Free Boundary Minimal Surfaces in the Unit 3-Ball

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- A. Fraser, M. Li 2012, Conjecture : The unique (up to congruences) free boundary minimal **annulus** in B^3 is the critical catenoid:

$$(s,\theta) \mapsto \frac{1}{s_* \cosh s_*}(\cosh s\, e^{i\theta}, s), \quad s_* \tanh s_* = 1$$

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Steklov eigenproblem

(M^2, g) - compact Riemannian manifold, with $\partial M \neq \emptyset$

$$u \in \mathcal{C}^\infty(\partial M) \rightsquigarrow \hat{u} : \quad \begin{cases} \Delta_g \hat{u} = 0 & \text{on} \quad M, \\ \hat{u} = u & \text{on} \quad \partial M \end{cases}$$

Dirichlet-to-Neumann operator :

$$L_g : \mathcal{C}^\infty(\partial M) \longrightarrow \mathcal{C}^\infty(\partial M), \quad L_g u = \frac{\partial \hat{u}}{\partial \eta}$$

non-negative, self-adjoint,

diskrete spectrum: $\sigma_0 < \sigma_1 \leq \sigma_2 \leq \cdots$ (**Steklov eigenvalues**)

Genus 0 free boundary minimal surfaces

$\sigma^*(0,n) := \sup \sigma_1(g)\,\mathrm{Length}_g(\partial M)$.

1. Weinstock theorem : $\sigma^*(0,1) = 2\pi$, Σ_0 - flat unit disk
2. A. Fraser, R. Schoen :
   - For all $n \in \mathbb{N}$ a maximizing metric is achieved by a free boundary minimal surface Σ_n in $\mathbf{B}^3$ (if $n = 2$, then $\sigma^*(0,2) = 4\pi/1.2$ and Σ_1 is a critical catenoid)
   - $n \to \infty$, Σ_n converges on compact sets of $\mathbf{B}^3$ to a double equatorial disk. ($\sigma^*(0,n)$ converges to 4π).
   - For large n, Σ_n is approximately a pair of nearby parallel plane disks joined by n boundary bridges $\sim$ scaled down versions of half-catenoids.

Main result

$\mathbb{R}^3 \longleftrightarrow \mathbb{C} \times \mathbb{R}$, $\quad \mathfrak{S}_n \subset O(3)$ a subgroup of isometries generated by

$$(z,t) \mapsto (\bar{z},t), \quad (z,t) \mapsto (z,-t), \quad (z,t) \mapsto (e^{\frac{2\pi i}{n}} z, t)$$

Theorem (A. Folha, F. Pacard, —)

For n large enough we

- *Give an alternative proof of existence of **genus 0** free boundary minimal surfaces with $\mathbf{n}$ boundary components (invariant under $\mathfrak{S}_n$),*

  $\Sigma_n \underset{n\to\infty}{\to}$ *double copie of* D^2.

- *Prove the xistence of **genus 1** free boundary minmal surfaces $\tilde{\Sigma}_n$ with $\mathbf{n}$ boundary components (invariant under $\mathfrak{S}_n$),*

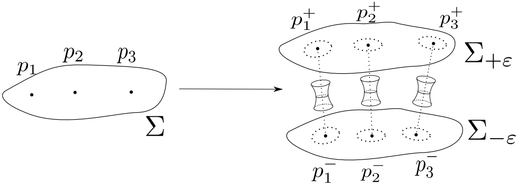
  $\tilde{\Sigma}_n \underset{n\to\infty}{\to}$ *double copie of* $D^2 \setminus \{0\}$.

Ingredients: An initial compact, oriented embedded minimal surface Σ and a set of points $p_1, \dots, p_n \in \Sigma$.

- Construct two "nearby copies" $\Sigma_{\pm\varepsilon}$ of Σ, which converge uniformly to Σ when $\varepsilon \to 0$:

  $$\Sigma_{\pm\varepsilon} = \Sigma \pm \varepsilon \Psi N_\Sigma, \qquad \Psi \in \mathcal{C}^2(\Sigma).$$

- Perform a connected sum of $\Sigma_{\pm\varepsilon}$ at $p_1, \dots, p_k$ and then deform it to a CMC surface.



