Level curves of constant mean curvature graphs over convex domains

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Abstract

We consider a compact graph of constant mean curvature $H \neq 0$ and with planar convex boundary. We prove that a level curve is convex provided its height with respect to the boundary plane is bigger than $1/(2|H|)$.

Keywords: mean curvature, level curve, convexity

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1. Introduction and statement of the result

Surfaces of constant mean curvature have the physical interpretation as the shapes taken up by soap films. The energy of a soap film, in absence of gravity, is proportional to its area. The soap film separates two regions of different pressure and, in equilibrium, this difference is, up a constant, the mean curvature of the film. When we pump air across a domain Ω, the initial

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surfaces are graphs on Ω and it is natural to discuss if the geometry of Ω, as for example convexity, imposes restrictions to the shape of the whole surface.

Mathematically we have a nonparametric compact smooth surface M of constant mean curvature H in Euclidean 3-space $\mathbb{R}^3$ and with planar boundary. Without loss of generality, we may assume that the boundary ∂M is included in the plane Π of equation $x_3 = 0$, where $x = (x_1, x_2, x_3)$ denote the usual coordinates of the ambient space. Consider $M = \{(x_1, x_2, u(x_1, x_2)); (x_1, x_2) \in \Omega\}$ as the graph of a function $u = u(x_1, x_2)$ and Ω is a bounded smooth domain of $\mathbb{R}^2$. Then $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies the constant mean curvature equation

$$\text{div } \mathsf{T}u = 2H, \qquad \mathsf{T}u = \frac{Du}{\sqrt{1 + |Du|^2}} \tag{1}$$

in Ω under the boundary condition

$$u = 0 \tag{2}$$

on $\partial\Omega$. In the case that M is a minimal graph, that is, $H = 0$, the maximum principle assures that M must be contained in the very plane Π, that is, $M = \Omega$. Thus we excluded the zero mean curvature case.

In this paper we consider the next problem:

> *Are the level curves of a compact graph of constant mean curvature spanning a convex planar boundary convex?*

This question remains open and only a few partial answers have been given. In this sense, Sakaguchi and McCuan gave an affirmative answer assuming, respectively, that the value of the mean curvature H or the enclosed volume by M is sufficiently small [11, 15].

The question is also related with a more general problem in the theory of elliptic partial differential equations asking whether the convexity of the domain Ω implies convexity of the solution (see for example [1, 6, 7, 8, 9, 12]. In the particular case of the constant mean curvature equation (1), we distinguish the type of boundary conditions. In the Dirichlet problem (1)-(2), the answer is no, as it shows some opens of unduloids: by cutting off an unduloid with a parallel plane to the axis of rotation in such way the compact part contains an inflection point of the generating curve, this surface is a graph with convex planar boundary but there exist points with negative Gaussian curvature K: [3, p. 189-191]. See also computer graphics that appear in [4] using the Brakke's Surface evolver program. In fact, the control of the set $M^- = \{p \in M; K(p) < 0\}$ allows to show that if M^- is not an empty set, then M^- must extend up to the boundary ∂M of the graph [6].

In the context of the capillarity, that is, if we change (2) by the Neumann condition $\mathsf{T}u \cdot \nu = \cos\gamma$ along $\partial\Omega$, where ν denotes the unit outer normal vector to $\partial\Omega$ and γ $(0 \leq \gamma < \pi/2)$, it is proved that the solution is convex if $\gamma = 0$ ([1]), but the result fails if $\gamma \neq 0$ [2]. In the latter case, it is interesting again to understand the set M^- ([5]). Assuming in the capillarity problem the existence of gravity, that is, replacing the constant H by κu, $\kappa > 0$, then the solution u for $\gamma = 0$ is convex [9] but if $\gamma \neq 0$, there are counterexamples

[8].

