

Stationary rotating surfaces in Euclidean space

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Abstract A stationary rotating surface is a compact surface in Euclidean space whose mean curvature H at each point x satisfies $2H(x) = ar(x)^2 + b$, where $r(x)$ denotes the distance from x to a fixed straight-line L , and a and b are constants. These surfaces are solutions of a variational problem that describes the shape of a drop of incompressible fluid in equilibrium by the action of surface tension when it rotates about L with constant angular velocity. The effect of gravity is neglected. In this paper we study the geometric configurations of such surfaces, focusing the relationship between the geometry of the surface and the one of its boundary. As special cases, we will consider two families of such surfaces: axisymmetric surfaces and embedded surfaces with planar boundary.

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1 Introduction

In the absence of gravity, we consider the steady rigid rotation of an homogeneous incompressible fluid drop which is surrounded by a rigidly rotating incompressible fluid. Our interest is the study of the shape of such drop when it attains a state of mechanical equilibrium. In such case, we will call it a rotating liquid drop, or simply, a rotating drop. Rotating drops have been the subject of intense study beginning from the work of Joseph Plateau [32]. Experimentally, he observed a variety of axisymmetric shapes that can summarized as follows: starting with zero angular velocity, we begin with a spherical shape. As we increase the angular velocity, the drop changes through a sequence of shapes which evolved from axisymmetric for slow rotation to ellipsoidal and two-lobed and finally toroidal at very large rotation. This was theoretically shown by Poincaré [33], Beer [7], Chandrasekhar [15] and Brown and Scriven [9, 10]. The experiments of Plateau inspired an scientific interest since they could be models in other areas of physics, such as, astrophysics, nuclear physics, fluid dynamics, amongst others. For example, they arise in celestial mechanics in the study of self-rotating stars and planets [14, 27, 39]. Experimentally, these figures have appeared in microgravity

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environments and experimental works in absence of gravity [29, 38]. From a theoretical viewpoint, there is a mathematical interest for rotating drops, focusing in subjects such as existence and stability. The literature is extensive, and without to be a complete list, we refer the reader to [2, 4, 5, 12, 16, 17, 24, 37]

Let us take usual coordinates (x_1, x_2, x_3) and assume the liquid drop and the surrounding fluid rotate about the x_3 -axis with a constant angular velocity ω . Let W be the bounded open set in \mathbb{R}^3 , which is the region occupied by the rotating drop. We set $S = \partial W$ as the free interface between the drop and the ambient liquid and that we suppose to be a smooth boundary surface. The energy of this mechanical system is given by

$$E = \tau|S| - \frac{1}{2}(\rho_1^2 - \rho_2)\omega^2 \int_W r^2 dx,$$

where τ stands for the surface tension on S , $|S|$ is the surface area of S and $r = r(x) = \sqrt{x_1^2 + x_2^2}$ is the distance from a point x to the x_3 -axis, the axis of rotation. Here ρ_1 and ρ_2 are the mass density of the liquid drop and its surrounding, respectively. The term $\tau|S|$ is the surface energy of the drop and $\frac{1}{2}(\rho_1 - \rho_2)\omega^2 \int_W r^2 dx$ is the potential energy associated with the centrifugal force. We assume that the volume V of the drop remains constant while rotates. If we assume that the liquid drop has more density than its surrounding, that is, $\rho_1 > \rho_2$, the physical setting corresponds with *heavy liquid drops*, whereas if $\rho_1 < \rho_2$, we have *air bubbles*.

We seek the shape of the liquid drop when the configuration is stationary, that is, the drop is a critical point of the energy for all volume preserving perturbations. The equilibrium is obtained by the balance between the capillary force that comes from the surface tension of S and the centrifugal force of the rotating liquid. In equilibrium, the interface S is governed by the Laplace equation

$$2\tau H(x) = \frac{1}{2}(\rho_1 - \rho_2)\omega^2 r(x)^2 + \lambda \quad x \in S,$$

where the mean curvature H is calculated with the inward direction. The constant λ results from the volume constraint. As consequence, the mean curvature H satisfies an equation of type

$$2H(x_1, x_2, x_3) = ar^2 + b,$$

where $a, b \in \mathbb{R}$. We say then that S is a *stationary rotating surface*.

In the case that the interface S is an embedded surface (no self-intersections), Wente showed that a rotating drop has a plane of symmetry perpendicular to the x_3 -axis and any line parallel to the axis and meeting the drop cuts it in a segment whose center lies on the plane of symmetry [42]. Moreover, the plane through the center of mass perpendicular to the axis of the rotation coincides with the plane of symmetry of the rotating liquid drop.

The first configurations considered in the literature are the axisymmetric shapes of rotating liquid drops, that is, surfaces of revolution with respect to the axis of rotation. In such case, the Laplace equation is a second order differential equation and a first integration can be done (see Section 4). The purpose of this paper is to present the study of rotating liquid drops in a general sense, assuming for example that the surface is not rotational or with possible non-empty boundary. The possible existence of non-axisymmetric configurations was suggested by Lichtenstein and Poincaré. See [34].

When the physical system does not move, that is, the angular velocity is zero, $\omega = 0$ (or $a = 0$ in the Laplace equation), the mean curvature of the surface is constant and we abbreviate by saying a cmc-surface. Although in this paper we discard this situation, the theory of stationary rotating surfaces share techniques and type of results with cmc-surfaces. Actually part of our work follows

the same scheme and methodology, although the Laplace equation in our setting is more difficult and the results are less definitive. This can clearly see in the case that the surface is embedded. In this sense, it is worthwhile saying two facts that makes different both settings:

1. There are stationary rotating surfaces with genus zero that are not embedded. In fact, there exist axisymmetric examples. See Figure 2 (right). However, the celebrated Hopf's theorem [22] asserts that round spheres are the only cmc-surfaces with genus zero.
2. There are toroidal rotating drops that are embedded. In the family of embedded closed cmc-surfaces, the only possibility is the round sphere (Alexandrov's theorem [3]).

Since it is rather difficult to consider the case in which the liquid does not form a surface of revolution and because the axisymmetric shapes are more suitable to study, a first question is whether an embedded closed stationary rotating surface must be a surface of revolution. As we have mentioned, Wente's theorem assures that there exists a plane of symmetry perpendicular to the axis of rotation. For this, he uses the so-called Alexandrov reflection method by horizontal planes. However, one cannot do a similar argument with vertical planes: in general, when one reflects the surface about these planes, it cannot compare the value of the mean curvature at the contact points, on the contrary what occurs for cmc-surfaces and capillary surfaces [3, 41].

Assume now that the boundary of the surface is a non-empty set. The simplest case to consider is that the boundary is a horizontal circle centered at the origin. A natural question is whether a compact rotating liquid drop in \mathbb{R}^3 bounded by a circle is necessarily a rotational surface. More generally, one can consider the problem whether a stationary rotating surface inherits the symmetries of its boundary.

In Section 2, we formulate the Laplace equation and we derive the second variation of the energy. In Section 3, a set of integral formulae will be obtained relating quantities between the surface and its boundary. As a consequence, we show that the geometry of a given closed curve Γ imposes restrictions to the possible configurations of a rotating surface. For example, we show:

Let Γ be a circle of radius R in a horizontal plane and centered at the x_3 -axis. Then any stationary rotating surface spanning Γ and with mean curvature $2H = ar^2 + b$ satisfies $|aR^2 + 2b| \leq 4/R$.

Section 4 considers axisymmetric rotating surfaces and we derive some estimates of the profile curve, the area and the volume of the surface. In Sections 5 we are devoted to analyse configurations of rotating liquid drops with possible boundary. Using the Alexandrov reflection method we show the following result:

Consider M an embedded closed surface whose mean curvature is $2H = ar^2 + b$ computed with respect to the unit normal vector that points to the domain bounded by M . If $a < 0$, then M is a surface of revolution about the x_3 -axis.

This result asserts that rotating air bubbles have axisymmetric configurations. Finally, in Section 6 we address with the problem of stability, obtaining that rotating graphs are strong stable, in particular, they are stable.

2 Preliminaries

Let \mathbb{R}^3 be the Euclidean three-space and let (x_1, x_2, x_3) be the usual coordinates. Let M be an oriented (connected) compact surface and we shall denote by ∂M the boundary of M . Consider a

smooth immersion $x : M \rightarrow \mathbb{R}^3$ and let N be the Gauss map. Denote by $\{E_1, E_2, E_3\}$ the canonical orthonormal base in \mathbb{R}^3 :

$$E_1 = (1, 0, 0), \quad E_2 = (0, 1, 0), \quad E_3 = (0, 0, 1).$$

We write $N_i = \langle N, E_i \rangle$ and $x_i = \langle x, E_i \rangle$ for $1 \leq i \leq 3$. Given a constant $a \in \mathbb{R}$, we define the energy functional (associated to a) as

$$E(x) = \int_M dM - a \int_M r^2 x_3 N_3 dM, \quad r = \sqrt{x_1^2 + x_2^2}$$

where dM is the area element on M . Here $\int_M r^2 x_3 N_3 dM$ represents the centrifugal force of the surface with respect to the x_3 -axis. Consider a smooth variation X of x , that is, a smooth map $X : (-\epsilon, \epsilon) \times M : M \rightarrow \mathbb{R}^3$ such that, by setting $x_t = X(t, -)$, we have $x_0 = x$ and $x_t - x \in C_0^\infty(M)$. Let $u \in C^\infty(M)$ be the normal component of the variational vector field of x_t ,

$$u = \left\langle \frac{\partial x_t}{\partial t} \Big|_{t=0}, N \right\rangle.$$

We take $E(t) := E(x_t)$ the value of the energy for each immersion x_t . The first variation of E at $t = 0$ is given by

$$E'(0) = - \int_M (2H - ar^2)u dM.$$

Here H is the mean curvature of the immersion x . We require that the volume $V(t)$ of each immersion x_t remains constant throughout the variation. The first variation of the volume functional is

$$V'(0) = \int_M u dM.$$

By the method of Lagrange multipliers, the first variation of E at $t = 0$ is to be zero relative for all volume preserving variations if there is a constant b so that $E'(0) + bV'(0) = 0$. This yields the condition (see [42] for details):

$$2H(x) = ar^2 + b, \quad a, b \in \mathbb{R}. \quad (1)$$

Definition 1 Let L be a straight-line of \mathbb{R}^3 . A stationary rotating surface (with respect to L) is an oriented surface M immersed in \mathbb{R}^3 such that the mean curvature of the immersion satisfies the Laplace equation (1), where $r = \text{dist}(x, L)$. If the surface M is embedded and compact, we also say that M is a rotating liquid drop, and we identify M with its image $x(M)$.