Question : Under what conditions is it possible to carry out a doubling construction based on given minimal surface?

Neck configuration : The number, the size and the positions of the necks.

Jacobi operator : $J_\Sigma = \Delta_\Sigma + |A_\sigma|^2 + \mathrm{Ric}(N_\Sigma, N_\Sigma)$.

Σ is said to be nondegenerate if

$$J_\Sigma\, w = 0, \quad w|_{\partial\Sigma} = 0 \quad \Rightarrow \quad w = 0.$$

F. Pacard, T. Sun : If Σ is a **nondegenerate minimal hypersurface** in a Riemannian manifold, one can choose Ψ. s.t. $H(\Sigma \pm \Psi\, N_\Sigma) = \pm 1$ and produce a CMC surface with $H = \varepsilon$ by doubling Σ at any nondegenerate critical point of Ψ (neck size $\sim \varepsilon$).

CMC surfaces by doubling constructions

Question : What if Σ is degenerate?

- Green's function method (R. Mazzeo, F. Pacard, D. Pollack) :

  consists to study the solutions to

$$J_\Sigma \, \Gamma = \sum_{i=1}^{k} c_i \, \delta_{p_i}, \qquad \Gamma|_{\partial\Sigma} = 0,$$

  and construct $\Sigma_{\pm\varepsilon}$ as a normal graphs about Σ of the functions $\pm\,\varepsilon\,\Gamma$

  ($\Sigma_{\pm\varepsilon}$ converge to Σ uniformly on compact sets when $\varepsilon \to 0$).

- Balancing considerations (using the first variation formula).

A. Butscher, F. Pacard : Surfaces with $H = \varepsilon$ in $\mathbf{S}^3$ can be constructed by doubling the minimal Clifford torus at the points of a square lattice that contains $2\pi\,\mathbb{Z}^2$ (neck size $\sim \varepsilon$).

Minimal surfaces by doubling constructions

- N. Kapouleas, S.D. Yang : Construction of minimal surfaces in $\mathbf{S}^3$ by doubling the Clifford torus at the points of the square lattice $\frac{2\pi}{n}\mathbb{Z}$, with n large enough.
- N. Kapoules : Construction of minimal surfaces in $\mathbf{S}^3$ by doubling the equatorial sphere
- D. Wiygul : Construction of minimal surfaces in $\mathbb{S}^3$ by stacking Clifford tori.

! A certain relation should be satisfied between the number of necks and the size of the necks $\Rightarrow$ Construction works only for large n !

The expansion of the Green's function in a neighborhood of the poles reads :

$$\varepsilon\,\Gamma = \varepsilon\, c(n) + \varepsilon \log r + \dots$$

On the other hand, catenoid scaled by a factor ε (neck size)

$$X_{cat}^{\varepsilon} : (s, \theta) \in \mathbb{R} \times S^1 \mapsto \varepsilon\,(\cosh s\ \cos\theta, \cosh s\ \sin\theta, s)$$

can be seen as a bi-graph of the function

$$\mathcal{G}_{\varepsilon} = \varepsilon\ \log\frac{2}{\varepsilon} + \varepsilon\ \log r + \mathcal{O}\left(\frac{\varepsilon^3}{r^2}\right)$$

! We need the constant terms to match exactly : $c(n) = \log\frac{2}{\varepsilon}$, which gives the relation between the size and the number of the necks.

