin{aligned}
\nabla_{e_1}\nabla_{e_2}e_2 &= \nabla_{e_1}(-e^{2z}E_3) = e^{2z}(\cos\theta E_2 - 2\sin\theta E_3).\\
\nabla_{e_2}\nabla_{e_1}e_2 &= \nabla_{e_2}(\sin\theta e^{z}E_1) = -\sin\theta e^{2z}E_3.\\
\nabla_{[e_1,e_2]}e_2 &= 0.
\end{aligned}$$

Thus we have

$$K(e_1 \wedge e_2) = \cos^2\theta - \sin^2\theta.$$

Consequently, the intrinsic curvature is

$$K_{int} = -\theta'\cos\theta - \sin^2\theta. \tag{2.2}$$

If we consider an invariant surface S of Sol, any condition of its curvature writes as an ordinary differential equation $\mathcal{E}(s,\theta,\theta') = 0$ on the function θ. In order to obtain S, we have to get the generating curve α, and so, we need to solve $\mathcal{E} = 0$ together with the system

$$y'(s) = e^{z(s)}\cos\theta(s) \tag{2.3}$$

$$z'(s) = \sin\theta(s). \tag{2.4}$$

Remark 2.1 We can also assume that α is locally a graph on the y-axis or on the z-axis. If α writes as $\alpha(y) = (0, y, z(y))$, the change of variables is

$$\theta'(s) \to e^{2z}\frac{z'' + z'^2}{(1 + z'^2e^{2z})^{3/2}},\ \sin\theta \to \frac{z'e^{z}}{\sqrt{1 + z'^2e^{2z}}},\ \cos\theta \to \frac{1}{\sqrt{1 + z'^2e^{2z}}}. \tag{2.5}$$

Depending on each case, we use interchangeably (2.3)–(2.4) by (2.5).

Remark 2.2 In what follows, we will omit the integration constants for the function $y(s)$, since it represents an isometry of the surface by translations of type $T_{2,c}$. Similarly, we omit the additive constants of the function z: in this case, the isometry $\phi(x,y,z) = (e^{\lambda}x, e^{-\lambda}y, z - \lambda)$ converts the generating curve $s \longmapsto (0, y(s), z(s) + \lambda)$ into $s \longmapsto (0, e^{-\lambda}y(s), z(s))$.

3 Surfaces with constant mean curvature

In this section we study T_1-invariant surfaces which are minimal or constant mean curvature.

Theorem 3.1 *The only T_1-invariant minimal surfaces of Sol are:*

1. *a leaf of the foliation $\mathcal{F}_2$ or,*
2. *a leaf of the foliation $\mathcal{F}_3$ or,*
3. *the surface generated by the logarithmic function.*

P r o o f. If S is minimal, then $\theta' = 0$, that is, $\theta(s) = \theta_0$ for some constant $\theta_0 \in \mathbb{R}$ and $z(s) = (\sin\theta_0)s$. If $\sin\theta_0 = 0$, then (2.3)-(2.4) gives $z(s) = \lambda$ and $y(s) = (\cos\theta_0)s$. This says that α is a horizontal straight-line and S is a leaf of $\mathcal{F}_3$. Similarly, if $\cos\theta_0 = 0$, then y is a constant function, α is a vertical straight-line and the surface belongs to the family $\mathcal{F}_2$.

If $\sin\theta_0 \neq 0$, we have from (2.4) that

$$y' = e^{(\sin\theta_0)s}\cos\theta_0 \Rightarrow y(s) = \cot\theta_0 e^{(\sin\theta_0)s}.$$

Then the generating curve is given by

$$\alpha(s) = \Big(\cot\theta_0 e^{(\sin\theta_0)s}, (\sin\theta_0)s\Big).$$

This means that α describes the graph of a logarithmic function: $z = \log((\tan\theta_0)y)$. $\Box$

The above result can be obtained also by using (2.5): if the surface is not a graph of $z = z(y)$, then y is a constant function and the surface is a leaf of $\mathcal{F}_2$.