In this article we address with the initial question studying the set M^- and we prove that this set, if exists, must locate near to the boundary. In order to give this estimate, we recall a result of Serrin that asserts that a compact graph in $\mathbb{R}^3$ with constant mean curvature $H \neq 0$ and with boundary included in a plane Π can rise at most $1/|H|$ above Π [16]. At the most distant point $p \in M$ of the plane Π, we have $K(p) > 0$ and thus, in a neighbourhood of p, the surface is convex and the level curves near to p are convex.

Our result is the following:

Theorem 1. *Let M be a compact graph in $\mathbb{R}^3$ of constant mean curvature $H \neq 0$ with convex boundary included in a plane Π. Let M^- be the set of points of M with nonpositive Gaussian curvature. Then*

$$dist(x, \Pi) \leq \frac{1}{2|H|}$$

for all $x \in M^-$.

As an immediate consequence, the region of the graph that is beyond the plane Π a distance $1/(2|H|)$ has positive Gaussian curvature. This allows us to give a partial answer to the initial question.

Corollary 2. *Let M be a compact graph in $\mathbb{R}^3$ of constant mean curvature $H \neq 0$ with convex boundary included in a plane Π. Let $M_t = \{x \in M; dist(x, \Pi) = t\}$ be the level curve at t-distance from Π. Then M_t is a convex curve provided $t > 1/(2|H|)$.*

Remark 1. If $\Omega \subset \mathbb{R}^2$ is a convex domain, the value $1/(2H)$ in the constant mean curvature equation (1) is significant. The best known result is due to Serrin [17] that asserts that if the radius of curvature of $\partial\Omega$ is less or equal than $1/(2H)$, the equation (1) has solution for *arbitrary* Dirichlet condition. Moreover, if $\Omega \subset \mathbb{R}^2$ is a (bounded or not bounded) domain included in a strip of width $1/H$, then there is always a (unique) solution of (1)-(2) and whose height is less than $1/(2H)$ [10].

2. Proof of Theorem 1

It should be noted here that the constant H in the Dirichlet problem (1)-(2) cannot be prescribed and it is implicitly determined by the size of Ω. Indeed, by applying the divergence theorem to (1)

$$2|H||\Omega| = \left| \int_{\partial\Omega} \mathsf{T}u \cdot \nu \right| < |\partial\Omega|,$$

where ν is the unit outer normal to $\partial\Omega$ and $|\Omega|$, $|\partial\Omega|$ denote the measures of Ω and $\partial\Omega$ respectively. Therefore a necessary condition for the existence of solutions of (1)-(2) is that

$$|H| < \frac{|\partial\Omega|}{2|\Omega|}.$$

On the other hand, and as we have noticed in the Introduction, the height of a graph of constant mean curvature H defined on Ω can not be arbitrarily big and it is determined by H and the values that takes u along $\partial\Omega$. If the boundary is planar (we may assume that $u = 0$ along $\partial\Omega$), then $|u| \leq 1/|H|$. A hemisphere of radius $1/|H|$ shows that this estimate is the best possible.

A simple proof of this fact is as follows (see [13]). Without loss of generality, we may assume that Π is the plane $x_3 = 0$ and that $H > 0$. The mean curvature equation (1) is computed with respect to the upwards orientation on the graph M of u and the maximum principle implies that $u < 0$ on Ω: here we use that $u = 0$ along $\partial\Omega$. If N denotes the orientation, then $\langle N, \vec{a}\rangle \geq 0$ on M, where $\vec{a} = (0,0,1)$. Let us write $x_3 = x \cdot \vec{a}$ and $N_3 = N \cdot \vec{a}$. The position vector x defined on M and N satisfy the equations

$$\Delta_M x = 2HN, \qquad \Delta_M N = -(4H^2 - 2K)N,$$

where Δ_M stands for the Beltrami-Laplace operator on M. The first equation holds for any surface of $\mathbb{R}^3$, but in the second one we need that H is constant. Combining now both equations, together that $H^2 \geq K$, we have

$$\Delta_M (Hx_3 + N_3) = 2(H^2 - K)N_3 \leq 0.$$

Then the function $Hx_3 + N_3$ is superharmonic and the maximum principle yields

$$Hx_3 + N_3 \geq \min_{\partial M}(Hx_3 + N_3) \geq \min_{\partial M} N_3 \geq 0,$$

that is

$$u = x_3 \geq -\frac{N_3}{H} \geq -\frac{1}{H},$$

which, together the fact that $u < 0$, gives the required estimate.