In all this article, our surfaces M are *compact* with possible boundary, which will denoted by ∂M . In the case that $\partial M = \emptyset$, we say that the surface is closed.

Remark 1 Throughout this work, we suppose that the straight-line L is the x_3 -axis. Thus, $r^2 = x_1^2 + x_2^2$. Moreover, we shall use the words "horizontal" and "vertical" with respect to L , that is, by "horizontal" we mean orthogonal to L and by "vertical", we mean parallel to L .

We need to precise the definition of the boundary of an immersion $x : M \rightarrow \mathbb{R}^3$. Let Γ be a closed curve in \mathbb{R}^3 . We say the Γ is the boundary of x if $x|_{\partial M} : \partial M \rightarrow \mathbb{R}^3$ is a diffeorphism. We abbreviate saying that Γ is the boundary of M .

Given the definition of a stationary rotating surface, we derive the second variation of the energy of critical points in order to give the notion of stability. A general formula of the second variation was obtained by Wente [40] (see also [31]). We give a different method for this computation following ideas of Koiso and Palmer [26]. Assume that x is a critical point of E and we calculate the second variation of the functional E . For this, we write $E''(0)$ in the form

$$E''(0) = - \int_M u \cdot L[u] dM,$$

where L is a linear differential operator acting on the normal component, which we want to find it. The computation of the operator L is as follows. Since the translations in the E_3 are symmetries of the energy functional E , then $L[N_3] = 0$. Moreover, the rotation with respect to the x_3 -axis is a symmetry of the functional and then $L[\phi] = 0$, where $\phi = \langle x \wedge N, E_3 \rangle$ and \wedge is the vector product of \mathbb{R}^3 . We now compute $L[N_i]$ and $L[\phi]$. The tension field of the Gauss map satisfies

$$\Delta N + |\sigma|^2 N = -2\nabla H,$$

where Δ is the Laplacian in the metric induced by x , σ is the second fundamental of the immersion and ∇ is the gradient operator. Since $2H = ar^2 + b$, we have

$$2\nabla H = 2a(x_1 \nabla x_1 + x_2 \nabla x_2) = 2a \left(x_1 E_1 + x_2 E_2 - (f - x_3 N_3) N \right). \quad (2)$$

Here $f = \langle N, x \rangle$ stands for the support function of M . Thus

$$\Delta N + \left(|\sigma|^2 - 2a(f - x_3 N_3) \right) N = -2a(x_1 E_1 + x_2 E_2). \quad (3)$$

In particular,

$$\Delta N_3 + \left(|\sigma|^2 - 2a(f - x_3 N_3) \right) N_3 = 0 \quad (4)$$

We now take the function ϕ . The following formula holds for any immersion:

$$\Delta \phi + |\sigma|^2 \phi = -2 \langle \nabla H, E_3 \wedge x \rangle.$$

It follows from (2) that

$$-2 \langle \nabla H, E_3 \wedge x \rangle = 2a(f - x_3 N_3) \langle N, E_3 \wedge x \rangle = 2a(f - x_3 N_3) \phi.$$

Therefore,

$$\Delta \phi + \left(|\sigma|^2 - 2a(f - x_3 N_3) \right) \phi = 0. \quad (5)$$

As a consequence of (4) and (5),

$$L = \Delta + |\sigma|^2 - 2a(f - x_3 N_3) = \Delta + |\sigma|^2 - 2a(x_1 N_1 + x_2 N_2).$$

Definition 2 Consider $x : M \rightarrow \mathbb{R}^3$ be a smooth immersion of a compact surface M that satisfies the Laplace equation (1). We say that x is strongly stable if

$$-\int_M u \left(\Delta u + (|\sigma|^2 - 2a(x_1 N_1 + x_2 N_2))u \right) dM \geq 0 \quad (6)$$

for all $u \in C_0^\infty(M)$.

Remark 2 In the literature there are different notions about the stability of rotating surfaces according to the physical problem that lies behind the mathematical formulation. This does that the admissible deformations of the surface changes with each problem (see [28]). Of course, the constraint volume is always assumed, which means that the functions u used in (6) satisfy the condition

$$\int_M u dM = 0. \quad (7)$$

Some authors consider variations with extra assumptions. For example, in [11,35] it is assumed that the center of mass of any surface of the variation lies in the x_3 -axis. In such case, the functions $u \in C_0^\infty(M)$ in the above Definition must satisfy (7) together the extra conditions $\int_M x_1 u dM = \int_M x_2 u dM = 0$. On the other hand, in the known article of Brown and Scriven [9], the authors study the stability for axisymmetric liquid drops, assuming axisymmetric variations of the surface. Anyway it is clear that strong stability implies stability.

3 Integral formulae for stationary rotating surfaces with boundary

In the context of surfaces with prescribed mean curvature, it appears in a natural way the problem of existence that in our setting of stationary rotating surfaces can be formulated as follows:

Problem. Let Γ be a closed curve in Euclidean space and $a, b \in \mathbb{R}$. Does exist a compact stationary rotating surface spanning Γ and with mean curvature $2H = ar^2 + b$?

In order to understand the question, in this section we develop a series of identities relating integrals on a rotating surface and integrals on its boundary, that will help to clarify the problem.

We define the vector valued 1-form $\omega_p(v) = x(p) \wedge v$, $p \in M$, $v \in T_p M$, being $T_p M$ the tangent plane at p . Then $d\omega = 2N$, and the Stokes formula gives

$$2 \int_M N dM = \int_{\partial M} x \wedge \alpha' ds, \quad (8)$$

where ds is the length-arc element and α is a parametrization by the length-arc of ∂M that orients ∂M by the induced orientation from M . Now, consider $\mu_i = x_i^2 \omega$, $1 \leq i \leq 3$. A straightforward computation leads

$$d\mu_i = 4x_i^2 N - 2fx_i E_i, \quad f := \langle N, x \rangle.$$

By integrating on M , we deduce

$$4 \int_M x_i^2 N dM - 2 \int_M fx_i E_i dM = \int_{\partial M} x_i^2 x \wedge \alpha' ds. \quad (9)$$

Finally we define the 1-form $\beta_p(v) = N(p) \wedge v$. Then $d\beta = -2HN$ and the Stokes formula yields

$$a \int_M r^2 N dM + b \int_M N dM = - \int_{\partial M} \nu ds, \quad (10)$$

where ν the inward pointing conormal vector to M along ∂M . Equation (10) can also be obtained by considering the equation

$$\Delta x = 2HN = (ar^2 + b)N, \quad (11)$$

where the first equality holds for any isometric immersion x . The divergence theorem yields (10) again.

A first consequence of the above formulae is that the center of mass of a rotating liquid drop lies in the x_3 -axis. This result was showed by Smith and Ross ([37]). For completeness sake we include the proof.

Theorem 1 *Assume that M is an embedded closed stationary rotating surface. Then the center of mass of the surface lies at the x_3 -axis.*

Proof Let $W \subset \mathbb{R}^3$ denote the enclosed domain by M , which do exist because M is a embedded closed surface. If Δ_0 denotes the Euclidean Laplacian operator, then $\Delta_0 x_i^3 = 6x_i$, $1 \leq i \leq 3$. The divergence theorem asserts

$$2 \int_W x_i dV = \int_M x_i^2 N_i dM, \quad 1 \leq i \leq 3, \quad (12)$$

where dV is the volume element of \mathbb{R}^3 and N is the orientation that points towards W . We multiply the equations (9) and (10) by E_j obtaining for $i, j \in \{1, 2\}$

$$2 \int_M x_i^2 N_j dM = \delta_{ij} \int_M f x_i dM, \quad \text{and} \quad \int_M r^2 N_i dM = 0.$$

As a consequence,

$$\int_M x_i^2 N_j dM = 0$$

for all $i, j \in \{1, 2\}$. Using the above equation together (12), we conclude

$$\int_W x_i dV = 0, \quad i = 1, 2,$$

which means that the first two coordinates of the center of mass of W vanish. This proves the theorem. \square

We extend the above result for rotating liquid drops orthogonally deposited on a horizontal plane.

Theorem 2 *Let M be an embedded compact stationary rotating surface. Assume that ∂M is contained in a horizontal plane P . If M lies in one side of P and M is orthogonal to P along ∂M , then the center of mass of the surface lies at the x_3 -axis*

Proof As ∂M is a planar curve and M is orthogonal to P , then $\langle x \wedge \alpha', E_i \rangle = 0$ and $\langle \nu, E_i \rangle = 0$, respectively, for $i = 1, 2$. Multiplying by E_i in (9) and (10), we obtain for each $i, j \in \{1, 2\}$

$$\begin{aligned} \int_M N_j dM &= 0, \\ 2 \int_M x_i^2 N_j dM - \delta_{ij} \int_M f x_i dM &= 0 \\ a \int_M r^2 N_j dM + b \int_M N_j dM - \int_{\partial M} \nu_j &= 0, \end{aligned}$$

It follows of the above three equations that

$$\int_M x_i^2 N_i dM = 0, \quad i = 1, 2. \quad (13)$$

The calculation of the center of mass of the surface follows the same steps as in Theorem 1. One begins by considering the closed surface $M \cup \Omega$, where $\Omega \subset P$ is the bounded domain by ∂M and let $W \subset \mathbb{R}^3$ be the bounded domain by $M \cup \Omega$. If η_Ω is the induced orientation on Ω , we use (13) and the fact that $\langle \eta_\Omega, E_i \rangle = 0$, $i = 1, 2$, to conclude

$$2 \int_W x_i dV = \int_M x_i^2 N_i dM + \int_\Omega x_i^2 \langle \eta_\Omega, E_i \rangle d\Omega = \int_M x_i^2 N_i dM = 0, \quad i = 1, 2.$$

The result now follows. \square

In the next theorem we obtain an integral formula where all integrals are evaluated on ∂M .

