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Surfaces with constant mean curvature in Sol geometry

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ABSTRACT

Among the eight geometries of Thurston, Sol_3 is the space with the smallest number of isometries, for example, there are no rotations. In this work, we classify all surfaces with constant mean curvature that either are invariant by a 1-parameter group of isometries or are the product of two planar curves.

1. Introduction

The theory of surfaces with constant mean curvature (briefly, CMC surfaces) in Euclidean space $\mathbb{R}^3$ is a classical topic in Differential Geometry starting in the XVIII century with the problems on elasticity studied by Sophie Germain, which motivated the consideration of the notion of mean curvature. Usually the mean curvature of a surface can be introduced when one considers one of the oldest problems in Geometry, namely the isoperimetric problem, which can be formulated in the following way: *among all compact surfaces in Euclidean space enclosing a fix volume, find the one of the biggest area*. The answer is the round sphere, such as it was showed by Schwarz using previous ideas of Steiner and Minkowski. The theory of CMC surfaces has been extensively studied, including minimal surfaces, in space forms.

Recently, many geometers have focused their interest in the theory of submanifolds in homogeneous three-manifolds, specially after the geometrization conjecture formulated by Thurston in 1982. This conjecture asserts that every compact orientable three-manifold has a canonical decomposition into pieces, each of which admits a canonical geometric structure from among the eight maximal simply connected homogeneous Riemannian 3-geometries [14]. These eight spaces are:

1. The three space forms $\mathbb{R}^3$, $\mathbb{H}^3$ and $\mathbb{S}^3$, where the dimension of the group of isometries is 6.
2. The product spaces $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{R}$, the Heisenberg group Nil and $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$. In each case, the dimension of the group of isometries is 4.
3. The Lie group Sol_3, whose group of isometries has dimension 3.

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The first problems were the generalization of the classical Hopf and Alexandrov theorems for CMC closed surfaces. Recall the two main theorems of classification which characterize the round sphere in the family of CMC closed surfaces, showing that it is the only one of genus 0 (Hopf) and the only one that is embedded (Alexandrov). The starting point was the article of Abresh and Rosenberg [1], interested into the Hopf theorem. For a recent accounts in the theory of CMC surfaces in homogeneous spaces we refer to the reader to [3,5] and references therein. In this work, we consider the Sol_3 space, the simply connected homogeneous 3-manifold whose isometry group is the smallest. The low dimension of $\mathrm{Iso}(\mathrm{Sol}_3)$ makes difficult to realize an extension of such theorems, such as it was pointed in [2]. Previously, it was known the geodesics of space [15] and later something on umbilical surfaces [13]. Finally, the extension of both theorems have been done very recently [4,11].

This article aims the search of examples of CMC surfaces in Sol_3 with some geometric properties. Following the Euclidean scheme, a first step that one could do is the study of CMC surfaces invariant by a 1-parameter group of isometries. Although in Sol_3 there are no rotations, we study in Section 3 CMC surfaces that are invariant by translations of Sol_3 and we will give a complete classification of invariant CMC surfaces. A second source of examples that we consider are the translation surfaces, obtaining in Section 4 examples with zero mean curvature. There are many possibilities to continue this work in Sol_3. For example, one could obtain a method to construct minimal surfaces, similar as in Euclidean space with the Weierstrass representation (see [6]), or consider problems of CMC compact surfaces with non-empty boundary asking whether the surface inherits the symmetries of its boundary, as it appears in [8].

2. The Sol_3 space

The space Sol_3 is the space $\mathbb{R}^3$ equipped with the metric

$$\langle\, ,\, \rangle = e^{2z}\,dx^2 + e^{-2z}\,dy^2 + dz^2,$$

where (x, y, z) are usual coordinates of $\mathbb{R}^3$. The space Sol_3 is a Lie group with the multiplication

$$(x, y, z) * \left(x', y', z'\right) = \left(x + e^{-z}x', y + e^{z}y', z + z'\right).$$

