Stable capillary hypersurfaces in slabs and half-spaces

Rabah Souam

CNRS
Institut de Mathématiques de
Jussieu - Paris Rive Gauche

Geometric aspects on capillary problems and related topics

Granada, December 14-17, 2015

Aim

The aim of this talk is to characterize stable immersed capillary hypersurfaces in slabs and half-spaces in the Euclidean spaces $\mathbb{R}^{n+1}$, $n \geq 2$.

Aim and reference

The aim of this talk is to characterize stable immersed capillary hypersurfaces in slabs and half-spaces in the Euclidean spaces $\mathbb{R}^{n+1}$, $n \geq 2$.

- A. Ainouz and R. Souam, *Stable capillary hypersurfaces in a half-space or a slab,* to appear in Indiana Univ. Math. J. (arXiv:1411.4241).

General setting, capillary hypersurfaces

Let $(M^{n+1}, \langle , \rangle)$ be an oriented Riemannian manifold of dimension $n+1, n \geq 2$ and $\mathcal{B} \subset M$ a domain with smooth boundary.

General setting, capillary hypersurfaces

Let $(M^{n+1}, \langle , \rangle)$ be an oriented Riemannian manifold of dimension $n+1, n \geq 2$ and $\mathcal{B} \subset M$ a domain with smooth boundary.

We let $\overline{N}$ be the outward unit normal to $\partial\mathcal{B}$.

General setting, capillary hypersurfaces

Let $(M^{n+1}, \langle , \rangle)$ be an oriented Riemannian manifold of dimension $n+1$, $n \geq 2$ and $\mathcal{B} \subset M$ a domain with smooth boundary.

We let $\overline{N}$ be the outward unit normal to $\partial\mathcal{B}$.

Fix an angle $\theta \in (0, \pi)$. Let Σ be a compact orientable manifold of dimension n. A capillary immersion $\psi : \Sigma \to \mathcal{B}$ with contact angle θ is:

- a proper immersion $\psi(\operatorname{int}\Sigma) \subset \operatorname{int}\mathcal{B}$ and $\psi(\partial\Sigma) \subset \partial\mathcal{B}$ with constant mean curvature

General setting, capillary hypersurfaces

Let $(M^{n+1}, \langle , \rangle)$ be an oriented Riemannian manifold of dimension $n+1$, $n \geq 2$ and $\mathcal{B} \subset M$ a domain with smooth boundary.

We let $\overline{N}$ be the outward unit normal to $\partial\mathcal{B}$.

Fix an angle $\theta \in (0, \pi)$. Let Σ be a compact orientable manifold of dimension n. A capillary immersion $\psi : \Sigma \to \mathcal{B}$ with contact angle θ is:

- a proper immersion $\psi(\operatorname{int}\Sigma) \subset \operatorname{int}\mathcal{B}$ and $\psi(\partial\Sigma) \subset \partial\mathcal{B}$ with constant mean curvature
- the unit normal N to Σ for which $H \geq 0$ makes an angle θ with $\overline{N}$ along $\partial\Sigma$.

General setting, the variational problem

Such an immersion is a critical point for the following variational problem:

An admissible variation of ψ is a smooth map $\Psi : (-\epsilon, \epsilon) \times \Sigma \to \mathcal{B}$ such that $\psi_t = \Psi(t, .)$ is a proper immersion for each t and $\psi_0 = \psi$.

General setting, the variational problem

Such an immersion is a critical point for the following variational problem:

An admissible variation of ψ is a smooth map $\Psi : (-\epsilon, \epsilon) \times \Sigma \to \mathcal{B}$ such that $\psi_t = \Psi(t, .)$ is a proper immersion for each t and $\psi_0 = \psi$.

The *volume function* $V : (-\epsilon, \epsilon) \to \mathbb{R}$ is defined by

$$V(t) = \int_{[0,t]\times\Sigma} \Psi^*\Omega$$

where Ω is the volume form on M.

The variation is *volume preserving* if $V(t) = 0$ for each t.