Parametrization of the unit ball

We choose the following parametrization of $\mathbf{B}^3$:

$$X : \mathbf{D}^2 \times \mathbb{R} \longrightarrow \mathbf{B}^3, \quad X(z,t) = A(z,t)\,(z\,,\,B(z)\,\sinh t)\,,$$

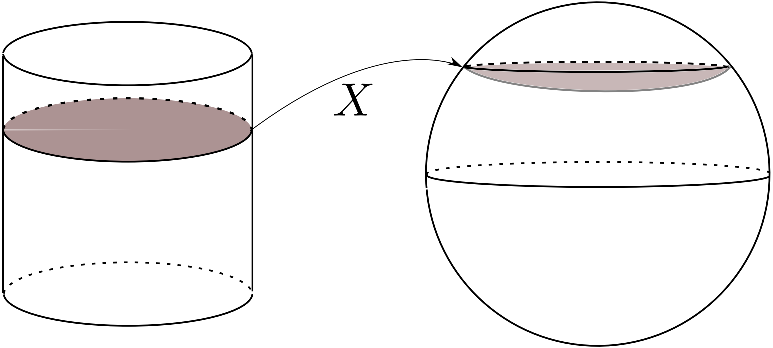
where $B(z) = \frac{1+|z|^2}{2}$ and $A(z,t) = \frac{1}{1+B(z)(\cosh t - 1)}$.

- $t = 0$ - horizontal disk $\mathbf{D}^2 \times \{0\}$
- $|z| = 1$ - unit sphere $\mathbf{S}^2$
- $t = t_0$ - CMC leaf with $H = \sinh t_0$ that meets $\mathbf{S}^2$ orthogonally.

Induced metric :

$$X^* g_{eucl} = A^2(z,t)\left(|dz|^2 + B^2(z)\,dt^2\right)$$

Parametrization of the unit ball



Take $w \in \mathcal{C}^2(D^2)$ and consider the image in $\mathbf{B}^3$ of the vertical graph of w :

$$z \mapsto X(z, w(z)).$$

Lemma

1. *$X(z, w(z))$ is orthogonal to $\mathbf{S}^2 = \partial\mathbf{B}^3$ at the boundary iff*

$$\partial_r w|_{r=1} = 0;$$

2. *Mean curvature of $X(z, w(z))$ is given by*

$$H(w) = \frac{1}{A^3(\cdot,w)\,B} \operatorname{div} \left(\frac{A^2(\cdot,w)\,B^2\,\nabla w}{\sqrt{1+B^2\,|\nabla w|^2}} \right) + 2\sqrt{1 + B^2\,|\nabla w|^2}\,\sinh w.$$

Linearized mean curvature operator

The linearized mean curvature operator is given by

$$L_{gr} w = \Delta(Bw) = \Delta\left(\frac{1+|z|^2}{2}\, w\right)$$

Remark:

- The operator L_{gr} has a kernel :

$$\mathrm{Ker}(L_{gr}) = \left\{\frac{2x_1}{1+|z|^2}, \frac{2x_2}{1+|z|^2}\right\}.$$

  This corresponds to tilting the unit disk $D^2 \times \{0\}$ in $\mathbf{B}^3$.

- One can eliminate the kernel by asking the function w to be invariant under a group of rotations around the coordinate axis $Ox_3 \Rightarrow$

  (We arrange the catenoidal bridges periodically along S^1)

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Take $\varepsilon,\ \tilde{\varepsilon} \in \mathbb{R}_+$ and $\quad z_m := e^{2\pi i m/n},\ m = 1, \ldots, n$

and consider the catenoids in $\mathbb{R}^3$ centered at $z = 0$ and $z = z_m$:

$$(s, \theta) \in \mathbb{R} \times \mathbb{S}^1 \mapsto \left(\tilde{\varepsilon} \cosh s\, e^{i\theta}\, ,\ \tilde{\varepsilon} s\right)$$

$$(s, \theta) \in \mathbb{R} \times \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \mapsto \left(\varepsilon \cosh s\, e^{i\theta} + z_m\, ,\ \varepsilon s\right)$$

which can be seen as bi-graphs of the functions

$$\mathcal{G}_{\tilde{\varepsilon},0}(z) = \left(z\, ,\ \pm \left(\tilde{\varepsilon} \log \frac{\tilde{\varepsilon}}{2} - \tilde{\varepsilon} \log |z| + \mathcal{O}\left(\tilde{\varepsilon}^3 / |z|^2\right)\right)\right)$$

$$\mathcal{G}_{\varepsilon,m}(z) := \left(z\, ,\ \pm \left(\varepsilon \log \frac{\varepsilon}{2} - \varepsilon \log |z - z_m| + \mathcal{O}\left(\varepsilon^3 / |z - z_m|^2\right)\right)\right)$$

in small neighborhoods of $z = 0$ and $z = z_m$.