We begin with the proof of Theorem 1. The proof uses the ideas of Chen and Huang and the comparison technique that appear in [1]. Consider the above notation. We may assume that the set $\{x \in M; \mathrm{dist}(x, \Pi) >$

$1/(2H)\}$ is not empty, otherwise there is nothing to prove. Since u must achieve an interior minimum, there exists a point in Ω at which the Gaussian curvature K is positive. In fact, by a result of Philippin [14] (see also [15]), the solution u of the Dirichlet problem (1)-(2) has exactly one critical point which corresponds with the above minimum of the function u. Therefore, in the set $\{x \in M; \mathrm{dist}(x, \Pi) > 1/(2H)\}$ there exist points of positive Gaussian curvature. Supposing the contrary, we may assume that there exist points of M^- whose distance from Π is bigger than $1/(2H)$. In particular, there exists a point $p_0 = (x_0, u(x_0))$ with $K(p_0) = 0$ and $u(x_0) < -1/(2H)$. We construct the comparison surfaces

$$v_\theta(x_1, x_2) = -\frac{1}{\cos\theta}\sqrt{\frac{1}{4H^2} - x_1^2} - (\tan\theta)x_2, \qquad -\frac{1}{2H} \leq x_1 \leq \frac{1}{2H},$$

for all $\theta \in (-\pi/2, \pi/2)$. The graph of v_θ is the lower half part of the cylinder $x_1^2 + x_3^2 = 1/(4H^2)$ of radius $1/(2H)$ rotated an angle θ with respect to the x_1-axis.

Let L be the line containing p_0 and tangent to a zero curvature direction on the graph M. If $\theta \in (-\pi/2, \pi/2)$ is the angle of L with respect to the plane Π, we consider the corresponding function $v := v_\theta$ and let $\mathcal{S}$ be the cylinder associated to v. After a horizontal translation followed of a rotation with respect to the x_3 axis, we move $\mathcal{S}$ until the position that $\mathcal{S}$ is tangent to M at the point p_0 and such that L is contained in $\mathcal{S}$. We point out that L is not necessarily a horizontal line. The surfaces M and $\mathcal{S}$ have at the point p_0 equal mean and Gaussian curvature. Thus the two surfaces have a contact of (at least) second order over p_0.

Denote Ω^* the projection of $\mathcal{S}$ onto the plane Π, that is, Ω^* is a strip of width $1/H$. The convexity of Ω implies that the boundary of $\Omega\cap\Omega^*$ consists of at most four arcs, each of which belong to $\partial\Omega$ or $\partial\Omega^*$ alternatively. Moreover, the arcs belonging to $\partial\Omega^*$ are segments of straight lines.

We consider $\Omega' = \{(x_1, x_2) \in \Omega^*; v(x_1, x_2) < 0\}$ and denote by the same symbol v the restriction of v into Ω'. Let $D = \Omega \cap \Omega'$. In the particular case that L is a horizontal line, $\Omega^* = \Omega'$. The difference function $w = u - v$ is defined on D. We wish to control the local behavior of w around the point x_0. Let $\mathcal{Z}$ be the zero set of w. The function w satisfies an elliptic equation without zero-order term. Then $\mathcal{Z}$ is a union of piecewise smooth arcs intersecting at x_0 and it divides a neighborhood $\mathcal{U}$ of x_0 into at least six components on which the signs of w alternate. For a proof, see e.g. [1, 2].