The minimality condition $H = 0$ can be written as $z'' + z'^2 = 0$. Then z is either constant and S is a leaf of $\mathcal{F}_3$ or after a first integration, it is given, up to a constant, by $\log(z') = -z + \text{constant}$ which yields $e^z = cy + \mu$, with $c > 0$.

Theorem 3.2 *Let S be a T_1-invariant surface in Sol with constant mean curvature $H \neq 0$. We write the generating curve of S as $\alpha(s) = (0, y(s), z(s))$. Then*

1. *The curve α is invariant by a discrete group of translation in the y-direction.*
2. *The z-coordinate is bounded and periodic.*
3. *The curve α has self-intersections.*
4. *The velocity vector of α turns around the origin such that it takes all values in the unit circle.*

P r o o f. From $\theta' = 2H$, we obtain, after a possible translation in parameter s (which is allowed), $\theta(s) = 2Hs$. Then $z'(s) = \sin(2Hs)$ and hence

$$z(s) = -\frac{1}{2H}\cos(2Hs).$$

In particular, z is a periodic function of principal period $T = \pi/H$, whose derivative vanishes in a discrete set of points, namely, $A = \{n\pi/(2H); n \in \mathbb{Z}\}$. From (2.3),

$$y' = \exp\Big(-\frac{1}{2H}\cos(2Hs)\Big)\cos(2Hs).$$

Then the function y' vanishes at $B = A + \pi/2$ and this means that α is not a graph on the y-axis, being the velocity of α vertical at each point of B. Moreover, z takes the same value at these points: with our choice of the integration constants, this value is $z = 0$.

It is easy to show that if $\{y(s), z(s), \theta(s)\}$ satisfy (2.3)–(2.4) and $\theta' = 2H$, with initial conditions $\{y_0, z_0, \theta_0\}$, then the functions $\{y(s+T) - y(T) + y_0, z(s), \theta(s)\}$ satisfy the same equations and initial conditions. By uniqueness of solutions of an ODE, they must agree with the first set of solutions. In particular, $y(s+T) = y(s) + y(T) - y_0$. Thus, we have proved that the generating curve α is invariant by translations of the group of translations generated by the vector $(0, y(T) - y_0, 0)$. In our notations, this group is $\{T_{2,n(y(T)-y_0)}; n \in \mathbb{Z}\}$.

Finally, the function $\theta(s)$ takes all real values, which means that the planar velocity vector $\alpha'(s) = \cos\theta(s)E_2(s) + \sin\theta(s)E_3(s)$ moves taking all the values of a unit circle $\mathbb{S}^1$ in a monotonic sense. $\Box$

We end this Section with two pictures, drawn by using Mathematica, representing the generating curve α.

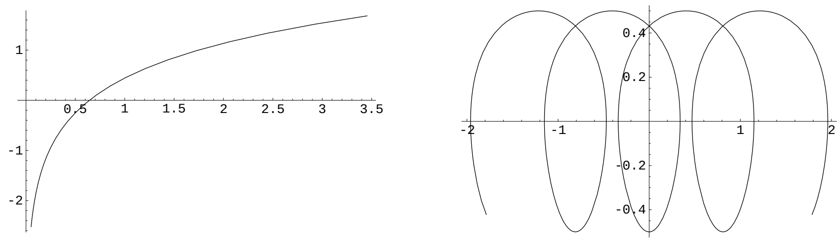

Fig. 1 Generating curves of T_1-invariant surfaces with constant mean curvature: case $H = 0$ (left); case $H = 1$ (right).

4 Surfaces with constant Gaussian curvature

In this section we study T_1-invariant surfaces in Sol with constant (intrinsic or extrinsic) Gaussian curvature.