With respect to this operation, the metric $\langle\, ,\, \rangle$ is left-invariant. As we have already pointed out, the isometry group $\mathrm{Iso}(\mathrm{Sol}_3)$ has dimension 3. The component of the identity is generated by the following families of isometries (see [13,15]):

$$T_{1,t}(x, y, z) := (x + t, y, z), \qquad T_{2,t}(x, y, z) := (x, y + t, z), \qquad T_{3,t}(x, y, z) := \left(e^{-t}x, e^{t}y, z + t\right)$$

where $t \in \mathbb{R}$ is a real parameter. These isometries are left multiplications by elements of Sol_3 and so, they are left-translations with respect to the structure of Lie group. Remark that the elements $T_{1,t}$ and $T_{2,t}$ are precisely Euclidean translations along horizontal directions parallel to the vertical coordinate planes and that the set of fixed points are totally geodesic surfaces in Sol_3. In general, the totally geodesic surfaces are the vertical planes, that is, $ax + by + c = 0$. On the other hand, the horizontal planes $x = ct$ are minimal surfaces.

The Killing vector fields associated to these isometries are, respectively,

$$\frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial y}, \qquad -x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{\partial}{\partial z}. \tag{1}$$

A left-invariant orthonormal frame $\{E_1, E_2, E_3\}$ in Sol_3 is given by

$$E_1 = e^{-z}\frac{\partial}{\partial x}, \qquad E_2 = e^{z}\frac{\partial}{\partial y}, \qquad E_3 = \frac{\partial}{\partial z}.$$

The Riemannian connection $\widetilde{\nabla}$ of Sol_3 with respect to $\{E_1, E_2, E_3\}$ is

$$\begin{aligned}
&\widetilde{\nabla}_{E_1}E_1 = -E_3, && \widetilde{\nabla}_{E_1}E_2 = 0, && \widetilde{\nabla}_{E_1}E_3 = E_1,\\
&\widetilde{\nabla}_{E_2}E_1 = 0, && \widetilde{\nabla}_{E_2}E_2 = E_3, && \widetilde{\nabla}_{E_2}E_3 = -E_2,\\
&\widetilde{\nabla}_{E_3}E_1 = 0, && \widetilde{\nabla}_{E_3}E_2 = 0, && \widetilde{\nabla}_{E_3}E_3 = 0.
\end{aligned}$$

Let M be an orientable surface and let $x : M \to \mathrm{Sol}_3$ be an isometric immersion. Consider N the Gauss map of M and denote by ∇ the induced Levi-Civita connection on M. The Gauss formula is

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y)N, \qquad \sigma(X, Y) = \langle \widetilde{\nabla}_X Y, N\rangle, \tag{2}$$

where X, Y are tangent vector fields on M and σ is the second fundamental form of the immersion. For each $p \in M$, we consider the Weingarten map $A_p : T_pM \to T_pM$, where T_pM is the tangent plane, defined by $A_p(v) = -\widetilde{\nabla}_X(N)$, with X a tangent vector field of M that extends v at p. The mean curvature H of the immersion is defined as $H(p) = (1/2)\mathrm{trace}(A_p)$. We know that A_p is a self-adjoint endomorphism with respect to the metric on M, that is, $\langle A_p(u), v\rangle = \langle u, A_p(v)\rangle$, $u, v \in T_pM$. Moreover,

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$$-\langle \widetilde{\nabla}_X N, Y\rangle = \langle \widetilde{\nabla}_X Y, N\rangle. \tag{3}$$

At each tangent plane T_pM we take a basis $\{e_1, e_2\}$ and let us write

$$A_p(e_1) = -\widetilde{\nabla}_{e_1} N = a_{11}e_1 + a_{12}e_2, \qquad A_p(e_2) = -\widetilde{\nabla}_{e_2} N = a_{21}e_1 + a_{22}e_2.$$

We multiply in both identities by e_1 and e_2 and denote by $\{E, F, G\}$ the coefficients of the first fundamental form:

$$E = \langle e_1, e_1\rangle, \qquad F = \langle e_1, e_2\rangle, \qquad G = \langle e_2, e_2\rangle.$$

We conclude with the classical formula for H:

$$H = \frac{1}{2}(a_{11} + a_{22}) = \frac{1}{2}\frac{G\langle N, \widetilde{\nabla}_{e_1} e_1\rangle - 2F\langle N, \widetilde{\nabla}_{e_1} e_2\rangle + E\langle N, \widetilde{\nabla}_{e_2} e_2\rangle}{EG - F^2}. \tag{4}$$