Theorem 3 (Flux formula) *Let M be a stationary rotating surface with non-empty boundary. Then*

$$\int_{\partial M} (ar^2 + 2b) \langle x \wedge \alpha', E_3 \rangle ds = -4 \int_{\partial M} \nu_3 ds. \quad (14)$$

In particular, we have

$$a \int_{\partial M} r^2 ds \leq 4 \left(L(\partial M) - b a(\partial M) \right), \quad (15)$$

where $L(\partial M)$ is the length of ∂M and $a(\partial M)$ is the algebraic area of ∂M .

Proof Multiplying by E_3 in (8) and (9), we know that

$$2 \int_M N_3 dM = \int_{\partial M} \langle x \wedge \alpha', E_3 \rangle ds. \quad (16)$$

$$4 \int_M x_i^2 N_3 dM = \int_{\partial M} x_i^2 \langle x \wedge \alpha', E_3 \rangle ds \quad i = 1, 2. \quad (17)$$

Then

$$4 \int_M r^2 N_3 dM = \int_{\partial M} r^2 \langle x \wedge \alpha', E_3 \rangle ds. \quad (18)$$

From (10) and (16), we obtain

$$a \int_M r^2 N_3 dM + \frac{b}{2} \int_{\partial M} \langle x \wedge \alpha', E_3 \rangle ds = - \int_{\partial M} \nu_3 ds.$$

By combination this equation with (18), we conclude (14), and this completes the proof. \square

Equation (14) (or (15)) can be viewed as a necessary condition for the existence of a stationary rotating surface with prescribed curve and mean curvature $2H = ar^2 + b$. In fact, in the simplest case of Γ , that is, a horizontal circle of radius R , we have

Corollary 1 *Let Γ be a horizontal circle of radius R centered at the x_3 -axis. If M is a stationary rotating surface bounded by Γ then*

$$|aR^2 + 2b| \leq \frac{4}{R}. \quad (19)$$

Proof In the flux formula (14) we do $\langle \nu, E_3 \rangle \leq 1$ and we use that $r|_{\partial M} = R$ is a constant function. \square

This corollary has the same flavor than a classical result due to Heinz [21], which asserts that a necessary condition for the existence of a compact surface with constant mean curvature H bounded by a circle of radius R is that $|H| \leq 1/R$. In our context, if H is constant, then $a = 0$ and $2H = b$. Then (19) reads as $|2b| = |4H| \leq 4/R$, rediscovering the Heinz's result.

This section finishes with an application of the flux formula in order to derive a height estimate for a stationary rotating graph.

Theorem 4 *Let M be a stationary rotating surface that is a graph on a horizontal plane P and $\partial M \subset P$. Denote $R = \max_{x \in \partial M} r(x)$. Assume that the mean curvature is $2H(x) = ar^2 + b$, where $a \neq 0$ and $ab \geq 0$. If $h = \max_{x \in M} \text{dist}(x, P)$, then*

$$h \leq \frac{|aR^2 + 2b|}{8\pi} \text{area}(M). \quad (20)$$

Proof After a vertical displacement, we assume that P is the plane $x_3 = 0$ and that $M = \text{graph}(u)$, where u is a smooth function on a domain $\Omega \subset P$. Because $a \neq 0$ and $ab \geq 0$, the mean curvature satisfies $H \geq 0$ on M or $H \leq 0$ on M . The maximum principle implies that M lies in one side of P . If it is necessary, after a reflection about P , we can suppose that M lies in the upper half-space determined by P and that the orientation points downwards (this implies $H \geq 0$ and $a, b \geq 0$). Let us introduce the following notation:

$$P(t) = \{x \in \mathbb{R}^3; x_3 = t\} \quad M(t) = \{x \in M; x_3 \geq t\}, \quad \Gamma(t) = M(t) \cap P(t),$$

and $A(t)$ and $L(t)$ denote the area and length of $M(t)$ and $\Gamma(t)$ respectively. Let $\Omega(t)$ be the planar domain of $P(t)$ bounded by $\Gamma(t)$. We apply the flux formula (14) for each surface $M(t)$ obtaining

$$\int_{\Gamma(t)} \nu_3^t ds_t \leq \frac{1}{4} \int_{\Gamma(t)} (ar^2 + 2b) \langle x \wedge \alpha', E_3 \rangle ds_t \leq \frac{1}{2} |\Omega(t)| (aR^2 + 2b), \quad (21)$$

where ν^t is the inward pointing conormal vector to $M(t)$ along $\Gamma(t)$, ds_t is the induced length arc of $\Gamma(t)$ and $|\Omega(t)|$ is the area of $\Omega(t)$. Using the Hölder inequality, the coarea formula, the isoperimetric inequality and (21), we derive

$$\begin{aligned} 4\pi |\Omega(t)| &\leq L(t)^2 = \left(\int_{\Gamma(t)} 1 ds_t \right)^2 \leq \int_{\Gamma(t)} \frac{1}{|\nabla u|} ds_t \int_{\Gamma(t)} |\nabla u| ds_t \\ &= -A'(t) \int_{\Gamma(t)} \nu_3^t ds_t \leq -\frac{1}{2} A'(t) |\Omega(t)| (aR^2 + 2b). \end{aligned}$$

Let us point out that along $\Gamma(t)$, $|\nabla u| = |\nu_3^t| = \nu_3^t \geq 0$. It follows that

$$\frac{8\pi}{aR^2 + 2b} \leq -A'(t).$$

Hence we may infer the estimate (20) by an integration between $t = 0$ and $t = h$. \square

Corollary 2 *Let M be a closed rotating liquid drop with mean curvature $2H(x) = ar^2 + b$. If $a \neq 0$ and $ab \geq 0$, then the diameter of M along the x_3 -direction, that is, $h = \max_{p,q \in M} |x_3(p) - x_3(q)|$, satisfies*

$$h \leq \frac{|aR^2 + 2b|}{8\pi} \text{area}(M), \quad (22)$$

where $R = \max_{x \in M} r(x)$.

Proof By the Wente's symmetry result, we know that there exists a horizontal plane P that is a plane of symmetry of M and each one of pieces of M in both sides of P is a graph on P . For each graph, we apply the inequality (20). \square

Remark 3 The above estimates (20) and (22) attain for cmc-surfaces. In this case, $a = 0$ and $2H = b$. The estimate (20) is an equality if M is a spherical graph bounded by a circle, whereas the inequality (22) is an equality if M is a round sphere, where h is the diameter of the sphere.

4 Estimates of axisymmetric configurations

In this section we address our attention into axisymmetric stationary rotating surfaces, that is, surfaces of revolution with respect to x_3 -axis. For the case of spheroidal rotating drops, we are going to obtain estimates of the size, as well as, the area and the volume. Part of this section follows ideas due to Finn in the theory of capillarity [18]. We write the generating curve α of a surface of revolution as the graph of a function $u = u(r)$ and we parametrize the surface as $x(r, \theta) = (r \cos \theta, r \sin \theta, u(r))$, $r \in [0, c]$, $\theta \in \mathbb{R}$. The Gauss map is

$$N(r, \theta) = \frac{1}{\sqrt{1 + u'(r)^2}} (-u'(r) \cos \theta, -u'(r) \sin \theta, 1).$$

With respect to this orientation, the mean curvature equation (1) takes the form

$$\frac{u''}{(1 + u'^2)^{3/2}} + \frac{1}{r} \frac{u'}{\sqrt{1 + u'^2}} = ar^2 + b. \quad (23)$$

or

$$\left(\frac{ru'}{\sqrt{1 + u'^2}} \right)' = r(ar^2 + b).$$

A first integration yields

$$v(r) := \frac{u'}{\sqrt{1 + u'^2}} = \frac{1}{4}r(ar^2 + 2b) + \frac{d}{r}.$$

The integration constant d describes the shape of the surface as follows. If $d \neq 0$, the solutions correspond with toroidal shapes that do not intersect the x_3 -axis. This family of surfaces have been studied, for example, in [6, 19, 23, 34].

From now on we will restrict our analysis to the case $d = 0$. In such case, the solutions correspond to surfaces of revolution that are topologically a sphere (*spheroidal shapes*). Consider initial conditions

$$u(0) = u_0, \quad u'(0) = 0. \quad (24)$$

If it is necessary we indicate the dependence of the solutions with respect to the parameters as usually. Some properties of the solutions of (23)-(24) are the following:

1. The existence is a consequence of standard theory.
2. A solution u is symmetric with respect to $r = 0$, that is, $u(-r; u_0) = u(r; u_0)$.
3. The surface is invariant by vertical displacements, that is, if $\lambda \in \mathbb{R}$, then $u(r; u_0) + \lambda = u(r; u_0 + \lambda)$.
4. We have $-u(r; u_0, a, b) = u(r; -u_0, -a, -b)$.

By the last property, we will assume in this section that $b \geq 0$ (this does that the function u is increasing near to $r = 0^+$).

We describe the geometry of the corresponding axisymmetric surfaces. By the symmetry properties of the solutions of (23)-(24), it suffices to know the curve u in the maximal interval of definition $[0, c_0)$. As we will see later, $c_0 < \infty$, u cannot be continued beyond $r = c_0$ but it is bounded at $r = c_0$ with derivative unbounded at the same point. Thus, the whole surface is obtained by rotating α with respect to x_3 -axis and reflecting about the horizontal plane $x_3 = u(c_0)$.

By differentiation the function $v(r)$, we obtain

$$\kappa(r) := v(r)' = \frac{u''(r)}{(1 + u'(r)^2)^{3/2}} = \frac{1}{4}(3ar^2 + 2b). \quad (25)$$

Here κ stands for the curvature of the planar curve α . Three types of axisymmetric rotating closed drops appear and we show the generating curves in Figures 1 and 2. The graphs correspond with $u_0 = 0$ in (24). The solution u is indicated by a bold line in the maximal domain $[0, c_0)$. Recall that $b \geq 0$.