Goal : To find the solution to the problem $(*)$ invariant under rotations by the angle $\frac{2\pi}{n}$.

$$(*)\quad \begin{cases} \Delta(B\Gamma) = c_0\,\delta_0 \quad \text{in} \quad D^2 \\ \partial_r\Gamma = \sum_{m=1}^{n} c_m\,\delta_{z_m} \quad \text{on} \quad S^1 \end{cases}$$

If $\quad \Gamma(z) = G(z^n)/B(z), \quad$ then G satisfies

$$(**)\quad \begin{cases} \Delta\, G = d_0\,\delta_0 \quad \text{in} \quad D^2 \\ \partial_r G - \frac{1}{n}\, G = d_1\,\delta_1 \quad \text{on} \quad S^1 \end{cases}$$

in the sense of distributions.

Green's function

We construct explicitly G_0 and G_1, such that

$$(1)\ \begin{cases} \Delta G_0 = d_0\,\delta_0 & \text{in} \quad D^2 \\ \partial_r G_0 - \frac{1}{n}\,G_0 = 0 & \text{on} \quad S^1 \end{cases} \qquad (2)\ \begin{cases} \Delta G_1 = 0 & \text{in} \quad D^2 \\ \partial_r G_1 - \frac{1}{n}\,G_1 = d_1\,\delta_1 & \text{on} \quad S^1 \end{cases}$$

- $G_0 := -\log|z| - n$ - satisfies (1)
- $\forall k \in \mathbb{N}, \quad H_k(z) = \frac{1}{n^k}\mathrm{Re}\sum_{j=1}^{\infty}\frac{z^j}{j^{k+1}}$

  $H_0(z) = -\log|1-z|, \qquad \partial_r H_k|_{r=1} = \frac{1}{n}H_{k-1}$

  $G_1(z) := -\frac{n}{2} + \sum_{k=0}^{\infty} H_k(z)$ - satisfies (2)
- $\forall \alpha, \beta \in \mathbb{R}, \quad \Gamma_n(z) := \frac{1}{B(z)}\left(\alpha\, G_0(z^n) + \beta\, G_1(z^n)\right)$ satisfies $(*)$

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Neck configurations

$$\Gamma_n(z) = \begin{cases} c_0(n) + 2n\,\alpha\,\log|z| + \mathcal{O}(\alpha\,|z|^2), & \text{as} \quad z \to 0 \\ c_1(n) + \beta\,\log|z - z_m| + \mathcal{O}(\beta\,|z - z_m|), & \text{as} \quad z \to z_m, \end{cases}$$

where $c_0(n) \sim c_1(n) \sim n$. We obtain :

$$\varepsilon \sim \tilde{\varepsilon}, \quad n \sim \log\varepsilon, \quad \alpha \sim \frac{\varepsilon}{n}, \quad \beta \sim \varepsilon.$$

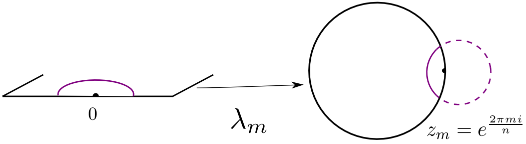
and do the "gluing" in the regions

$$|z| \sim \varepsilon^{1/2}, \quad |z - z_m| \sim \varepsilon^{2/3}, \; m = 1, \ldots, n$$

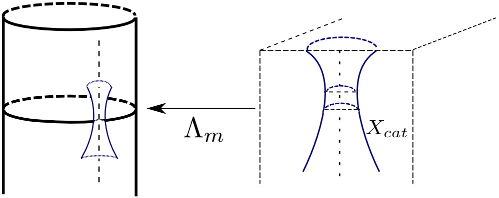
Catenoidal bridges orthogonal to S^2

$\mathbb{C}_- := \{\zeta \in \mathbb{C} : \mathrm{Re}(\zeta) \leq 0\}$. Consider the diffeomorphisms

$$\lambda_m(\zeta) : \mathbb{C}_- \longrightarrow \bar{D}^2, \quad \lambda_m(\zeta) = e^{2\pi i m/n}\,\frac{1+\zeta}{1-\zeta}$$