Using (3), we obtain

$$a_{11} = \frac{\begin{vmatrix} -\langle \widetilde{\nabla}_{e_1} N, e_1\rangle & F \\ -\langle \widetilde{\nabla}_{e_1} N, e_2\rangle & G \end{vmatrix}}{EG - F^2} = \frac{\begin{vmatrix} \langle N, \widetilde{\nabla}_{e_1} e_1\rangle & F \\ \langle N, \widetilde{\nabla}_{e_1} e_2\rangle & G \end{vmatrix}}{EG - F^2}, \qquad a_{22} = \frac{\begin{vmatrix} E & -\langle \widetilde{\nabla}_{e_2} N, e_1\rangle \\ F & -\langle \widetilde{\nabla}_{e_2} N, e_2\rangle \end{vmatrix}}{EG - F^2} = \frac{\begin{vmatrix} E & \langle N, \widetilde{\nabla}_{e_2} e_1\rangle \\ F & \langle N, \widetilde{\nabla}_{e_2} e_2\rangle \end{vmatrix}}{EG - F^2}.$$

3. Invariant CMC surfaces in Sol_3

The interest in this section is to find examples of CMC surfaces that are invariant by a 1-parameter group of isometries. In contrast to the other Thurston geometries, in Sol_3 there are no rotations. However, we can consider the three 1-parameter groups $\{T_{i,t}\}_{t\in\mathbb{R}}$ defined by (1). We pay our attention in the first two ones:

Definition 3.1. A surface M in Sol_3 is called invariant if it is invariant under the action of one of the 1-parameter groups of isometries $\mathcal{G}_i := \{T_{i,t};\ t \in \mathbb{R}\}$, with $i = 1, 2$.

After an isometry of the ambient space, an invariant surface under the group $\mathcal{G}_2$ converts into an invariant surface under the group $\mathcal{G}_1$: this can be achieved by taking the isometry of Sol_3 given by $\phi(x, y, z) = (y, x, -z)$. Thus we restrict to invariant surfaces under the first group $\mathcal{G}_1$. Following [9], in this section we classify all invariant surfaces in Sol_3 with constant mean curvature.

We compute the curvatures of an invariant surface M. The surface M is determined by the intersection curve α obtained by the orbits generated by the group with any orthogonal plane. Any such curve α is called a *generating curve* of the surface. We will take α to be the intersection of M with the plane $x = 0$. Let us take a parametrization of α given by $\alpha(s) = (0, y(s), z(s))$, $s \in I$, where s is the arc-length parameter. Thus

$$e^{-z(s)} y'(s) = \cos\theta(s), \qquad z'(s) = \sin\theta(s)$$

where $\theta = \theta(s)$ is a certain smooth function. We can parametrize M by $X(s, t) = (t, y(s), z(s))$, $s \in I \subset \mathbb{R}$, $t \in \mathbb{R}$. We have

$$e_1 := X_s = \big(0, y', z'\big) = \cos\theta E_2 + \sin\theta E_3,$$
$$e_2 := X_t = (1, 0, 0) = e^z E_1.$$

We choose as Gauss map $N = -\sin\theta E_2 + \cos\theta E_3$. The coefficients of the first fundamental form are $E = 1$, $F = 0$ and $G = e^{2z}$ and the values of $\nabla_{e_i} e_j$ are given by

$$\nabla_{e_1} e_1 = \big(\theta' + \cos\theta\big)(-\sin\theta E_2 + \cos\theta E_3),$$
$$\nabla_{e_1} e_2 = \nabla_{e_2} e_1 = \sin\theta e^z E_1,$$
$$\nabla_{e_2} e_2 = -e^{2z} E_3.$$

The computation of H in (4) gives $H = \theta'/2$. This means that the constancy of the mean curvature reduces to consider a system of ODE, namely,

$$y'(s) = e^{z(s)} \cos\theta(s), \tag{5}$$
$$z'(s) = \sin\theta(s), \tag{6}$$
$$\theta'(s) = 2H. \tag{7}$$