- I *Case* $ab \geq 0$, that is, $a > 0$. The function $v(r)$ is positive and this means that u is a strictly increasing function. Moreover, $\kappa(r) > 0$ and u is a convex function. The value c_0 of the maximal domain satisfies $v(c_0) = 1$. The surface is embedded. See Figure 1, left.
- II *Case* $ab < 0$, that is, $a < 0$. As $\kappa(0) = b/2 > 0$, the function u is increasing on r near to $r = 0^+$. The function $v(r)$ attains a maximum at $r_1 = \sqrt{\frac{-2b}{3a}}$ since $v''(r_1) < 0$. Thus u increasing until to reach the value $v(c_0) = 1$ if $v(r_1) \geq 1$. After some manipulations, this occurs iff $a \leq \frac{-2b^3}{27}$. We distinguish two subcases.
 - II (a) Let $a \geq \frac{-2b^3}{27}$. The function u is a strictly increasing function defined in $[0, c_0)$, with $v(c_0) = 1$. Moreover u is a convex function. The surface is embedded. See Figure 1, right.
 - II (b) Let $a < \frac{-2b^3}{27}$. The function u is increasing in the interval $[0, r_2)$, with $r_2 = \sqrt{\frac{-2b}{a}}$ and with an inflection at $r_1 = \sqrt{\frac{-2b}{3a}}$. Next, u decreases until c_0 , where $v(c_0) = -1$. See Figure 2, left. The graph of the generating curve has self-intersections iff $u_0 \leq u(c_0)$. See Figure 2, right.

We point out that for $r = c_0$, the value where $v(c_0) = \pm 1$, the function u is finite. This is due to the following ([23]):

Lemma 1 *Let $g \in C^2[c_0 - \epsilon, c_0]$ with $g(c_0) = \pm 1$ and $|g(r)| < 1$. Then the improper integral*

$$\left| \int_{c_0 - \epsilon}^{c_0} \frac{g(r)}{\sqrt{1 - g(r)^2}} dr \right|$$

is finite if and only if $g'(c_0) \neq 0$.

In our case, $g(r) = \frac{1}{4}r^2 + 2b$ and $g'(c_0) = \frac{1}{4}(3c_0^2 + 2b) \neq 0$. As conclusion,

$$u(c_0) = \int_{c_0 - \epsilon}^{c_0} u'(r) dr = \int_{c_0 - \epsilon}^{c_0} \frac{g(r)}{\sqrt{1 - g(r)^2}} dr < \infty.$$

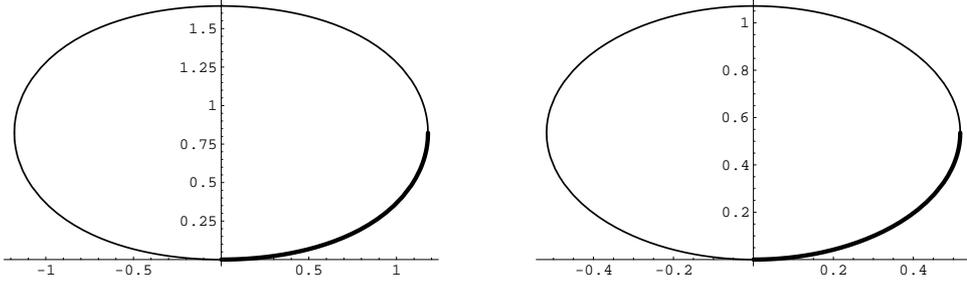


Fig. 1 (left) Surface of type I, for $a = 1$ and $b = 1$; (right) Surfaces of type II (a), for $a = -1$ and $b = 4$.

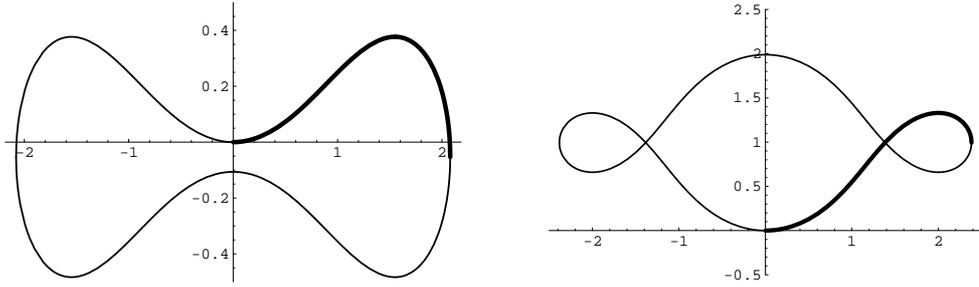


Fig. 2 (left) Surface of type II (b), for $a = -1$ and $b = 1.2$; (right) Surfaces of type II (b), for $a = -1$ and $b = 2$: non-embedded case.

After the description of all rotating drops with spheroidal shapes, we now compare these surfaces with appropriate spheres. Suppose that u is solution of (23)-(24) and that $I = [0, c]$ is an interval where u is defined. Consider the sphere obtained by rotating about the x_3 -axis a piece of circle $y = y(r)$ with the same slope than u at $r = c$ and that coincides with u at the origin. This circle is given by

$$y(r) = R + u_0 - \sqrt{R^2 - r^2}, \quad R = c \frac{\sqrt{1 + u'(c)^2}}{u'(c)}.$$

The graph of y is a piece of a lower halfcircle with $y(0) = u_0$ and $y'(0) = 0$. The choice of the radius R is such that $y'(c) = u'(c)$. Thus $y(r)$ is a solution of (23)-(24) for $a = 0$ and $b = 2/R$.

Theorem 5 *Let u be a solution of (23)-(24) defined in the interval $[0, c]$. Suppose that $a > 0$, $b \geq 0$. Then*

$$u(r) < y(r), \quad 0 < r \leq c.$$

Proof Let $\psi(r)$ be the angle that makes the graph of u with the r -axis at each point r , that is $\tan \psi(r) = u'(r)$. By the definition of the function $v(r)$,

$$\sin \psi(r) = \frac{u'(r)}{\sqrt{1 + u'(r)^2}} = \frac{1}{4} r(ar^2 + 2b).$$

Then $\sin \psi(r)$ is positive at $(0, c]$. In particular, $u'(c) \neq 0$ and the radius R is well-defined. This means that u is a strictly increasing function on r . The curvature κ of α is also an increasing

function on r since $\kappa'(r) = \frac{3}{2}ar > 0$. The angle $\psi^y(r)$ and the curvature κ^y of the graph of $y(r)$ are respectively

$$\sin \psi^y(r) = \frac{r}{R}, \quad \kappa^y(r) = \frac{1}{R}.$$

We compare the curves u and y at the point $r = 0$. First we have $\kappa(0) < \kappa^y(0)$ since this inequality is equivalent to

$$\frac{b}{2} < \frac{1}{R} = \frac{\sin \psi(c)}{c} = \frac{ac^2 + 2b}{4}, \quad (26)$$

which is trivial. As $\kappa(0) < \kappa^y(0)$, $y(0) = u(0)$ and $y'(0) = u'(0)$, the graph of y lies above of u around the point $r = 0$. Theorem 5 claims that this occurs in all the domain $(0, c]$. Suppose, by way of contradiction, that the graph of u crosses the graph of y at some point. Let $r = \delta \leq c$ the first value where this occurs, that is, $u(r) < y(r)$ for $r \in (0, \delta)$ and $u(\delta) = y(\delta)$. Then $u'(\delta) \geq y'(\delta)$ and so, $\sin \psi(\delta) \geq \sin \psi^y(\delta)$. As $u'(0) = y'(0)$, we have

$$\int_0^\delta \left(\kappa(t) - \kappa^y(t) \right) dt = \int_0^\delta \left((\sin \psi(t))' - (\sin \psi^y(t))' \right) dt = \sin \psi(\delta) - \sin \psi^y(\delta) \geq 0. \quad (27)$$

On the other hand, as $\kappa(0) < \kappa^y(0)$ and the above integral is non-negative, the integrand in (27) is positive at some point. Then there exists $\bar{r} \in (0, \delta)$ such that $\kappa(\bar{r}) > \kappa^y(\bar{r})$. Because κ is increasing on r , we have for $r \in [\bar{r}, c]$

$$\kappa(r) > \kappa(\bar{r}) > \kappa^y(\bar{r}) = \kappa^y(r).$$

Since $\bar{r} \leq \delta \leq c$, we have

$$\begin{aligned} 0 &< \int_{\bar{r}}^c \left(\kappa(t) - \kappa^y(t) \right) dt \leq \int_{\delta}^c \left(\kappa(t) - \kappa^y(t) \right) dt \\ &= \int_{\delta}^c \left((\sin \psi(t))' - (\sin \psi^y(t))' \right) dt = \sin \psi^y(\delta) - \sin \psi(\delta). \end{aligned}$$

This leads to a contradiction with (27) and we have proved the theorem. \square

For the next result, we lower the circle $y(r)$ vertically until it touches with the graph of u at $r = c$. We call $w = w(r)$ the new position of y , that is, $w(r) = y(r) - y(c) + u(c)$.