A component of $\overline{D} \setminus \mathcal{Z}$ that contains a component of $\mathcal{U} \cap (\overline{D} \setminus \mathcal{Z})$ must intersect ∂D because, in otherwise we have two functions, namely w and 0, defined in some domain, both satisfy the Dirichlet problem (1)-(2), violating the uniqueness of such problem. In the same way, two distinct components of $\mathcal{U} \cap (\overline{D} \setminus \mathcal{Z})$ cannot be included in the same component of $\overline{D} \setminus \mathcal{Z}$. Thus there are at least six arcs of ∂D where w alternates the sign. In particular, this shows that $\partial\Omega \cap \partial\Omega' \neq \emptyset$ because on the contrary, $D = \Omega$, where $w > 0$. In fact, we have that ∂D has arcs in $\partial\Omega^*$ and configurations as shows Fig. 1 are no possible since the number of arcs of ∂D where w changes of sign is at most four.

Depending on the angle θ that defines v, the boundary of D consists

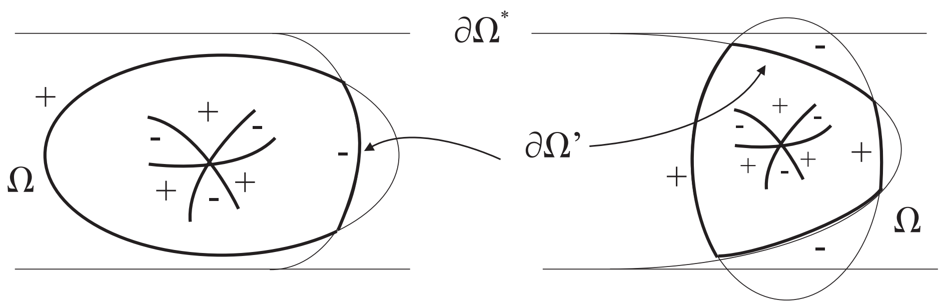

Figure 1: Configurations of D that are not possible by the existence of at least six arcs of ∂D where w changes of sign.

of arcs included in $\partial(\Omega \cap \Omega')$ and at most two arcs A_1 and A_2 included in $\Omega \cap \Omega'$ whose ends points belong to $\partial\Omega$ and $\partial\Omega^*$: see Fig. 2. Because $u(x_0) < -1/(2H)$, at least one of the two arcs of $\partial(\Omega \cap \Omega^*)$ belonging to $\partial\Omega^*$ lies below the height $x_3 = -1/(2H)$. Therefore this arc, which we denote by C_1, belongs to ∂D and ∂D contains at least two segments T_1 and T_2 of $\partial\Omega^*$. This means that ∂D consist at least of C_1, T_1 and T_2 and, possibly, A_1 and A_2 and other arc $C_2 \subset \partial\Omega$.

We analyze the sign of w in each of the arcs that define ∂D. Along C_1, $u = 0$ and $v < 0$ and so, $w > 0$. Then there exists one component of $\mathcal{U} \cap (\overline{D} \setminus \mathcal{Z})$ that contains C_1 in its boundary. Thus one component of $\partial D - (\partial D \cap \mathcal{Z})$ contains C_1 together two segments of $\partial\Omega^*$ where the function w is positive: denote q_1 and q_2 the end points of this component: see Fig. 2.