Theorem 3.2. *The only minimal surfaces invariant by $\mathcal{G}_1$ are the planes $y = ct$, the planes $z = ct$ and the surfaces $z = z(x, y)$, where $z(x, y) = \log(y + \lambda) + \mu$, $\lambda, \mu \in \mathbb{R}$.*

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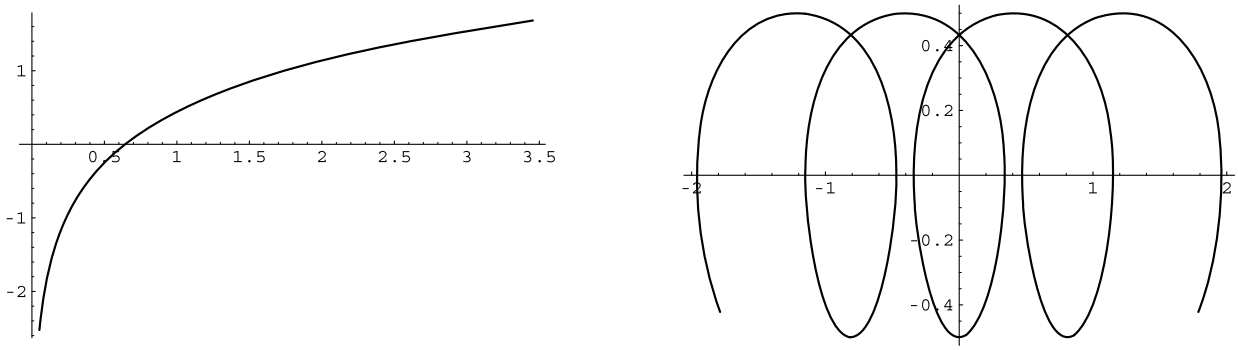

Fig. 1. Generating curves of invariant surfaces with constant mean curvature: case $H = 0$ (left); case $H = 1$ (right).

Proof. Since $H = 0$, then $\theta(s) = \theta_0$ for some constant $\theta_0 \in \mathbb{R}$. By (6), $z(s) = (\sin\theta_0)s + z_0$, $z_0 \in \mathbb{R}$. If $\sin\theta_0 = 0$, then (5) gives $y(s) = (\cos\theta_0)s + y_0$. This says that α is a horizontal straight-line and M is the horizontal plane $z = z_0$. If $\cos\theta_0 = 0$, then $y(s) = y_0$ for some constant y_0, α is a vertical straight-line and the surface is $y = ct$.

If $\sin\theta_0 \neq 0$, we have from (6) that (5) writes as

$$y' = e^{(\sin\theta_0)s+z_0}\cos\theta_0 \quad \Rightarrow \quad y(s) = (\cot\theta_0)e^{(\sin\theta_0)s+z_0} + y_0, \quad y_0 \in \mathbb{R}.$$

Then the generating curve is

$$\alpha(s) = \left(y_0 + (\cot\theta_0)e^{(\sin\theta_0)s+z_0}, (\sin\theta_0)s + z_0\right).$$

This means that α describes the graphic of a logarithmic function $z = z(y) = \log((\tan\theta_0)(y - y_0))$. □

For the case $H \neq 0$, we describe the shape of the surface.

Theorem 3.3. *Let M be an invariant surface in Sol_3 with constant mean curvature $H \neq 0$. We write the generating curve of M as $\alpha(s) = (0, y(s), z(s))$. Then*

1. *The curve α is invariant by a discrete group of translation in the y-direction.*
2. *The z-coordinate is bounded and periodic.*
3. *The curve α has self-intersections.*
4. *The velocity vector of α turns around the origin such that it takes all values in the unit circle.*

Proof. From $\theta' = 2H$, we obtain, after a possible translation in parameter s, $\theta(s) = 2Hs$. Then $z'(s) = \sin(2Hs)$ and hence

$$z(s) = -\frac{1}{2H}\cos(2Hs).$$

In particular, z is a periodic function of principal period $T = \pi/H$, whose derivative vanishes in a discrete set of points, namely, $A = \{n\pi/(2H);\ n \in \mathbb{Z}\}$. From (5),

$$y'(s) = \exp\left(-\frac{1}{2H}\cos(2Hs)\right)\cos(2Hs).$$

Then the function y' vanishes at the set $B = A + \pi/2$. In particular, this means that α is not a graph on the y-axis, being the velocity of α vertical at each point of B. Moreover, z takes the same value at these points: with our choice of the integration constants, this value is $z = 0$.