Theorem 6 *Let u be a solution of (23)-(24) defined in the interval $[0, c]$. Suppose that $a > 0$, $b \geq 0$. Then*

$$w(r) < u(r), \quad 0 \leq r < c.$$

Proof As in the preceding proof, we begin by comparing the curvatures of u and w at $r = c$. Exactly, we have

$$\kappa^w(c) = \frac{1}{R} = \frac{1}{4}(ac^2 + 2b) < \frac{1}{4}(3ac^2 + 2b) = \kappa(c).$$

As $\kappa(c) > \kappa^w(c)$, $w(c) = u(c)$ and $w'(c) = u'(c)$, the graph of u lies above than the circle w around $r = c^-$. Thus $w(r) < u(r)$ in some interval (δ, c) . The proof is by contradiction again and with a similar argument. We suppose that the graph of w crosses the graph of u at some point. Denote by δ the largest number such that $w(r) < u(r)$ for $r \in (\delta, c)$ and $w(\delta) = u(\delta)$. For this value, $w'(\delta) = y'(\delta) \leq u'(\delta)$ and $\sin \psi^y(\delta) \leq \sin \psi(\delta)$. Then

$$\int_{\delta}^c \left(\kappa(t) - \kappa^w(t) \right) dt = \int_{\delta}^c \left((\sin \psi(t))' - (\sin \psi^y(t))' \right) dt = \sin \psi^y(\delta) - \sin \psi(\delta) \leq 0. \quad (28)$$

Here we have used that $u'(c) = w'(c) = y'(c)$. As $\kappa(c) - \kappa^w(c) > 0$ and the integral in (28) is non-positive, then there would be $\bar{r} \in (\delta, c)$ such that $\kappa(\bar{r}) < \kappa^w(\bar{r})$. Because κ is an increasing function on r , for any $r \in [0, \bar{r}]$ we would have

$$\kappa(r) < \kappa(\bar{r}) < \kappa^w(\bar{r}) = \kappa^w(r).$$

Since $\delta < \bar{r}$ and using that $u'(0) = y'(0)$, we obtain

$$0 > \int_0^\delta \left(\kappa(t) - \kappa^w(t) \right) dt = \int_0^\delta \left((\sin \psi(t))' - (\sin \psi^y(t))' \right) dt = \sin \psi(\delta) - \sin \psi^y(\delta),$$

which contradicts the inequality (28) and the theorem is proved. \square

As conclusion of the two previous theorems, the solution u of (23)-(24) lies between two pieces of circles, namely, y and w , such that the slopes of the three functions agree at the points $r = 0$ and $r = c$ and the graph of u coincides with y and w at $r = 0$ and $r = c$, respectively. See Figure 3.

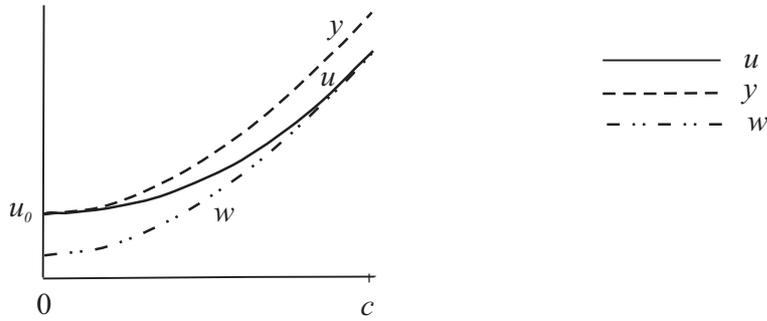


Fig. 3 The solution u lies sandwiched between the circles y and w .

Finally we point out that, with appropriate modifications, the conclusions of both theorems hold even if u is defined in the maximal interval $[0, c_0)$.

Corollary 3 Suppose that $a > 0$, $b \geq 0$. Let u be a solution of (23)-(24) defined in the interval $[0, c_0)$, where c_0 is the unique positive root of $x(ax^2 + 2b) - 4 = 0$. Then

$$u(r) - u_0 < c_0 - \sqrt{c_0^2 - r^2},$$

An easy exercise shows that if $a > 0$, $b \geq 0$, the equation $x(ax^2 + 2b) - 4 = 0$ has a unique positive root.

Corollary 4 Let M be an axisymmetric rotating liquid closed drop. Suppose that $2H(x) = ar^2 + b$, where $a > 0$, $b \geq 0$. If c_0 is the unique positive root of $x(ax^2 + 2b) - 4 = 0$, then the volume of the liquid drop is less than the volume of a sphere of radius c_0 .

Proof With the above notation, it is sufficient to point out that the volume(u) < volume(w) and that $R = c_0$. \square

This estimate can also be obtained as follows. After an integration by parts, the volume of the drop is

$$V = 2\pi c_0^2 u(c_0) - 4\pi \int_0^{c_0} r u(r) dr = 2\pi \int_0^{c_0} r^2 u'(r) dr.$$

From the expression of $\sin \psi(r)$ and since $ar^2 + 2b \leq ac_0^2 + 2b$, we have

$$u'(r) = \frac{r}{\sqrt{\left(\frac{4}{ar^2+2b}\right)^2 - r^2}} < \frac{r}{\sqrt{\left(\frac{4}{ac_0^2+2b}\right)^2 - r^2}} := \eta(r). \quad (29)$$

An explicit integration of $2\pi \int_0^{c_0} r^2 \eta(r) dr$ and using the fact $ac_0^3 + 2bc_0 = 4$, we conclude that $V < \frac{4\pi}{3} c_0^3$.

The next theorem establishes bounds for the height and the area of a rotating surface.

Theorem 7 *Let M be an axisymmetric rotating surface of type I with $a > 0$, $b \geq 0$ given by a solution u of (23)-(24) and defined in the interval $[0, c]$. Then*

$$\frac{1}{b} \left(2 - \sqrt{4 - b^2 r^2} \right) \leq u(r) - u_0 \leq \frac{4 - \sqrt{16 - r^2 (ar^2 + 2b)^2}}{ar^2 + 2b}, \quad r \in (0, c]. \quad (30)$$

If $A(c)$ denotes the area of M , we have

$$\frac{4\pi}{b^2} \left(2 - \sqrt{4 - b^2 c^2} \right) \leq A(c) \leq \frac{8\pi(4 - \sqrt{16 - c^2 (ac^2 + 2b)^2})}{(ac^2 + 2b)^2}. \quad (31)$$

Proof We have the following expressions for $u(r)$ and $A(c)$:

$$u(r) - u_0 = \int_0^r u'(t) dt, \quad A(c) = 2\pi \int_0^c r \sqrt{1 + u'(r)^2} dr.$$

The estimates (30) are obtained by appropriate bounds for the derivative $u'(r)$. The upper bound or u' is given in (29). The lower bound is obtained from the first equality in (29): since $ar^2 + 2b \geq 2b$, it follows that

$$u'(r) \geq \frac{rb}{\sqrt{4 - b^2 r^2}}.$$

The estimates follow now by simple integrations. We remark that the inequality in the right hand-side of (30) is also a consequence of Theorem 5. \square

Remark 4 In the case that the surface M is closed, Theorem 7 says

$$u(c_0) \leq u_0 + c_0, \quad \text{area}(M) < 2A(c_0) = 4\pi c_0^2,$$

where c_0 is the unique positive root of $x(ax^2 + 2b) - 4 = 0$. Here $u(c_0) - u_0$ measures the half of the difference of heights between the highest and the lowest points of M . In particular, the area of M is less than the area of the sphere of radius c_0 : this sphere is given by the function w , which contain M in its interior, see Figure 3.

Remark 5 If H is constant, then $a = 0$ and $b = 2H$ and u describes a spherical cap of radius $2/|b|$. The estimates (30) are now identities.

We employ Theorem 4 in the axisymmetric case.

Corollary 5 *Under the same hypothesis and notation as in Theorem 7, we have*

$$u(r) - u_0 \leq \frac{ar^2 + 2b}{8\pi} A(r) < \frac{4 - \sqrt{16 - r^2(ar^2 + 2b)^2}}{ar^2 + 2b}.$$

Proof If the function u is defined in the interval $[0, r]$, the boundary ∂M of M is given by the level $x_3 = u(r)$. With the notation of Theorem 4, $R = r$ and $h = u(r) - u_0$, which gives the first inequality. The second one is a consequence of (31). \square

For cmc-graphs, a classical result due to Serrin [36] asserts that if $M = \text{graph}(u)$ with boundary in a plane $P = \{x_3 = 0\}$, then $|u(x)| \leq 1/|H|$. For stationary rotating graphs, we generalize this estimate as follows:

Theorem 8 *Let u be solution of (23)-(24) with $a, b > 0$. Then*

$$u(r) \leq \frac{b}{ar^2 + b} \left(u_0 + \frac{2}{b} \right). \quad (32)$$

Proof Let M be the corresponding axisymmetric surface generated by u . From (4) and (11), we have

$$\begin{aligned} \Delta(Hx_3 + N_3) &= (2H^2 - |\sigma|^2)N_3 + 2a(f - x_3N_3)N_3 \leq 2a(f - x_3N_3)N_3 \\ &= -\frac{2atu'(t)}{1 + u'(t)^2} \leq 0, \quad t \geq 0, \end{aligned}$$

where we have used the parametrization of M as surface of revolution, the fact that $2H^2 - |\sigma|^2 \leq 0$ everywhere and that a and $u'(t)$ are positive. Consider u defined on $[0, r]$ and thus, ∂M is given by the level $u = u(r)$. Then the maximum principle yields

$$Hx_3 + N_3 \geq \min_{\partial M}(Hx_3 + N_3) \geq \min_{\partial M}(Hx_3) = H(r)u(r).$$

Hence that for each $t \in [0, r]$ it follows

$$u(t) \geq \frac{H(r)}{H(t)}u(r) - \frac{N_3(t)}{H(t)} \geq \frac{H(r)}{H(t)}u(r) - \frac{1}{H(t)}.$$

If we choose $t = 0$ then $u(0) = u_0$ and $H(0) = b/2$ and from above inequality we derive (32). \square

We end this section with a new application of the flux formula (14). Regarding the axisymmetric case, there exist special situations about the behaviour of the surface M with respect to the plane P containing ∂M . For example, for the surfaces of type I, the surface M is orthogonal to P along ∂M if the radius of ∂M is $R = c_0$ with $v(c_0) = 1$. The same occurs for surfaces of type II (a). If the surface is of type II (b), the surface is orthogonal to P if $R = c_0$ with $v(c_0) = -1$. Analogously, M is tangent to P if the boundary is a circle of radius R , with $aR^2 + 2b = 0$ (only for surfaces of type II). We prove that this can generalize for any non necessarily axisymmetric stationary rotating surface.

Corollary 6 *Let M be a stationary rotating surface bounded by a horizontal circle of radius R centered at the x_3 -axis. Denote by P the plane containing the boundary. Let $2H(x) = ar^2 + b$ be the mean curvature of M .*

1. *Suppose that M is orthogonal to P along ∂M . Then the radius R satisfies*

- (a) $R(aR^2 + 2b) = 4$ if the boundary is running according to the counterclockwise direction.
 (b) $R(aR^2 + 2b) = -4$ if the boundary is running according to the clockwise direction.