Theorem 4.1 *Let S be a T_1-invariant surface in Sol with constant intrinsic Gaussian curvature $K_{int} = c$. Up to integration constants, we have the next classification:*

(i) *If $c = 0$, the surface is a leaf of $\mathcal{F}_3$ or the generating curve α of S is*

$$\alpha(s) = \Big(0, \frac{1}{2}\Big(s\sqrt{s^2-1} - \log(s + \sqrt{s^2-1})\Big), \log(s)\Big),\ s^2 \geq 1.$$

(ii) *If $c = -1$, the surface is a leaf of the foliation $\mathcal{F}_2$ or the generating curve α of S is the graph of $z(y) = \log(\cosh(y))$.*

(iii) *If $c \in (-1, 0)$, then α is a graph of type $z(y) = \log(y)$, or $z(y)$ is defined in all the real line $\mathbb{R}$ with a single minimum, or $z(y)$ is a monotonic function defined in some interval (a, ∞).*

(iv) *If $c > 0$ or $c < -1$, $z(y)$ is a bounded function defined in a bounded interval (a, b) with a single maximum or minimum and it is vertical at the end points of (a, b).*

Moreover, except when S is a leaf of $\mathcal{F}_2$, the generating curve is a graph of a function $z = z(y)$.

Proof. Equation (2.2) becomes $\theta' \cos\theta + \sin^2\theta = -c$ or, equivalently

$$(\sin\theta)' + \sin^2\theta + c = 0. \tag{4.1}$$

Notice that if $\cos\theta = 0$ at some point, then $c = -1$. In this case, if $\cos\theta \equiv 0$ (on an open interval I), then y is a constant function (on I). This means that α is a vertical straight–line and S is a leaf of $\mathcal{F}_2$. Otherwise, if $c \neq -1$ (on an open interval) then $\cos\theta \neq 0$ and from (2.3), α is the graph of $z = z(y)$.

If we put $p = \sin\theta$, then (4.1) can be written as $p' + p^2 + c = 0$, that is,

$$\frac{p'}{p^2 + c} = -1. \tag{4.2}$$

This equation makes sense only if $p^2 + c \neq 0$ and then, it can be integrated. In contrast, that is, if $\sin^2\theta + c = 0$, then $c \in [-1, 0]$.

We will distinguish all possible cases:

1. Case $c = -1$. We know that S is a leaf of $\mathcal{F}_2$ if $\cos\theta \equiv 0$. On the contrary, after a possible translation in parameter s, a first integration gives $\sin\theta = \tanh(s)$ and from (2.4) one gets $z(s) = \log(\cosh(s))$. Remark that a translation along z-axis implies a homothetical transformation along y-axis, and hence one can suppose that the initial value of $z(s)$ is 0. Then $y'(s) = \pm 1$, that is, $y(s) = \pm s$ (translations along y-axis are allowed).

   This means that α is the graph of $z(y) = \log(\cosh(y))$.