It is easy to show that if $\{y(s), z(s), \theta(s)\}$ satisfy (5)–(7), with initial conditions $\{y_0, z_0, \theta_0\}$, then the functions $\{y(s+T) - y(T) + y_0, z(s), \theta(s)\}$ satisfy the same equations and initial conditions. By uniqueness of solutions of ODE, both solutions must agree. In particular, $y(s+T) = y(s) + y(T) - y_0$. Thus, we have proved that the generating curve α is invariant by translations of the group of translations generated by the vector $(0, y(T) - y_0, 0)$. In our notation, this group is $\{T_{2,n(y(T)-y_0)};\ n \in \mathbb{Z}\}$.

Finally, the function $\theta(s)$ takes all real values, which means that the planar velocity vector $\alpha'(s) = \cos\theta(s)E_2(s) + \sin\theta(s)E_3(s)$ moves taking all the values of a unit circle in a monotonic sense. □

We end this section with two pictures (see Fig. 1) the generating curve α of invariant CMC surfaces.

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Remark 3.4. A similar work can do for invariant surfaces in Sol_3 with constant Gaussian curvature K. Again, the constant Gauss curvature equation is an ODE, which can be studied and solved in particular cases of K. See [9]. Here, we point out two special cases for K.

1. The only invariant surfaces with $K=0$ are horizontal planes $z=ct$ or the generating curve α is
$$\alpha(s)=\left(0,\frac{1}{2}\left(s\sqrt{s^2-1}-\log\left(\left|s+\sqrt{s^2-1}\,\right|\right)\right),\log\left(|s|\right)\right),\quad s^2\geqslant 1.$$
2. The only invariant surfaces with $K=-1$ are vertical planes $y=ct$ or the generating curve α is $\alpha(s)=(0,s,\log(\cosh(s)))$.

4. Minimal translation surfaces in Sol_3

In 1835, Scherk [12] found all minimal surfaces in $\mathbb{R}^3$ that are graphs of functions $z=z(s,t)$, with $z(s,t)=f(s)+g(t)$, and f and g are defined in open sets of $\mathbb{R}$. These surfaces are called translation surfaces since its parametrization $X(s,t)=(s,t,f(s)+g(t))$ can be written as the sum (translation) of two curves, namely, $X(s,t)=(s,0,f(s))+(0,t,g(t))$. Besides planes, the only minimal translation surfaces are
$$z(s,t)=\frac{1}{a}\log\left|\cos(as)\right|-\frac{1}{a}\log\left|\cos(at)\right|=\frac{1}{a}\log\left|\frac{\cos(as)}{\cos(at)}\right|,$$
where $a\neq 0$. Later, Liu obtained that the only translation surfaces with non-zero constant mean curvature are right cylinders [7]. We propose a similar problem in Sol_3 changing the additive operation $+$ in Euclidean space by the group operation $*$ of Sol_3.

Definition 4.1. A translation surface $M(\alpha,\beta)$ in Sol_3 is a surface parameterized by $X(s,t)=\alpha(s)*\beta(t)$, where $\alpha: I\to\mathrm{Sol}_3$, $\beta: J\to\mathrm{Sol}_3$ are curves in coordinate planes of $\mathbb{R}^3$.