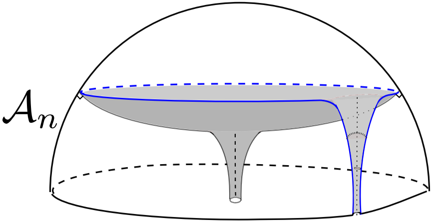


and $\quad \Lambda_m(\zeta, t) : \mathbb{C}_- \times \mathbb{R} \longrightarrow \bar{D}^2 \times \mathbb{R}, \quad \Lambda_m(z,t) = (\lambda_m(z), 2t)$



Approximate solution $\mathcal{A}_n$

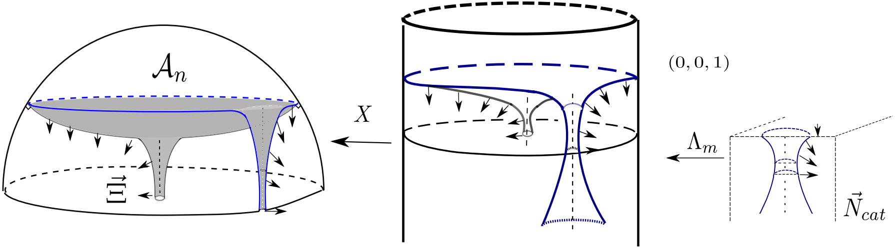
$\mathcal{A}_n$ = connected sum of the graph of the Green's function Γ_n with n half-catenoidal bridges and 1 catenoidal neck.



Orthogonality to $\mathbf{S}^2$ at the boundary:

- The half-catenoid X_{cat} is foliated by horizontal leafs orthogonal to $\partial\mathbb{C}_- \times \mathbb{R}$ at the boundary.
- The restriction $X \circ \Lambda_m$ to the horizontal planes is conformal.
- The surface parametrized by $X \circ \Lambda_m \circ X_{cat}$ is foliated by spherical-cap leafs orthogonal to $\mathbf{S}^2$ at the boundary.

Perturbation argument



Define a vector field $\vec{\Xi}$ transverse to $\mathcal{A}_n$

(smoothly interpolating between the "normal" to the catenoids and the "vertical" vector field), s.t.

if ξ_t is the associated flow : $\xi\left(\mathcal{A}_n, 0\right) = \mathcal{A}_n$ and $\frac{\partial \xi}{\partial t} = \Xi(\xi(\cdot, t))$

then $\mathcal{A}_{n,t} := \xi_t(\mathcal{A}_n)$ meets $\mathbf{S}^2$ orthogonally along $\partial \mathcal{A}_{n,t}$.

Take $w \in \mathcal{C}^2(\mathcal{A}_n)$ and put $\mathcal{A}_n(w) = \{\xi(p, w(p)), \quad p \in \mathcal{A}_n\}$.

- The Taylor expansion of the mean curvature of $\mathcal{A}_n(w)$ satisfies

$$H(\mathcal{A}_n(w)) = H(\mathcal{A}_n) + \mathcal{L}_n\, w + \mathcal{Q}_n(w, \nabla w, \nabla^2 w),$$

$\mathcal{L}_n$ - the linearized mean curvature operator,

$\mathcal{Q}_n$ - smooth nonlinear function, s.t. $Q(0,0,0) = DQ(0,0,0) = 0$.

- If w satisfies the Neumann boundary condition : $\frac{\partial w}{\partial \eta} = 0$ on $\partial \mathcal{A}_n$

$\Rightarrow \quad \mathcal{A}_n(w)$ meets $\mathbf{S}^2$ orthogonally at the boundary.