2. Case $c = 0$. If $\sin\theta \equiv 0$, then (2.4) shows that z is a constant function, α is a horizontal curve and S is a leaf of $\mathcal{F}_3$. On the other case, we easily get $\sin\theta = 1/s$ (with $|s| \geq 1$) and by (2.4), we obtain $z(s) = \log(|s|)$. It is possible to solve (2.3) obtaining
$$y(s) = \pm\frac{1}{2}\Big(s\sqrt{s^2-1} - \log(s + \sqrt{s^2-1})\Big).$$
3. Case $c \in (-1, 0)$. Then, there exists θ_0 such that $\sin\theta_0 = \pm\sqrt{-c}$. If $c + \sin^2\theta = 0$ on an open interval, the solution of (2.3)–(2.4) is up to constants $\theta(s) = \theta_0$, $y(s) = \cot\theta_0 e^{(\sin\theta_0)s}$ and $z(s) = (\sin\theta_0)s$. This means that α is the graph of $z(y) = \log((\tan\theta_0)y)$. Finally, we assume that $\sin^2\theta + c \neq 0$ at some point (for example, at $s = 0$) and hence it preserves constant sign on an interval I. A first integration in (4.2) depends on the sign of $\sin^2\theta + c$ on the interval I.
   (a) Assume $\sin^2\theta + c < 0$. Then (4.2) gives $\sin\theta = \sqrt{-c}\tanh(\sqrt{-c}\ (s + \lambda))$, for a certain constant λ. Letting $s \to \infty$, we conclude that $\sin\theta$ vanishes at some point. Without loss of the generality, we suppose that this occurs at $s = 0$. By integration, one gets $z(s) = \log(\cosh(\sqrt{-c}s))$. Moreover, z' vanishes only at one point, namely, $s = 0$, and $z''(s) > 0$. This means that $z = z(s)$ is a convex function with only a single minimum at $s = 0$. Finally,
$$|y'(s)| = \cosh(\sqrt{-c}s)\sqrt{1 + c\tanh^2(\sqrt{-c}s)} \geq \sqrt{1+c}$$
   which means that the function y is defined in all $\mathbb{R}$. Thus $z = z(y)$ with $y \in \mathbb{R}$.
   Since $z'(y) = z'(s)/y'(s)$, we know that $y = 0$ is the only extremum of $z(y)$ and from (2.5), we conclude $z''(0) = -c > 0$, that is, $z = z(y)$ has a minimum at $y = 0$, after a translation along y-axis.
   (b) Assume $\sin^2\theta + c > 0$. Now (4.2) gives $\sin\theta = \sqrt{-c}\coth(\sqrt{-c}(s + \lambda))$. We can have $0 < \sqrt{-c} < \sin\theta < 1$ or $-1 < \sin\theta < -\sqrt{-c} < 0$. In the first situation we have that s belongs to an interval of the form $(a, +\infty)$ while in the second situation s belongs to $(-\infty, a)$. If we choose $\lambda = 0$ we obtain $\sin\theta = \sqrt{-c}\coth(\sqrt{-c}s)$, $s \in (a, +\infty)$ or $s \in (-\infty, -a)$, with $a = \frac{1}{2\sqrt{-c}}\log\frac{1+\sqrt{-c}}{1-\sqrt{-c}}$.
   Then $z(s) = \log(\sinh(\sqrt{-c}s))$ which is a monotonic function. Moreover, it follows
$$y(s) = \pm\int^s \sqrt{\sinh^2(\sqrt{-c}s) + c\cosh^2(\sqrt{-c}s)}\, ds.$$
4. Case $c > 0$. Now $\sin\theta = -\sqrt{c}\tan(\sqrt{c}\ s)$ and $z(s) = \log(|\cos(\sqrt{c}\ s)|)$. Since $\sin\theta \in [-1, 1]$, the values of $\sqrt{c}s$ lie in intervals of the form $\big[-\arctan\frac{1}{\sqrt{c}}, \arctan\frac{1}{\sqrt{c}}\big] + k\pi$, $k \in \mathbb{Z}$. Moreover z' vanishes at exactly these points ($\sqrt{c}s = k\pi$). The same reasoning as above shows that $z = z(y)$ has a maximum at that points. The values of $y'(s)$ are bounded because
$$|y'(s)| = |\cos(\sqrt{c}s)|\sqrt{1 - c\tan^2(\sqrt{c}s)} \leq |\cos(\sqrt{c}s)| \leq 1.$$
   Without loss of the generality we restrict the interval for $\sqrt{c}s$ to an interval, e.g. $\big[-\arctan\frac{1}{\sqrt{c}}, \arctan\frac{1}{\sqrt{c}}\big]$. We are able to express the function y in terms of elliptic functions (cf. e.g. [1]), namely
$$y(s) = \pm\frac{1}{\sqrt{c}}\ EllipticE(\sqrt{c}s, 1 + c)$$
   where $EllipticE(\phi, m) = \int\limits_0^\phi \sqrt{1 - m\sin^2\tau}d\tau$, with $\phi \in \big(-\frac{\pi}{2}, \frac{\pi}{2}\big)$ is the elliptic function of second kind. Then the function y takes values in some bounded domain $(-y_M, y_M)$. Finally, notice that
$$\lim_{s\to\pm M}|z'(s)| = 1, \qquad \lim_{s\to\pm M}|y'(s)| = 0,$$
   and this means that α is vertical at the points $\pm y_M$.