2. Suppose that M is tangent to P along ∂M . Then $R = \sqrt{\frac{-2b}{a}}$. In particular $ab < 0$.

Proof We remark that the given mean curvature H orients the surface M and thus, its boundary ∂M with the induced orientation, which is the one considered in the hypothesis of the corollary. For the proof, it is sufficient to look (14). By distinguishing the fact that $\nu_3 = \pm 1$ or $\nu_3 = 0$, we are going establishing the statements of the corollary. \square

5 Rotating liquid drops with boundary

This section is devoted to study embedded stationary rotating surfaces with non-empty boundary. The main tools that we will use are the maximum principle, the so-called Alexandrov reflection method and the integral formulae of section 3. The reflection method was employed to prove a classical result due to Alexandrov [3] that asserts that round spheres are the only embedded closed cmc-surfaces in Euclidean space \mathbb{R}^3 . In the context of rotating liquid drops, the method was used again in the cited Wente's theorem [42]. In fact, our first result refers to closed rotating drops and it asserts that air bubbles are surfaces of revolution.

Theorem 9 *Let M be a closed embedded stationary rotating surface. Assume that with the choice of the Gauss map that points inside, the mean curvature is $2H(x) = ar^2 + b$. If $a < 0$, then M is an axisymmetric surface (with respect to the x_3 -axis). In particular, air bubbles have axisymmetric shapes.*

Proof We apply the Alexandrov reflection method by using reflection with respect to vertical planes. For completeness, and since we will use it in the rest of this section, we describe the process. See [42] for details. We prove that any plane containing the x_3 -axis is a plane of symmetry of M . Without loss of generality, we suppose that such plane is the plane $Q = \{x_2 = 0\}$. Denote by $W \subset \mathbb{R}^3$ the bounded domain by M and consider the orientation N that points inside, that is, towards W : this means that $a < 0$.

We introduce the following notation. Let $Q(t)$ be the 1-parameter family of translated copies of Q , where we choose the parameter t such that $Q(t) = \{x_2 = t\}$. Let $A_t^+ = \{x \in \mathbb{R}^3; x_2 \geq t\}$ and $A_t^- = \{x \in \mathbb{R}^3; x_2 \leq t\}$. Also, let $M_t^+ = A_t^+ \cap M$, $M_t^- = A_t^- \cap M$ and M_t^* the reflection of M_t^+ about the plane $Q(t)$. Because M is a compact surface, for t large, $Q(t)$ is disjoint from M . Now, if we approach M by $Q(t)$ by moving left $Q(t)$ (letting $t \searrow 0$), one gets a the first plane $Q(t_0)$, $t_0 > 0$, that reaches M , that is, $Q(t_0) \cap M \neq \emptyset$, but if $t > t_0$ then $Q(t) \cap M = \emptyset$. We point out that this occurs because the center of mass lies in the x_3 -axis, and so $M_0^+ \neq \emptyset$. Thus $Q(t_0)$ is tangent to M at some point and M is contained in one side of $Q(t_0)$: $M \subset A_{t_0}^-$. Decreasing t , let consider M_t^* . Since M is embedded, the reflected surface M_t^* lies inside W , at least, near t_0 : there exists at least a small $\epsilon_0 > 0$ such that $M_{(t_0-\epsilon)}^* \subset W$ for $0 < \epsilon < \epsilon_0$ and $M_{t_0-\epsilon}$ is a graph on $Q(t_0 - \epsilon)$.

Continuing the movement of $Q(t_0)$ by parallel translations toward $Q(0)$, we search the first time $t_1 < t_0$ where there is a point of tangential contact of M with the reflection of $M \cap (\cup_{t_1 \leq t \leq t_0} Q(t) \cap M)$ in $Q(t_1)$. Exactly, consider

$$t_1 = \inf\{t < t_0; M_a^* \subset W, a \in (t, t_0)\}.$$

We claim that $t_1 \leq 0$. On the contrary, we assume that $t_1 > 0$. The surfaces $M_{t_1}^-$ and $M_{t_1}^*$ are two surfaces tangent at p and because reflections invert normal vectors, the Gauss maps of both $M_{t_1}^*$

and $M_{t_1}^-$ agree at such point p . Then $M_{t_1}^*$ is over $M_{t_1}^-$ with respect to the direction indicated by $N(p)$. Set $q \in M_{t_1}^+$ the point whose reflection about $Q(t_1)$ is p . We have two cases:

1. If p is an interior point of both surfaces $M_{t_1}^-$ and $M_{t_1}^*$, then $H(q) \geq H(p)$, that is, $ar(q)^2 \geq ar(p)^2$. As $a < 0$, $r(q) \leq r(p)$. However this is not possible as we see now. Denote \mathcal{R}_t the reflection about $Q(t)$, that is, $\mathcal{R}_t(x_1, x_2, x_3) = (x_1, -x_2 + 2t, x_3)$. Then for any point (x_1, x_2, x_3) with $x_2 > t$, we have that $r(x_1, x_2, x_3) > r(\mathcal{R}_t(x_1, x_2, x_3))$ provided $t > 0$, since

$$r(x_1, x_2, x_3) > r(\mathcal{R}_t(x_1, x_2, x_3)) \Leftrightarrow x_2^2 > (-x_2 + 2t)^2 \Leftrightarrow t^2 < tx_2,$$

which is true. We apply this for $q = (x_1, x_2, x_3)$ and $p = \mathcal{R}_{t_1}(q)$.

2. If $p \in \partial M_{t_1}^- \cap \partial M_{t_1}^*$, then $q^* = q = p$. The maximum principle says that $H(x) \geq H(y)$ for $x \in M_{t_1}^*$, $y \in M_{t_1}^-$ near to p with $\pi(x) = \pi(y)$, being π the orthogonal projection onto the common tangent plane $T_p M_{t_1}^- = T_p M_{t_1}^*$. Then $ar(z)^2 \geq ar(y)^2$, with $x = z^*$, $z \in M_{t_1}^+$, that is, $r(z) \leq r(y)$, obtaining a contradiction again.

We observe that the Alexandrov reflection process works until that $Q(t)$ arrives at $t = 0$: if we would follow far away with negative values of t , we would not assure that $r(p) < r(q)$.

Therefore the number t_1 can not be positive. Moreover, this shows that M_0^* lies on the right to M_0^- . After proving that $t_1 \geq 0$, we now start with vertical planes coming from $t = -\infty$. With a similar reasoning, we show that $t_1 \geq 0$, obtaining $t_1 = 0$. Furthermore, the reflection of M_0^- about $Q(0)$ lies on the left to M_0^+ . By combining both facts, we conclude that $M_0^* = M_0^-$ and $Q(0) = Q$ is a plane of symmetry of M . \square

From the above proof, we obtain the version of Theorem 9 for the case that $\partial M \neq \emptyset$.

Corollary 7 *Let P a horizontal plane and let Q be a plane containing the x_3 -axis. Set $R = P \cap Q$. Consider $\Gamma \subset P$ a Jordan curve and denote by $\Omega \subset P$ the bounded domain by Γ . Assume that Γ is symmetric with respect to the reflections about Q and that R divides Γ into two pieces that are graphs on R . Let M be a rotating liquid drop bounded by Γ such that M lies in one side of P and that with the choice of the Gauss map that points inside of $M \cup \Omega$, the mean curvature $2H(x) = ar^2 + b$ satisfies $a < 0$. Then Q is a plane of symmetry of M . In the particular case that $\Gamma \subset P$ is a circle centered at that x_3 -axis, then M is an axisymmetric surface.*

Proof The proof is similar as above and we only point out the differences. Now the domain W is bounded by $M \cup \Omega$. Without loss of generality, we assume that $Q = Q(0)$ and we use the notation as in the previous theorem. We begin with the process of reflection until that we arrive to the time $t = t_1$ and we discuss about the contact point p . Again, we claim that $t_1 \leq 0$.

By contradiction, we assume that $t_1 > 0$. The difference happens in the discussion of cases about the contact point. The case 1 is impossible. The case 2 changes now assuming that $p \in (\partial M_{t_1}^- \cap \partial M_{t_1}^*) - \Gamma$. The reasoning is the same, proving that this can not happen. Before continuing with the reflection method, it is important to remark that it is not possible that $q \in M - \partial M$ and $p \in \partial M$ or that $q \in \partial M$ and $p \in M - \partial M$ since the surface lies in one side of P . As conclusion, the (only) new case that can occur is that $q \in \partial M$ and (necessary), $p \in \partial M$. However, the hypothesis about the boundary curve Γ ensures that the plane $Q(t_1)$ must be a plane of symmetry of Γ and then $t_1 = 0$: contradiction, since we are assuming that $t_1 > 0$. This contradiction shows that $t_1 \leq 0$ and we finish the proof as in the previous theorem. \square

This corollary can be extended for liquid bridges (liquid drops between two parallel plates) in a natural manner. We only present the case that the boundary is formed by two coaxial circles and that complements the Corollary 9.

Corollary 8 *Any rotating liquid bridge between two coaxial circular discs (respect to the x_3 -axis) has axisymmetric shape if $a < 0$, being $2H = ar^2 + b$ the mean curvature computed with the unit normal vector pointing inside.*

The next result gives sufficient conditions to assure that a rotating surface with boundary is a graph over a horizontal plane. As in [42], the result holds for embedded surfaces whose mean curvature H depends only on the x_1 and x_2 coordinates, and we state the theorem in this sense.

Theorem 10 *Let Γ be a Jordan curve contained in a horizontal plane $P = \{x_3 = d\}$ and $\Omega \subset P$ the corresponding bounded domain by Γ . Let M be an embedded compact surface with boundary Γ and suppose that M satisfies:*

1. *The mean curvature H depends only on the x_1, x_2 coordinates.*
2. *The surface M does not intersect the cylinder $\Omega \times (-\infty, d]$.*
3. *In a neighbourhood of $\partial M = \Gamma$ in M , the surface M is a graph above Ω .*

Then M is a graph on Ω . See Figure 4 (a).