Comparing with the Euclidean ambient space, a first problem that appears in Sol_3 is the noncommutativity of the group operation $*$. Moreover, there is no equivalence of the coordinate planes so that we have to distinguish the three possibilities of choice of pairs of planes. As conclusion, there are exactly six different types of translations surfaces in Sol_3. The next step to do is the computation of the mean curvature and impose that H is constant. The authors of the present work have been able to determine all translation *minimal* surfaces when one of the planar curves in the above definition lies in the plane $z=0$. In order to show here the techniques that we used in this classification, only we consider that the surface $M(\alpha,\beta)$ is of type
$$X(s,t)=\alpha(s)*\beta(t)=\big(s,f(s),0\big)*\big(t,0,g(t)\big)=\big(s+t,f(s),g(t)\big). \tag{8}$$
A similar work can be carried for the surface $M(\beta,\alpha)$, which it does not present extra difficulties. Because we are interested on minimal surfaces, in the computation of H given in (4), we can change N by other proportional vector $\overline{N}$. Then $M(\alpha,\beta)$ is a minimal surface if and only if
$$G\langle\overline{N},\widetilde{\nabla}_{e_1}e_1\rangle-2F\langle\overline{N},\widetilde{\nabla}_{e_1}e_2\rangle+E\langle\overline{N},\widetilde{\nabla}_{e_2}e_2\rangle=0. \tag{9}$$
The local computations for the surface parametrized as in (8) are
$$\begin{aligned}
e_1&=X_s=\big(1,f',0\big)=e^gE_1+f'e^{-g}E_2,\\
e_2&=X_t=\big(1,0,g'\big)=e^gE_1+g'E_3,\\
\overline{N}&=\big(f'g'e^{-g}\big)E_1-g'e^gE_2-f'E_3.
\end{aligned}$$
The coefficients of the first fundamental form are
$$E=e^{2g}+f'^2e^{-2g},\qquad F=e^{2g},\qquad G=e^{2g}+g'^2.$$
On the other hand,
$$\begin{aligned}
\widetilde{\nabla}_{e_1}e_1&=f''e^{-g}E_2+\big(f'^2e^{-2g}-e^{2g}\big)E_3,\\
\widetilde{\nabla}_{e_1}e_2&=g'e^gE_1-f'g'e^{-g}E_2-e^{2g}E_3,\\
\widetilde{\nabla}_{e_2}e_2&=2g'e^gE_1+\big(g''-e^{2g}\big)E_3
\end{aligned}$$
and

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$$\langle N, \widetilde{\nabla}_{e_1} e_1\rangle = -f''g' - f'^3 e^{-2g} + f'e^{2g},$$
$$\langle N, \widetilde{\nabla}_{e_1} e_2\rangle = 2f'g'^2 + f'e^{2g},$$
$$\langle N, \widetilde{\nabla}_{e_2} e_2\rangle = 2f'g'^2 - f'g'' + f'e^{2g}.$$

According to (9), the surface is minimal if and only if

$$-f''g'^3 - e^{2g}\big(f''g' + f'g'^2 + f'g''\big) + e^{-2g} f'^3\big(g'^2 - g''\big) = 0. \tag{10}$$

We begin to analyze Eq. (10) in simple cases. If f is constant, then $f(s) = y_0$ and $M(\alpha, \beta)$ is the plane $y = y_0$. If g is constant, $g(t) = z_0$ and the surface is the plane $z = z_0$.

Remark 4.2. If we write the curves α and β as $\alpha(s) = (f(s), s, 0)$ and $\beta(t) = (g(t), 0, t)$, then the parametrization of $M(\alpha, \beta)$ is $X(s,t) = (f(s) + g(t), s, t)$. Eq. (10) becomes now

$$f''g'^3 - e^{2g}\big(-f''g' + f'^2 g'^2 + f'^2 g''\big) + e^{-2g}\big(g'^2 - g''\big) = 0.$$

Then if f and g are constant, then the surface is minimal. This means that the planes $x = ct$ are also minimal translation surfaces.

From now on, we assume in (10) that $f'g' \neq 0$. We divide (10) by $f'^3 g'^3$:

$$-\frac{f''}{f'^3} - e^{2g}\left(\frac{f''}{f'^3}\frac{1}{g'^2} + \frac{1}{f'^2}\frac{1}{g'} + \frac{g''}{g'^3}\frac{1}{f'^2}\right) + e^{-2g}\frac{g'^2 - g''}{g'^3} = 0. \tag{11}$$

In (11), the first and the third summands are functions depending only on s and only on t, respectively. Then, we differentiate with respect to s and t, obtaining

$$\frac{\partial^2}{\partial s \partial t}\left[e^{2g}\left(\frac{f''}{f'^3}\frac{1}{g'^2} + \frac{1}{f'^2}\frac{1}{g'} + \frac{g''}{g'^3}\frac{1}{f'^2}\right)\right] = 0.$$