5. Case $c < -1$. Now we are reasoning in a similar way as in the case $c > 0$. We have $\sin\theta = \sqrt{-c}\tanh(\sqrt{-c}s)$ and $z(s) = \log(\cosh(\sqrt{-c}s))$, with $s \in (-M, M)$, $M = \frac{1}{2\sqrt{-c}}\log\frac{\sqrt{-c}+1}{\sqrt{-c}-1}$. The function $z = z(y)$ is convex with a minimum at the origin. Also, the function $z = z(y)$ is defined in some bounded domain $(-y_M, y_M)$ and the generating curve α is vertical at $\pm y_M$.

□

Remark 4.2 In the cases $c < -1$ and $c > 0$ the derivatives of the functions $y(s)$ and $z(s)$ are bounded at the end points of the maximal domain $(-M, M)$. However one can not extend the solutions because $\cos\theta \to 0$ and $\sin^2\theta \to 1$ as $s \to \pm M$ and so, from (4.1), the function θ' goes to ∞ as $s \to \pm M$.

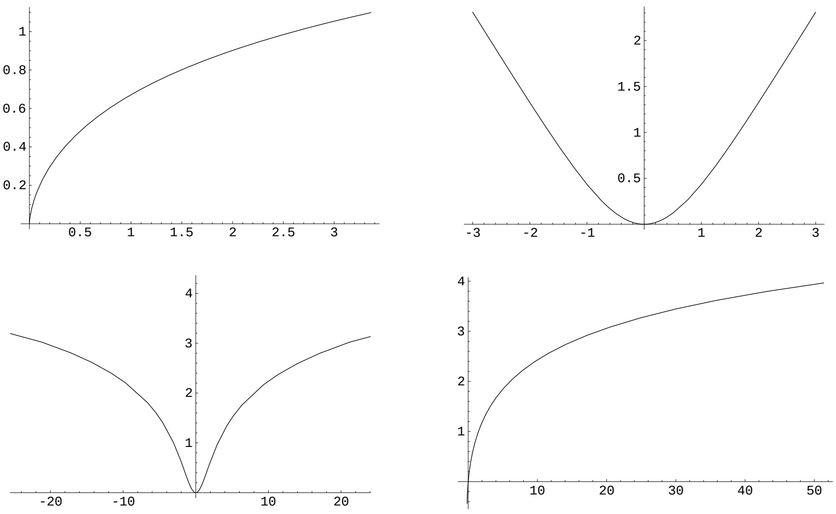

Fig. 2 Generating curves of T_1-invariant surfaces with $K_{int} = 0$ (top, left); $K_{int} = -1$ (top, right); $K_{int} = c \in (-1, 0)$, case $\sin\theta_0 + c < 0$ (bottom, left); and $K_{int} = c \in (-1, 0)$, case $\sin\theta_0 + c > 0$ (bottom, right).

Theorem 4.3 *Let S be a T_1-invariant surface in Sol with constant extrinsic Gaussian curvature $K_{ext} = c$. Up to integration constants, we have the following classification:*

1. *If $c = 0$, the surface is a leaf of $\mathcal{F}_2$ or the generating curve α of S is*
$$\alpha(s) = (0, \tanh(s), -\log(\cosh(s))).$$

2. *If $c = -1$, the surface is a leaf of $\mathcal{F}_3$ or*
$$\alpha(s) = \Big(0, -\frac{\sqrt{s^2-1}}{s} + \log(s + \sqrt{s^2-1}), -\log(|s|)\Big) \text{ with } |s| > 0.$$

3. *If $c \in (-1, 0)$, then α is the graph $z(y) = \log(y)$; or $z(y)$ is defined in a bounded interval $(-M, M) \subset \mathbb{R}$ and it is asymptotic to vertical lines $y = \pm M$; or $z(y)$ is defined in a bounded interval (m, M) being asymptotic to the vertical line $y = m$.*

4. *If $c > 0$ or $c < -1$, the function $z(y)$ is defined in a bounded interval $I = (a, b)$ with a single maximum or minimum, it is bounded and it is vertical at the end points of I.*