Proof In this case, we use the Alexandrov reflection method with horizontal planes. Without loss of generality, we assume that P is the plane $x_3 = 0$ and we define the embedded surface $T = M \cup (\Gamma \times (-\infty, 0])$. This surface divides the ambient space \mathbb{R}^3 in two components. We denote by W the component that contains Ω and we orient M with the Gauss map N that points towards W .

Let $P(t)$ be the 1-parameter family of horizontal planes $P(t) = \{x_3 = t\}$. We consider an analogous notation as in the proof of Theorem 9. For big values of t , the plane $P(t)$ is disjoint from M . We approach M by $P(t)$ by moving down $P(t)$ (letting $t \searrow 0$), one gets a the first plane $P(t_0)$, $t_0 > 0$, that reaches M , that is, $P(t_0) \cap M \neq \emptyset$, and next we continue until the first time point of contact point of M_t^* with M .

We claim that $t_1 = 0$. On the contrary, that is, $t_1 > 0$, $M_{t_1}^*$ and $M_{t_1}^-$ are two surfaces with $\partial M_{t_1}^* \subset \partial M_{t_1}^- = \partial M_{t_1}^* \cup \Gamma$. Moreover, $M_{t_1}^*$ and $M_{t_1}^-$ touch at an interior point p or touch at a boundary point $p \in \partial M_{t_1}^*$. We remark that $p \notin \partial \Omega$ because the surface is a graph on Ω around $\partial \Omega$ and $M_a^* \subset W$ for $t_1 < a \leq t_0$. Anyway, $M_{t_1}^*$ and $M_{t_1}^-$ are one in a side of the other in a neighbourhood of p and because the mean curvature depends only the x_1 and x_2 , the mean curvatures of both surfaces agree at p . Moreover, the mean curvature of a point of $M_{t_1}^*$ agrees with the mean curvature of the point of $M_{t_1}^-$ where vertically projects. At last, one applies the maximum principle to infer that $M_{t_1}^* = M_{t_1}^-$ and $P(t_1)$ is a plane of symmetry of M . Because $\Gamma \subset M_{t_1}^- \setminus M_{t_1}^*$, we derive a contradiction.

As conclusion, $t_1 = 0$ and this means that we can go reflecting M_t^+ until to arrive at $t = 0$, maintaining the property that $M_t^* \subset W$ for all $t > 0$. The procedure shows that in each time $t > 0$, M_t^+ is a graph over $P(t)$, and so this implies that M is a graph on Ω . See Figure 4 \square

In the context of Theorem 10, we obtain a similar result as the one obtained by Wente in [42] for the case that the surface is bounded by two curves in parallel planes. This is motivated by the physical problem of a rotating drop of liquid trapped between two parallel plates in absence of gravity.

Corollary 9 *Let P_1 and P_2 be two horizontal planes. Consider $\Gamma_1 \cup \Gamma_2$ two Jordan curves, $\Gamma_i \subset P_i$, $i = 1, 2$, such that Γ_2 is the vertical translation of Γ_1 to the plane P_2 . Denote by Ω the bounded domain determined by Γ_1 in P_1 . Let M be an embedded compact surface with boundary $\Gamma_1 \cup \Gamma_2$ whose mean curvature depends only on the x_1, x_2 coordinates. Suppose that one of the following two conditions is satisfied:*

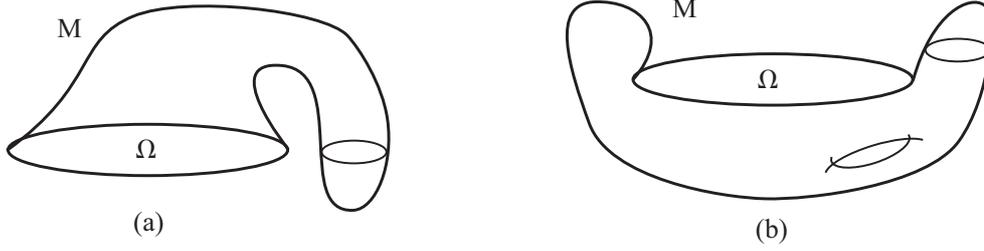


Fig. 4 Case (a) is impossible by Theorem 10; case (b) is impossible by Theorem 13.

1. M does not intersect the solid cylinder $\Omega \times \mathbb{R}$.
2. M is included in the solid cylinder $\Omega \times \mathbb{R}$.

Then the horizontal plane P equidistant from P_1 and P_2 is a plane of symmetry of M . Moreover, each one of the parts of M that lie in the two half-spaces determined by P is a graph on P .

Proof Assume that $P_i = \{x_3 = d_i\}$, $i = 1, 2$, with $d_1 < d_2$. In both cases, we construct a closed surface of \mathbb{R}^3 and let us apply the reflection method with horizontal planes. It is sufficient to consider $M \cup (\Omega_0 \times [d_1, d_2])$, where Ω_0 is the orthogonal projection of Ω onto the plane $x_3 = 0$. \square

The following result compares rotating graphs with stationary rotating surfaces that lie in solid vertical cylinders, one of them is a graph.

Theorem 11 *Let Γ be a Jordan curve contained in a horizontal plane P and Ω the corresponding bounded planar domain. Suppose that there exists a stationary rotating graph G on Ω with $\partial G = \Gamma$ and assume that the mean curvature $2H_G(x) = ar^2 + b$ does not vanish at any point of G . Let M be a stationary rotating surface bounded by Γ with the following conditions:*

1. The mean curvature H of M satisfies $|2H(x)| = |ar^2 + b|$.
2. The surface $x(M)$ lies in $\Omega \times \mathbb{R}$.

Then $x(M)$ either coincides with G or with its reflection about P .

Proof Denote by G^* the reflection of G about the plane P . Without loss of generality, we suppose that the mean curvature H_G is positive with the orientation N_G on the graph G pointing downwards. By the maximum principle together the fact that $H_G \neq 0$, we know that G lies above the plane P . We choose the orientation N on M whose mean curvature is exactly H_G . We move G upwards so it does not touch M and then we drop it until it reaches a contact point p with M for the first time. Denote by G' the translated graph at this time. If $p \notin \Gamma$, then M and G' are tangent at p and $N(p) = \pm N_G(p)$. If $N(p) = N_G(p)$, and as the mean curvatures of G' and M agree for the same choice of normal vector fields at p , the maximum principle implies that M and G' should be coincide. This is a contradiction, since the boundaries $\partial M = \Gamma$ and $\partial G'$ lie at different heights. If $N(p) = -N_G(p)$, we change the orientation of M , namely, $N' = -N$, and the mean curvature $H' = -H < 0$. Then $N'(p) = N_G(p)$, but $H'(p) < H_G(p)$. Invoking the maximum principle again, we assert that this situation is impossible.

As conclusion, we can move G downwards until that G returns into its original position. Working now with the graph G^* , the same reasoning shows that M lies above G^* . If ν_M and ν_G denote the inner conormal unit vectors of M and G respectively along their common boundary Γ , we have concluded that

$$|\langle \nu_M(p), E_3 \rangle| \leq \langle \nu_G(p), E_3 \rangle, \quad \forall p \in \Gamma.$$

If the equality holds at some point $p_0 \in \Gamma$, this means that M is tangent to G or G^* at p_0 . Now we use the boundary maximum principle which implies that either $M = G$ or $M = G^*$, proving the result.

Suppose, by way of contradiction, that we have

$$|\langle \nu_M(p), E_3 \rangle| < \langle \nu_G(p), E_3 \rangle$$

for each $p \in \Gamma$. Integrating this inequality along the common boundary $x(\partial M) = \Gamma = \partial G$, we have

$$\left| \int_{\partial M} \langle \nu_M, E_3 \rangle ds \right| \leq \int_{\partial M} |\langle \nu_M, E_3 \rangle| ds < \int_{\partial G} \langle \nu_G, E_3 \rangle ds. \quad (33)$$

We employ the flux formula (14) for both surfaces M and G :

$$\begin{aligned} \int_{\partial M} \langle \nu_M, E_3 \rangle ds &= -\frac{1}{4} \int_{\partial M} (ar^2 + 2b) \langle \alpha \wedge \alpha', E_3 \rangle ds. \\ \int_{\partial G} \langle \nu_G, E_3 \rangle ds &= -\frac{1}{4} \int_{\partial G} (ar^2 + 2b) \langle \alpha \wedge \alpha', E_3 \rangle ds. \end{aligned}$$

Since the integrands in the right-hand sides of the above two identities depend only on Γ and they are the same or of reverse sign depending of the direction of $\alpha|_{\partial M}$ and $\alpha|_{\partial G}$, we derive a contradiction with the inequality (33). \square

We now extend Theorem 4 for rotating liquid drops with boundary.

Theorem 12 *Let P be a horizontal plane of \mathbb{R}^3 . Consider M an embedded stationary rotating surface with $\partial M \subset P$. Assume that the mean curvature is $2H(x) = ar^2 + b$, where $a \neq 0$ and $ab \geq 0$. If M lies in one side of P , then*

$$\max_{x \in M} \text{dist}(x, P) \leq \frac{|aR^2 + 2b|}{8\pi} \text{area}(M), \quad R = \max_{x \in M} r(x).$$

Proof Without loss of generality, we suppose that M lies in the upper half-space determined by P . Consider $W \subset \mathbb{R}^3$ the domain that encloses $M \cup \Omega$, being Ω the bounded planar domain by ∂M . We orient M by the Gauss map that points inside W . By considering the highest point of M and the maximum principle, we deduce that the mean curvature is positive. Thus $a, b > 0$. We apply the reflection method with horizontal planes $P(t)$ coming from infinity as in the proof of Theorem 10. Following the same notation used there, we enumerate the next possibilities at the time t_1 :

1. Case $t_1 = 0$. Then M is a graph on Ω and Theorem 4 proves the result.
2. Case $t_1 > 0$ and there exists a smooth tangent point between the surfaces $M_{t_1}^-$ and $M_{t_1}^*$. Then the maximum principle says that $P(t_1)$ is a plane of symmetry of M , which it is a contradiction because the boundary ∂M lies below $P(t_1)$.
3. Case $t_1 > 0$ and $M_{t_1}^*$ and ∂M contact at some point. Then $M_{t_1}^+$ is a graph over a domain of $P(t_1)$. Theorem 4 asserts now

$$\begin{aligned} \frac{1}{2} \max_{x \in M} \text{dist}(x, P) &\leq \max_{x \in M_{t_1}^+} \text{dist}(x, P(t_1)) \leq \frac{|aR^2 + 2b|}{8\pi} \text{area}(M_{t_1}^+) \\ &\leq \frac{|aR^2 + 2b|}{8\pi} \text{area}(M), \end{aligned}$$

and the theorem is proved.