Our goal is to solve the equation $H(\mathcal{A}_n(w)) = 0$ or

$$\mathcal{L}_n\, w = -\big(H(\mathcal{A}_n) + \mathcal{Q}_n(w, \nabla w, \nabla^2 w)\big)$$

- If $H(\mathcal{A}_n) \underset{n\to\infty}{\to} 0$ in a suitable topology
- and if $\mathcal{L}_n$ were invertible with its inverse bounded uniformly in n

$\Rightarrow$ we could apply Banach fixed point theorem in a ball of an appropriate Banach space (the radius of the ball being defined by $\|H(\mathcal{A}_n)\|$).

Weighted Banach spaces

Weight function : $\gamma(x) = |x| \prod_{m=1}^{n} |x - e^{\frac{2\pi i m}{n}}|, \quad x \in \mathcal{A}_n.$

For $\nu \in \mathbb{R}, \quad w \in L^\infty_\nu(\mathcal{A}_n)$ iff $|\gamma^{-\nu}\, w| < \infty.$

Then, we find $\|H(\mathcal{A}_n)\|_{L^\infty_{\nu-2}} \leq c\, \varepsilon^{\frac{5}{3}-\nu} \leq c\, e^{-n(\frac{5}{3}-\nu)}$

and take $\nu \in (0,1)$. Moreover we have :

$$\mathcal{L}_n \;:\; L^\infty_\nu(\mathcal{A}_n) \longrightarrow L^\infty_{\nu-2}(\mathcal{A}_n).$$

In the same manner we define Hölder weighted spaces $\mathcal{C}^{k,\alpha}_\nu$.

Properties of the linearized operator

As $n \to \infty$,

In the "graph" region : $\quad \mathcal{L}_n \sim L_{gr} = \Delta(B\cdot)$,

In the "catenoidal" regions : $\quad \mathcal{L}_n \sim J_{cat} = \frac{1}{\varepsilon^2 \cosh^2 s}\left(\partial_s^2 + \partial_\theta^2 + \frac{2}{\cosh^2 s}\right)$.

(*Jacobi operator about the catenoid*)

We should study the corresponding operators in noncompact domains $\bar{D}^2 \setminus \{0, z_1, \dots, z_n\}$, $\mathbb{R} \times S^1$, and $\mathbb{R} \times \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

Problem : The catenoid is degenerate $\Rightarrow$ there are small eigenvalues of the operator $\mathcal{L}_n$ (eigenvalues that tend to 0 as fast as $n \to \infty$).

Linear analysis on the half-catenoid

Study the homogeneous problem

$$\begin{cases} \frac{1}{\varepsilon^2 \cosh^2 s}\left(\partial_s^2 + \partial_\theta^2 + \frac{2}{\cosh^2 s}\right) w = 0 & \mathbb{R} \times \left[\frac{\pi}{2}, \frac{3\pi}{2}\right], \\ \partial_\theta w = 0 & \mathbb{R} \times \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}. \end{cases}$$

Fourier series $w = \sum_{j \in \mathbb{Z}} w_j(s)\, e^{i\theta j}$.

- $|j| \geq 2$, maximum principle $\Rightarrow w_j = 0$;
- $|j| = 1$ (rotations + horizontal translations),

  Neumann boundary condition + symmetry $w(s,\theta) = w(s, 2\pi - \theta)$

  $\Rightarrow w_{\pm 1} = 0$;
- $j = 0$ (dilatation + vertical translation),

  symmetry $w(s,\theta) = w(-s,\theta)$ + suitable function space $\Rightarrow w_0 = 0$.

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Lemma

For $|\nu| < 1$ *and for all* $f \in (\varepsilon \cosh s)^{\nu-2} L^\infty(\mathbb{R} \times [\frac{\pi}{2}, \frac{3\pi}{2}])$ *there exist*

$v_{cat} \in (\varepsilon \cosh s)^{\nu} L^\infty(\mathbb{R} \times [\frac{\pi}{2}, \frac{3\pi}{2}])$ *and* $c_{cat} \in \mathbb{R}$ *, s.t.*