Proof. From (2.1) one gets $-\cos\theta(\theta' + \cos\theta) = c$. Similar to the previous theorem, we denote $p = \sin\theta$. Now we have
$$\frac{p'}{p^2 - c - 1} = 1$$

if the denominator is nowhere vanishing. In the sequel, the reasoning is similar as in Theorem 4.1. First, we consider that $\sin^2\theta - c - 1 \equiv 0$ on an interval, with the consequence that $c \in [-1, 0]$. Then the sectional curvature $K(e_1 \wedge e_2)$ is $-1 - 2c$ and $K_{int} = -1 - c$. But this problem was previously studied in Theorem 4.1 (see the first three cases when $\sin\theta$ is constant). More precisely, if $c \in (-1, 0)$ the solution is $z(y) = \log((\tan\theta_0)y)$; if $c = 0$, then S is a leaf of $\mathcal{F}_2$ and finally, if $c = -1$, then S is a leaf of $\mathcal{F}_3$. The rest of cases are the following:

1. Case $c = 0$. Then $z(s) = -\log(\cosh(s))$ and $y(s) = \pm\tanh(s)$.
2. Case $c = -1$. Then $z(s) = -\log(|s|)$ and
$$y(s) = -\frac{\sqrt{s^2-1}}{s} + \log\left(s + \sqrt{s^2-1}\right).$$
3. Case $c \in (-1, 0)$. The function θ is given by $\sin\theta = -\sqrt{c+1}\tanh(\sqrt{c+1}s)$.
   (a) If $\sin^2\theta_0 - c - 1 = 0$, then, up to constants, z is the logarithmic function.
   (b) If $\sin^2\theta_0 - c - 1 < 0$, $z(s) = -\log(\cosh(\sqrt{c+1}s))$ and it is defined on whole $\mathbb{R}$. Again, $z(s)$ has a maximum at $s = 0$. Subsequently,
   $$|y(\infty)| = |y(0)| + \int_0^\infty |y'(t)|dt \le |y(0)| + \int_0^\infty 2e^{-\sqrt{(c+1)}t}dt < \infty.$$
   This shows that the function $y(s)$ takes values in some interval $(-M, M)$. Thus the generating curve $z = z(y)$ is defined in this bounded interval and since $|z(s)|$ takes values arbitrary big, the graph of α tends asymptotically to the two vertical lines $y = \pm M$.
   (c) If $\sin^2\theta_0 - c - 1 > 0$, we obtain $\sin\theta = -\sqrt{c+1}\coth\left(\sqrt{c+1}(s+\lambda)\right)$, $\lambda \in \mathbb{R}$, and $z(s) = -\log|\sinh(\sqrt{c+1}(s+\lambda)|$. Assuming for example that $\sin\theta_0 > 0$, the constant λ must be negative with $\sin\theta_0 = -\sqrt{c+1}\coth(\sqrt{c+1}\lambda)$. The function z is monotonic on s and it is defined in some interval of type $(-\infty, M)$ or $(M, +\infty)$, where $1 = (c+1)\coth^2(\sqrt{c+1}M)$. As
   $$y'(s) = \frac{1}{\sinh(-\sqrt{c+1}(s+\lambda))}\sqrt{1-(c+1)\coth^2(\sqrt{c+1}(s+\lambda))}$$
   the value of $y'(M)$ is bounded and
   $$|y(-\infty)| < |y(0)| + \int_{-\infty}^0 |y'(s)|ds \le |y(0)| + \int_{-\infty}^0 \frac{1}{\sinh(-\sqrt{c+1}(s+\lambda))} < \infty.$$
   This shows that the value of y belongs an interval of type (m, M). It ensues that the line $y = m$ is a vertical asymptote for α.
4. Case $c > 0$. Now $\sin\theta = -\sqrt{c+1}\tanh(\sqrt{c+1}\,s)$ and $z(s) = -\log(\cosh(\sqrt{c+1}\,s))$. The curve has a single maximum at $s = 0$. From the expression of $\sin\theta$ and since $\sqrt{c+1} > 1$, the variable s can not take arbitrary values: more precisely, θ exists whenever $(c+1)\tanh^2(\sqrt{c+1}s) \le 1$. Then θ is defined in some bounded domain $(-M, M)$. On the other hand, and because
$$y'(s) = \pm\frac{1}{\cosh(\sqrt{c+1}s)}\sqrt{1-(c+1)\tanh^2(\sqrt{c+1}\,s)},$$
the values $y'(\pm M)$ vanish. Since the domain of s is bounded, y takes values in some interval $(-y_M, y_M)$. Because $y'(\pm M) = 0$ and $z'(\pm M) = 1$, we conclude that the generating curve α is vertical at the points $\pm y_M$. Finally it is obvious that the function $z(y)$ is bounded in the maximal domain.
5. Case $c < -1$. Then $\sin\theta = \sqrt{-c-1}\tan(\sqrt{-c-1}\,s)$ and $z(s) = -\log(|\cos(\sqrt{-c-1}\,s)|)$. The function z has minimum at every $s = \frac{1}{\sqrt{-c-1}}\mathbb{Z}\pi$. Now the conclusions are similar as the case $c > 0$, and we omit the details.