□

We end this section with a result motivated by what happens in the theory of cmc-surfaces with boundary. If M is an embedded cmc-surface with boundary, one asks under what conditions the symmetries of the boundary of a cmc-surface inherit to the whole surface. For example, if Γ is a circle and $\partial M = \Gamma$, is M a spherical cap? If M lies in one side of the plane containing Γ , the Alexandrov reflection method proves that M is a spherical cap. Thus, one seeks conditions that assure that the surface lies in one side of P . Two results stand out in this setting. Assume that Γ is a closed curve contained in a plane P . The first one is due to Koiso [25] and shows that if M does not intersect the outside of Γ in P , then M lies in one side of P . The second result, due to Brito et. al. [8], shows that if Γ is strictly convex and M is transverse to P along the boundary ∂M , then M is entirely contained in one of the half-spaces of \mathbb{R}^3 determined by P . Here, transversality means that the surface M is not tangent to P at each point of ∂M .

For stationary rotating surfaces, one asks whether a stationary rotating surface bounded by a (horizontal) circle must be a surface of revolution. A first difference is that even if M lies in one side of P , P the plane containing ∂M , one cannot apply the Alexandrov reflection method for surfaces that have arbitrary values of a and b (an exception is Theorem 9). With respect to Koiso's theorem, actually her result holds assuming that the mean curvature does not change of sign and so, the theorem is true for stationary rotating surfaces where H does not vanish. Finally, the result cited in [8] does not hold for stationary rotating surfaces even if the surface is axisymmetric: surfaces of type II (b) provide counterexamples as it can see in Figure 5. The last theorem in this section is related with both results.

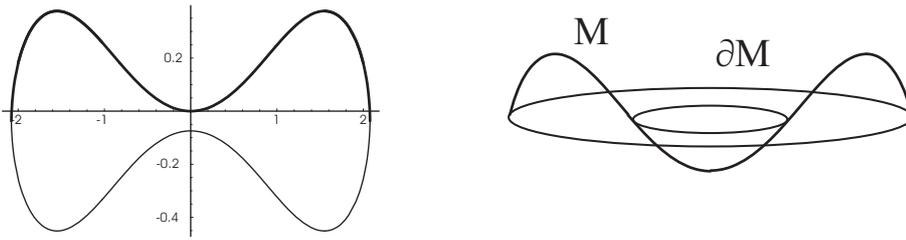


Fig. 5 Rotating drop for $a = -1$ and $b = 1.2$. The bold line represents the generating curve of a rotating drop that is embedded and intersects the outside of the boundary in the plane P containing the boundary. We observe that the surface is transverse to P , but the surface does not lie in one side of P .

Theorem 13 *Let Γ be a closed curve contained in a horizontal plane P and star-shaped with respect to O , the intersection point between P and the x_3 -axis. Let $\Omega \subset P$ be the corresponding bounded planar domain by Γ . Let M be an embedded stationary rotating surface with boundary Γ and suppose that M satisfies:*

1. *The mean curvature does not vanish on M .*
2. *The surface M does not intersect the domain Ω .*
3. *The surface M is transverse to P along Γ .*

Then M lies in one side of P . In particular, the result holds for horizontal convex curves whose inside intersects the x_3 -axis. See Figure 4 (b).

Proof Without loss of generality, we suppose that P is the plane $x_3 = 0$ and H is positive on M . By transversality, in a neighborhood of the boundary ∂M the surface M is contained in one of the two connected components of $\mathbb{R}^3 \setminus P$, which can be assumed to be the upper half-space. In this situation, we will prove that M is above P . We attach the domain Ω to M , obtaining a closed surface $M' = M \cup \Omega$. Thus, M' encloses a domain W of \mathbb{R}^3 . We orient M' by the mean curvature vector of M and let us denote by η_Ω the induced orientation on Ω .

We claim that N' points to the domain W . For this, we take the highest point p of M with respect to the plane P . In particular, p is an interior point of M and $N(p) = \pm E_3$. As the mean curvature H of M is positive, the maximum principle implies that $N(p) = -E_3$, and so, N points to W .

We show the theorem by way of contradiction: assume that M has points below P . We use the flux formula (14) and the notation that appears there. Because Γ is a star-shaped curve with respect to the origin, the function $\langle x \wedge \alpha', E_3 \rangle$ has sign along $\partial\Omega$. We show that this sign is positive. Since M does not intersect Ω and N points towards W , the orthogonal projection of the restriction $N|_{\partial M}$ onto the plane P points outside Ω . This means that $\alpha' = \nu \wedge N|_{\partial\Omega}$ follows the counterclockwise direction along ∂M . Thus $\langle x \wedge \alpha', E_3 \rangle > 0$ along ∂M . Once proved this, we remark that the function $\nu_3 = \langle \nu, E_3 \rangle$ is also positive because the surface lies in the upper half-space in a neighbourhood of ∂M . Using the flux formula (14), we arrive to a contradiction and this completes the proof. \square

6 Stability

The stability of rotating liquid drops has been studied for many authors. In Physics the literature is big: see the Chapter 5 in [28] and references therein. We only point out that Brown and Scriven did numerical analysis of stability of rotating drops [9] (see new results in [21]). Wang et. al. did experiments in low gravity environments in order to confirm the numerical results [38]. Mathematically, the problem has been studied in different articles: [1, 4, 9, 11, 13, 15, 30, 31].

In this section we study strongly stability and we prove that any stationary rotating graph is strongly stable. This generalizes a well known result of the theory of cmc-surfaces in Euclidean space. According to Section 2, we define the energy of a stationary rotating surface as follows. Let $x : M \rightarrow \mathbb{R}^3$ be an immersed compact surface with mean curvature $2H(x) = ar^2 + b$ for any $x \in M$. We define the energy of the immersion x as

$$E(x) = \int_M 1 \, dM + a \int_M r^2 x_3 N_3 \, dM + b \int_M x_3 N_3 \, dM.$$

The description of each one of the integrals that appears in the right-hand side is the following. The first one represents the area of the surface and is proportional to the surface tension energy; the second integral is the energy of the centrifugal force of the surface with respect to the x_3 -axis; and the last one is the algebraic volume between the surface and the plane $x_3 = 0$.

Theorem 14 *Let $x : M \rightarrow \mathbb{R}^3$ be a stationary rotating surface that is a compact graph on a horizontal domain $\Omega \subset \mathbb{R}^2$. Then there holds the following property about the energy of M . Let $y : M' \rightarrow \mathbb{R}^3$ be an immersion of a compact oriented surface M' with the same boundary as M in such that the immersion obtained by gluing x to y along the boundaries defines an oriented 3-chain W . If M' is included in the vertical solid cylinder $\Omega \times \mathbb{R}$, then $E(x) \leq E(y)$. As a consequence, M is strongly stable.*

Proof For the first part of Theorem, we use an argument of calibration type. Assume that $M = \text{graph}(u)$, where u is a smooth function defined on Ω . Consider the orientation on M pointing upwards, that is,

$$N(x) = \frac{1}{\sqrt{1 + |Du|^2}} \left(-\frac{\partial u}{\partial x_1}, -\frac{\partial u}{\partial x_2}, 1 \right)(x), \quad x \in M$$

On $\Omega \times \mathbb{R}$, we define the vector field

$$Z(x_1, x_2, x_3) = N(x_1, x_2) + (ar^2 + b)x_3E_3,$$

where the mean curvature of M is $H = ar^2 + b$. Then

$$\text{div}_{\mathbb{R}^3} Z = \text{div}_{\mathbb{R}^3} N + (ar^2 + b) = -2H + (ar^2 + b) = 0.$$

Using the divergence theorem in the chain W , one has

$$0 = \int_M \langle Z, N \rangle dM - \int_{M'} \langle Z, N' \rangle dM',$$

where N' is the orientation on M' induced by W . The first integral in the right-hand side is

$$\int_M \langle Z, N \rangle dM = \int_M \left(1 + (ar^2 + b)x_3N_3 \right) dM = E(x).$$

On the other hand, the second integral is

$$\begin{aligned} \int_{M'} \langle Z, N' \rangle dM' &= \int_{M'} \left(\langle N, N' \rangle + (ar^2 + b)x_3N'_3 \right) dM' \\ &\leq \int_{M'} \left(1 + (ar^2 + b)x_3N'_3 \right) dM' = E(y). \end{aligned}$$

where here we have used $\langle N, N' \rangle \leq 1$. Hence it follows the result.

Once proved the part of Theorem on energy minimization, we show that M is strongly stable. According to the definition given in Section 2, let $v \in C_0^\infty(M)$ and we are going to show that $-\int_M v \cdot L[v] dM \geq 0$. Consider the variation $x_t = x + tvN$, whose normal component of the variational vector field is $\langle (\partial x_t) / (\partial t)_{t=0}, N \rangle = v$. By the energy minimization property proved in this result, we have

$$-\int_M v \cdot L[v] dM = \frac{d^2}{dt^2} \Big|_{t=0} E(x_t) \geq 0.$$

□

We end this paper with several natural questions that could and should be addressed within this theory of stationary rotating surfaces.

1. Let M be an embedded rotating surface with non-empty boundary. Assume that ∂M lies in a horizontal plane and that the center of mass of ∂M lies in the x_3 -axis. Does the center of mass of M lies in the x_3 -axis? For example, when ∂M is a horizontal circle centered at x_3 -axis.
2. Let M be an embedded rotating surface whose boundary is a circle in a horizontal plane. Is M a surface of revolution? The same if the boundary are two coaxial circles in horizontal planes.
3. What axisymmetric stationary rotating surfaces bounded by a circle are stable?

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