This means

$$\left(\frac{f''}{f'^3}\right)'\left(\frac{1}{g'} - \frac{g''}{g'^3}\right) - 2\frac{f''}{f'^3} - \left(\frac{f''}{f'^3}\right)\left(\left(\frac{g''}{g'^3}\right)' + \frac{g''}{g'^2}\right) = 0. \tag{12}$$

1. Assume $f'' = 0$. Then $f(s) = as + b$, with $a, b \in \mathbb{R}$. Eq. (10) implies

   $$e^{2g}\big(g'' + g'^2\big) = a^2 e^{-2g}\big(-g'' + g'^2\big).$$

   We do the change $g(t) = h(t) + m$, with $e^{4m} = a^2$ and next, $\zeta(t) = 2h(t)$. Then we obtain $2\zeta''(e^{\zeta} + e^{-\zeta}) = -\zeta'^2(e^{\zeta} - e^{-\zeta})$, or

   $$2\zeta''\cosh(\zeta) = -\zeta'^2\sinh(\zeta).$$

   A first integration implies

   $$\zeta'^2 = \frac{c^2}{\cosh(\zeta)}, \quad c > 0.$$

   A second integration yields $\int^t \sqrt{\cosh\zeta(\tau)}\zeta'(\tau)\,d\tau = ct + d$, $d \in \mathbb{R}$. Consider $I(t) = \int^t \sqrt{\cosh\tau}\,d\tau$, which is a strictly increasing function. Hence, the equation $I(\zeta(t)) = ct$ has a unique solution $\zeta(t) = I^{-1}(ct)$.
2. Assume $g'' - g'^2 = 0$. Since g is not constant, the function g is $g(t) = -\log|t + \lambda| + \mu$, $\lambda, \mu \in \mathbb{R}$. Then (10) implies

   $$\big(1 + e^{2\mu}\big)(t + \lambda)f''(s) - 2e^{2\mu}f'(s) = 0.$$

   This is a polynomial on t. Then $f' = f'' = 0$: contradiction.
3. Consider $f''(g'' - g'^2) \neq 0$. From (12), we conclude that there exists $a \in \mathbb{R}$ such that

   $$\frac{\left(\frac{f''}{f'^3}\right)'}{\left(\frac{f''}{f'^3}\right)} = a = \frac{\left(\frac{g''}{g'^3}\right)' + \frac{g''}{g'^2} + 2}{\frac{1}{g'} - \frac{g''}{g'^3}}. \tag{13}$$

(a) Assume $a=0$. Then $f''=bf'^3$ for some constant $b\neq 0$. Then $1/f'^2=-2bs+c$, $c\in\mathbb{R}$. On the other hand, the second equation in (13) writes as

$$\left(\frac{g''}{g'^3}-\frac{1}{g'}\right)'+2=0. \tag{14}$$

Then

$$\frac{g''}{g'^3}-\frac{1}{g'}=-2t+d,\quad d\in\mathbb{R}.$$

With this information about f and g, Eq. (11) writes as

$$-b\left(1+\frac{e^{2g}}{g'^2}\right)+(2bs-c)e^{2g}\left(\frac{g''}{g'^3}+\frac{1}{g'}\right)-e^{-2g}\left(\frac{g''}{g'^3}-\frac{1}{g'}\right)=0. \tag{15}$$

Since this expression is a polynomial equation on s, and because $b\neq 0$, the leading coefficient corresponding to s implies

$$\frac{g''}{g'^3}+\frac{1}{g'}=0.$$

In combination with (14), we have $1/g'=t-d/2$ and $g(t)=\log(t-d/2)+\mu$, $\mu\in\mathbb{R}$. Now the independent coefficient in (15) is now

$$-b\left(1+e^{2\mu}\left(t-\frac{d}{2}\right)^4\right)+\frac{2e^{-2\mu}}{t-\frac{d}{2}}=0.$$

After some manipulations, we have a polynomial equation on t whose leading coefficient is $be^{2\alpha}$. As it mush vanish, we arrive to a contradiction.