Similarly, along C_2, $u = 0$ and $v < 0$, that is, $w > 0$. Moreover, in the points $q_3 = C_2 \cap A_1$ and $q_4 = C_2 \cap A_2$, $w = 0$, that is, two arcs of $\mathcal{Z}$ arrive just in q_3 and q_4: see Fig. 3. This means that there exists other component

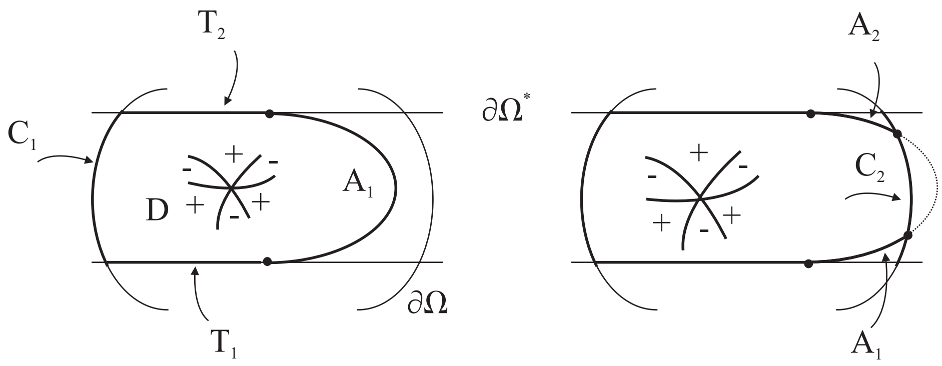

Figure 2: Two possible configurations of the domain D. On the left, $\partial D \cap \partial\Omega$ is one arc C_1 where the value of v is less than $-1/(2H)$; on the right, $\partial D \cap \partial\Omega$ has two components C_1 and C_2.

of $\mathcal{U} \cap (\overline{D} \setminus \mathcal{Z})$ containing C_2 in its boundary and where w is positive.

On the other hand, $v = 0$ along $A_1 \cup A_2$ where u is negative, which implies that $w < 0$ in $A_1 \cup A_2$. Then two components of $\partial D - (\partial D \cap \mathcal{Z})$ where $w < 0$ contains segments of $\partial\Omega^*$. Denote by q_5 and q_6 the ends points of these segments (see Fig. 3). In fact, $q_2 \neq q_6$ or $q_1 \neq q_5$ because on the contrary, there would be exactly four arcs of ∂D where w alternates the sign.

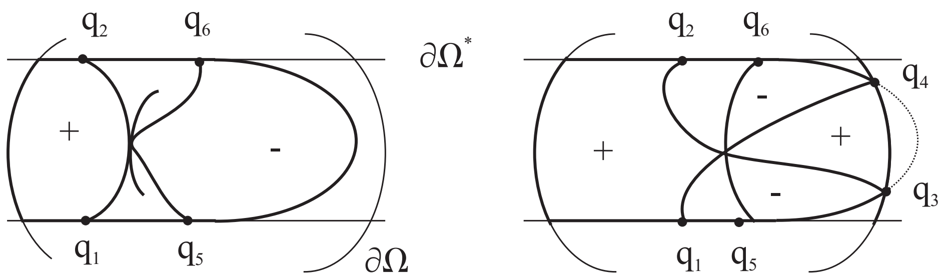

Figure 3: The discussion of the sign of w along ∂D.

As conclusion, there exists another component $\mathcal{K}$ of $\overline{D} \setminus \mathcal{Z}$ where w is positive and such that $\partial\mathcal{K} \cap \partial D$ lies completely in $\partial\Omega^*$. We show that this is impossible.

We decompose $\partial K = \Gamma \cup \Gamma^*$, where $\Gamma^* \subset \partial\Omega^*$ and $\Gamma \subset D$. Then an integration of (1) over $\mathcal{K}$ and the divergence theorem yields

$$\begin{aligned} 0 &= \int_{\mathcal{K}} (\text{div } \mathsf{T}u - \text{div } \mathsf{T}v) \\ &= \int_{\partial\mathcal{K}} (\mathsf{T}u - \mathsf{T}v)\cdot\nu = \int_{\Gamma} (\mathsf{T}u - \mathsf{T}v)\cdot\nu + \int_{\Gamma^*} (\mathsf{T}u - \mathsf{T}v)\cdot\nu, \end{aligned} \tag{3}$$

where ν is the unit outer normal on the boundary $\partial\mathcal{K}$.