□

For the cases $c < -1$ and $c > 0$ we can apply the same comments as in Remark 4.2.

5 Linear Weingarten surfaces

A generalization of umbilical surfaces, as well as, of surfaces with constant mean curvature, are the Weingarten surfaces. A *Weingarten surface* is a surface that satisfies a smooth relation of type $W(\kappa_1, \kappa_2) = 0$, where κ_i are the principal curvatures of the surface. Equation $W(\kappa_1, \kappa_2) = 0$ gives other relation of type $U(H, K_{ext}) = 0$. Among the choices of W and U, the simplest case is that they are linear on its variables. So, we say that S is a *linear Weingarten surface* if satisfies one of the two (non-equivalent) conditions:

$$a\kappa_1 + b\kappa_2 = c \tag{5.1}$$

or

$$aH + bK_{ext} = c \tag{5.2}$$

where a, b and c are constant. In particular, if $a = -b$, $c = 0$ in (5.1) we have umbilical surfaces, whereas if $a = b$, the surface has constant mean curvature. On the other hand, in (5.2), the choice $a = 0$ or $b = 0$ gives rise to surfaces with constant mean curvature or constant extrinsic Gaussian curvature, respectively. Consequently we will consider both a and b being different from zero. In terms of the angle function θ, equations (5.1) and (5.2) may be written as

$$a\theta' + (a - b)\cos\theta = c \quad \text{and} \quad (a - 2b\cos\theta)\theta' - 2b\cos^2\theta = 2c$$

respectively.

A complete study of the solutions of above two equations is not difficult, although the number of cases depending on the constants a, b and c makes lengthy the statements of results. For example, a simple case is the choice $a = 0$ in (5.1): the generating curve α satisfies that $\cos\theta$ is a constant function, that is, θ is a constant function θ_0. Then the generating curve is $\alpha(s) = \big(0, (\cot\theta_0)e^{(\sin\theta_0)s}, (\sin\theta_0)s\big)$.

In order to simplify the proofs, in this section we are only going to consider the linear relation (5.1) with $c = 0$. We will give a classification theorem assuming that $\kappa_1 = m\kappa_2$.

Theorem 5.1 *Let S be a T_1-invariant surface in Sol satisfying $\kappa_1 = m\kappa_2$. Then S is a leaf of $\mathcal{F}_2$ or it belongs to one of the following situations according to the values of the parameter m:*

1. *If $m = 1$, S is an umbilical surface.*
2. *If $m = -1$, S is a minimal surface.*
3. *If $m > -1$ or $m < -2$, then the generating curve α is a graph of $z = z(y)$, with a single minimum $(m < -2)$ or single maximum $(m > -1)$. Moreover, α is asymptotic to the two vertical lines.*
4. *If $m \in (-2, -1)$, α is a graph of $z = z(y)$ defined on whole $\mathbb{R}$ and it presents a single minimum.*
5. *If $m = -2$, α is given by the graph of $z(y) = \log(\cosh(y))$.*

Proof. The generating curve α is given by

$$\theta' + (1 + m)\cos\theta = 0. \tag{5.3}$$

We discard the case $m = 1$ that gives umbilical surfaces, which were studied in [16, Proposition 19], and the case $m = -1$, which corresponds with the minimal case studied in Theorem 3.1. If θ' vanishes at some point s_0, then $\cos\theta(s_0) = 0$. By uniqueness of solutions, $\theta(s) = \pm\pi/2$, that is, θ is a constant function. Moreover, $z(s) = \pm s$ and $y(s)$ is a constant function. This means that α is a vertical straight-line and S is a leaf of $\mathcal{F}_2$. In other words, each leaf of $\mathcal{F}_2$ satisfies the relation $\kappa_1 = m\kappa_2$ for any m, since $\kappa_1 = \kappa_2 = 0$ on S.