(b) Assume $a\neq 0$. From the first equation in (13), we obtain a first integral: there exists $b\neq 0$ such that

$$\frac{f''}{f'^3}=be^{as}. \tag{16}$$

Then we have that for some $c\in\mathbb{R}$,

$$\frac{-1}{2f'^2}=\frac{b}{a}e^{as}+c. \tag{17}$$

Plugging (16) and (17) in (11), we have for any s

$$-be^{as}\left[1+e^{2g}\left(\frac{1}{g'^2}-\frac{2}{a}\left(\frac{1}{g'}+\frac{g''}{g'^3}\right)\right)\right]+2ce^{2g}\left(\frac{1}{g'}+\frac{g''}{g'^3}\right)+e^{-2g}\left(\frac{1}{g'}-\frac{g''}{g'^3}\right)=0.$$

This is a polynomial on e^{as} and thus the two coefficients must vanish. It follows that g satisfies the next two differential equations:

$$\begin{aligned}&1+e^{2g}\left(\frac{1}{g'^2}-\frac{2}{a}\left(\frac{1}{g'}+\frac{g''}{g'^3}\right)\right)=0,\\&2ce^{2g}\left(\frac{1}{g'}+\frac{g''}{g'^3}\right)+e^{-2g}\left(\frac{1}{g'}-\frac{g''}{g'^3}\right)=0.\end{aligned} \tag{18}$$

If $c=0$, then $g''-g'^2=0$, which it is impossible. Therefore, we assume that $c\neq 0$. We study the function g. From (13), we have a linear equation for $\varphi=\frac{1}{g'}-\frac{g''}{g'^3}$, namely,

$$\varphi'+a\varphi-2=0.$$

The solution is

$$\varphi=\frac{1}{g'}-\frac{g''}{g'^3}=\frac{2}{a}+\lambda e^{-at},\quad \lambda\in\mathbb{R}. \tag{19}$$

Combining (19) with (18), we have

$$2ce^{2g}\left(\frac{2}{g'}-\frac{2}{a}-\lambda e^{-at}\right)+e^{-2g}\left(\frac{2}{a}+\lambda e^{-at}\right)=0.$$

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We deduce

$$\frac{1}{g'} = \frac{1}{4ac} e^{-at-4g}\left(-1 + 2ce^{4g}\right)\left(2e^{at} + a\lambda\right). \tag{20}$$

Putting this value in (19) again, we have

$$a\lambda + 4c^2 e^{8g(t)}\left(2e^{at} + a\lambda\right) - 4ce^{4g(t)}\left(3e^{at} + a\lambda\right) = 0.$$

This implies

$$e^{4g(t)} = \frac{3e^{at} + a\lambda \pm \sqrt{9e^{2at} + 4a\lambda e^{at}}}{2c(2e^{at} + a\lambda)}.$$

From here, we have two values for g. Without loss of generality, we take the sign $+$ in the above expression (the reasoning is analogous with the choice $-$). Together (20), we have:

$$24e^{at} + 11a\lambda + 4\sqrt{9e^{2at} + 4a\lambda e^{at}} + 3a\lambda e^{-at}\sqrt{9e^{2at} + 4a\lambda e^{at}} = 0.$$

This identity can be viewed as a polynomial equation on e^{at}:

$$108e^{3at} + 62a\lambda e^{2at} - 14a^2\lambda^2 e^{at} - 9a^3\lambda^3 = 0.$$

As the leading coefficient must vanish, we get a contradiction.

As conclusion, we have (see [10]):

Theorem 4.3. *The only minimal translation surfaces in Sol_3 parametrized as $X(s,t) = \alpha(s)\beta(t)$, where α is included in the plane $z = 0$ and β in $y = 0$ are the planes $y = ct$, the planes $x = ct$, the planes $z = ct$ and the surfaces whose parametrization is $X(s,t) = (s + t, f(s), g(t))$ where $f(s) = as + b$, $a, b \in \mathbb{R}$, $a \neq 0$ and*

$$g(t) = \frac{1}{2} I^{-1}(ct) + m, \qquad I(t) = \int^t \sqrt{\cosh\tau}\, d\tau, \qquad c > 0, \qquad e^{4m} = a^2.$$

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