$w_{cat} = v_{cat} + c_{cat}$ *satisfies*

$$\begin{cases} \frac{1}{\varepsilon^2 \cosh^2 s}\left(\partial_s^2 + \partial_\theta^2 + \frac{2}{\cosh^2 s}\right) w_{cat} = f & \mathbb{R} \times \left[\frac{\pi}{2}, \frac{3\pi}{2}\right], \\ \partial_\theta w_{cat} = 0 & \mathbb{R} \times \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}, \end{cases}$$

and $\|(\varepsilon \cosh s)^{-\nu} v_{cat}\|_{L^\infty} + |c_{cat}| \le C \, \|(\varepsilon \cosh s)^{-\nu+2} f\|_{L^\infty}.$

Deficiency space : $\mathfrak{D}_{cat} = \mathrm{span}\{1\}.$

We are interested in the problem

$$(*)\begin{cases} \Delta(Bw) = f \quad \text{in} \quad D^2 \setminus \{0\}, \\ \partial_r w = 0 \quad \text{on} \quad S^1 \setminus \{z_1, \ldots, z_n\}. \end{cases}$$

Suppose that f and w are invariant under rotations by the angle $\frac{2\pi}{n}$

$\Rightarrow$ no bounded kernel.

Put $w(z) = W(z^n)/B(z)$, then $(*)$ is equivalent to

$$(**)\begin{cases} \Delta W = F \quad \text{in} \quad D^2 \setminus \{0\}, \\ \partial_r W - \frac{1}{n} W = 0 \quad \text{on} \quad S^1 \setminus \{1\}. \end{cases}$$

Linear analysis in the punctured disk

Fix a cut-off function $\chi \in \mathcal{C}^\infty(D^2)$, which is identically equal to 1 in a neighborhood of $z = 1$ and to 0 in a neighborhood of $z = 0$.

Weight function : $\gamma(z) = |z|\,|z-1|$.

Lemma

Assume that $\nu \in (0,1)$, then $\forall n$ large enough and for all F, such that $F \in L^\infty_{\nu-2}(D^2)$, there exists a unique function V_{gr} and unique constants c_0 and c_1, such that $W_{gr} = V_{gr} + c_0\, n + c_1\, \chi$ is a solution to $(**)$ and

$$\|V_{gr}\|_{L^\infty_\nu} + |c_0| + |c_1| \leq C\, \|F\|_{L^\infty_{\nu-2}}$$

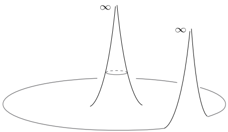
Deficiency space : $\mathfrak{D}_{gr} = \operatorname{span}\{n, \chi\}$.

Gluing the parametrices together

We need to solve

$$\mathcal{L}_n w = f, \quad \text{for} \quad f \in L^\infty_{\nu-2}(\mathcal{A}_n).$$

We divide $\mathcal{A}_n$ in 2 symmetric parts, extend f to



and solve the problem in $\mathbb{R} \times S^1$, $\mathbb{R} \times \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ and $D^2 \setminus \{0, z_1, \dots, z_n\}$, restrict the solutions to $\mathcal{A}_n$, and glue the restricted solutions together to produce :

$$w_n \; : \; \|\mathcal{L}_n w_n - f\| \ll \|f\|.$$

(We need to treat the terms decaying at infinity and deficiency terms separately).

Conclusion

So, we have

$$\mathcal{M}_n : f \mapsto w_n, \quad \mathcal{L}_n \circ \mathcal{M}_n - \mathrm{Id} = R_n, \quad \|R_n\| \ll 1$$

Finally, $\mathcal{L}_n^{-1} := \mathcal{M}_n \circ (\mathrm{Id} - R_n)^{-1}$.

Remark: $\|\mathcal{L}_n^{-1}\| \sim n$ - explodes when $n \to \infty$.

Conclusion: $\mathcal{L}_n^{-1} \circ \mathcal{Q}_n$ - a contraction mapping in a ball of radius $\|\mathcal{L}_n^{-1} H(\mathcal{A}_n)\|$ of $\mathcal{C}_\nu^{2,\alpha}(\mathcal{A}_n)$. Banach fixed point theorem is applied.

Thank you for your attention!