In the boundary Γ^*, we have $\mathsf{T}v\cdot\nu = 1$. Because $|\mathsf{T}u| < 1$, then

$$(\mathsf{T}u - \mathsf{T}v)\cdot\nu \leq 0, \quad \text{on } \Gamma^*. \tag{4}$$

On the other hand, and because $u > v$ on $\mathcal{K}$, we have $\langle \eta_u, \vec{a}\rangle \geq \langle \eta_v, \vec{a}\rangle$ along Γ, where $\langle , \rangle$ is the Euclidean product in $\mathbb{R}^3$ and η_u and η_v denote the inner conormal unit vectors along Γ of the graphs of u and v, respectively. Since

$$\langle \eta_u, \vec{a}\rangle = -\mathsf{T}u\cdot\nu, \quad \langle \eta_v, \vec{a}\rangle = -\mathsf{T}v\cdot\nu,$$

we conclude that $(\mathsf{T}u - \mathsf{T}v)\cdot\nu \leq 0$ in Γ. In virtue of (3) and (4), we have $(\mathsf{T}u - \mathsf{T}v)\cdot\nu = 0$ on $\partial\mathcal{K}$. We conclude that u and v satisfy (1) on the domain $\mathcal{K}$ and the same boundary condition $\mathsf{T}u\cdot\nu = \mathsf{T}v\cdot\nu$ along $\partial\mathcal{K}$. The uniqueness of the Neumann problem of (1) in $\mathcal{K}$ asserts $u = v$ in $\mathcal{K}$, that is, $w = 0$ on $\mathcal{K}$, a contradiction.

Remark 2. We analyze the estimate given in Theorem 1 for unduloids. Consider an ellipse of half axis $a > b > 0$ that rolls without slipping along the

x-axis. If we rotate with respect to the x-axis the curve $y = y(x)$ that describes one focus of the ellipse, we obtain a surface of revolution of constant mean curvature $H = 1/(2a)$ called unduloid. By varying the parameter b from $b = 0$ to $b = a$, we obtain a uniparametric family of surfaces with the same mean curvature starting from a stack of spheres of radius $2a$ (if $b = 0$) until a cylinder of radius a (if $b = a$). The equation of y is $1 + y'^2 - yy'' = 2Hy(1 + y'^2)^{3/2}$ and a first integral is

$$\frac{y}{\sqrt{1+y'^2}} = \frac{1}{2a}y^2 + \frac{b^2}{2a}.$$

The inflection point of $y(x)$ occurs when $y'' = 0$ and combining the above two equations, its height is $y = b$. The part of the unduloid obtained by the points with $y < b$ has negative Gaussian curvature while if $y > b$, the Gaussian curvature is positive. The minimum height of y is $y_m = a - \sqrt{a^2 - b^2}$. If we cut the unduloid by a parallel plane Π to the axis of rotation at distance $y = y_0 \geq y_m$ from the axis, the halfspace determined by Π that does not contains the x-axis defines a compact graph M over Π. The highest point of $\{p \in M; K(p) \leq 0\}$ is just the inflection point of y and is height with respect to Π is $b - y_0$. As y_0 can reach the value y_m, then maximum height is $b - y_m$. If we see $f(b) := b - y_m$ as a function on b, then f goes from 0 (for a sphere) until 0 again when the limit surface is a cylinder. The maximum of f occurs when $b = a/\sqrt{2}$, being $f(b/\sqrt{2}) = (\sqrt{2} - 1)a$. This value is strictly less than $1/(2H) = a$.

Remark 3. The author has been informed by the referee that recently Xu-Jia Wang (Centre for Mathematics and its Applications, ANU) has announced a counterexample that the level curves of the solution of (1)-(2) in a convex domain are not all convex [18]. But his example shows that the non-convexity of the level curves appear at height $h = 1/(4|H|)$ such as expected by Theorem 1.

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