On the contrary, we assume that $\theta' \neq 0$. Up to constants, an integration of (5.3) gives

$$\theta(s) = -2\arctan\Big(\tanh\big(\frac{m+1}{2}\,s\big)\Big)$$

with the initial condition $\theta(0) = 0$. Taking the limits, we obtain

$$\lim_{s\to\pm\infty}\theta(s) = \mp\frac{\pi}{2}.$$

As $y' = e^z \cos\theta$, this means that $y' \neq 0$ and α is a graph $z = z(y)$. Now, we have

$$\sin\theta(s) = -\tanh((m+1)s), \qquad \cos\theta(s) = \pm\frac{1}{\cosh((m+1)s)}$$

$$z(s) = -\frac{1}{m+1}\log(\cosh((m+1)s)), \qquad y'(s) = \pm\Big(\cosh((m+1)s)\Big)^{-\frac{m+2}{m+1}}.$$

We distinguish three cases.

1. Assume $\frac{m+2}{m+1} > 0$, that is, $m > -1$ or $m < -2$, then

$$y(\infty) - y(0) \leq \int_0^\infty |y'(s)| ds \leq \frac{1}{|m+1|}\int_0^\infty (e^{-t})^{\frac{m+2}{m+1}} dt < \infty.$$

   This shows that the the function y is bounded. Then the function $z(y)$ is defined in a bounded domain $I = (-M, M)$. As $z(\pm\infty) = \infty$, the graph of the generating curve α is asymptotic to the two vertical lines at $y = \pm M$. On the other hand, z' only vanishes at $s = 0$ and $z''(s) = \theta' \cos\theta = -(m+1)\cos^2\theta$. This implies that z (or α) has an absolute minimum or absolute maximum depending if $m+1 < 0$ or $m+1 > 0$, respectively.

2. Case that $m \in (-2, -1)$. The function $z(s)$ takes arbitrary values with a minimum at $s = 0$. On the other hand, $|y'(s)| \geq 1$, and so $y = y(s)$ takes values in all $\mathbb{R}$. Thus the generating curve α is a graph of the function $z = z(y)$ defined for any $y \in \mathbb{R}$.

3. Case $m = -2$. We find that $z(s) = \log(\cosh(s))$ and $y(s) = \pm s$. Thus α is the graph of $z(y) = \log(\cosh(y))$.

$\square$

Remark 5.2 If we put $m = 1$, the linear Weingarten condition says that S is an umbilical surface. As we have pointed out, umbilical surfaces have been studied in [16]. The analytic properties obtained there do agree with our results corresponding to the more general case $m > -1$ in Theorem 5.1.

We end the Section with some picture illustrating the results.

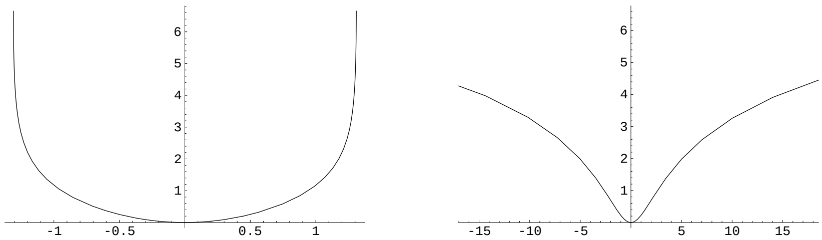

Fig. 3 Linear Weingarten surfaces with $\kappa_1 = m\kappa_2$: $m = -3$ (left) and $m = -3/2$ (right).

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