General setting, the variational problem

Such an immersion is a critical point for the following variational problem:

An admissible variation of ψ is a smooth map $\Psi : (-\epsilon, \epsilon) \times \Sigma \to \mathcal{B}$ such that $\psi_t = \Psi(t, .)$ is a proper immersion for each t and $\psi_0 = \psi$.

The *volume function* $V : (-\epsilon, \epsilon) \to \mathbb{R}$ is defined by

$$V(t) = \int_{[0,t]\times\Sigma} \Psi^*\Omega$$

where Ω is the volume form on M.

The variation is *volume preserving* if $V(t) = 0$ for each t. We have:

$$V'(0) = \int_{\Sigma} \langle X, N \rangle$$

where $X = \frac{\partial \Psi}{\partial t}(0, .)$ is the variation field of Ψ.

General setting, the variational problem

So, for a volume preserving variation the function $f := \langle X, N \rangle$ verifies $\int_\Sigma f = 0$.
Conversely any $f \in \mathcal{C}^\infty(\Sigma)$ with $\int_\Sigma f = 0$ is induced by a volume preserving admissible variation.

General setting, the variational problem

So, for a volume preserving variation the function $f := \langle X, N \rangle$ verifies $\int_\Sigma f = 0$.

Conversely any $f \in \mathcal{C}^\infty(\Sigma)$ with $\int_\Sigma f = 0$ is induced by a volume preserving admissible variation.

The *wetted area function* $W : (-\epsilon, \epsilon) \to \mathbb{R}$ is defined by

$$W(t) = \int_{[0,t]\times\partial\Sigma} \Psi^* \omega$$

where ω is the volume form on $\partial\mathcal{B}$.

General setting, the variational problem

So, for a volume preserving variation the function $f := \langle X, N \rangle$ verifies $\int_\Sigma f = 0$.
Conversely any $f \in \mathcal{C}^\infty(\Sigma)$ with $\int_\Sigma f = 0$ is induced by a volume preserving admissible variation.

The *wetted area function* $W : (-\epsilon, \epsilon) \to \mathbb{R}$ is defined by

$$W(t) = \int_{[0,t]\times\partial\Sigma} \Psi^* \omega$$

where ω is the volume form on $\partial\mathcal{B}$.

The capillary immersion ψ is a critical point for volume preserving admissible variations of the energy function:

$$E(t) = |\psi_t(\Sigma)| - \cos\theta\, W(t).$$

General setting, stability

ψ is said to be stable if $E''(0) \geq 0$ for all admissible volume preserving variations.

General setting, stability

ψ is said to be stable if $E^{''}(0) \geq 0$ for all admissible volume preserving variations.

Second variation formula for the energy:

$$E^{''}(0) = -\int_{\Sigma} f\left(\Delta f + (|\sigma|^2 + \mathrm{Ric}(N))f\right) + \int_{\partial\Sigma} f\left(\frac{\partial f}{\partial \nu} - q\,f\right),$$

General setting, stability

ψ is said to be stable if $E^{''}(0) \geq 0$ for all admissible volume preserving variations.

Second variation formula for the energy:

$$E^{''}(0) = -\int_{\Sigma} f\left(\Delta f + (|\sigma|^2 + \mathrm{Ric}(N))f\right) + \int_{\partial\Sigma} f\left(\frac{\partial f}{\partial \nu} - q\, f\right),$$

Δ : Laplacian on Σ,
σ : second fundamental form of ψ,
Ric is the Ricci curvature of M and

$$q = \frac{1}{\sin\theta}\mathrm{II}(\bar{\nu}, \bar{\nu}) + \cot\theta\, \sigma(\nu, \nu).$$

II: second fundamental form of $\partial\mathcal{B}$ associated to the unit normal $-\bar{N}$, that is, for $X, Y \in T(\partial\mathcal{B})$, $\mathrm{II}(X, Y) = \langle \nabla_Y X, -\overline{N} \rangle$.

General setting, stability

The index form $\mathcal{I}$: symmetric bilinear form on $H^1(\Sigma)$

$$\mathcal{I}(f,g) = \int_\Sigma \langle \nabla f, \nabla g \rangle - (|\sigma|^2 + \mathrm{Ric}(N)) fg - \int_{\partial\Sigma} q\, fg,$$

where ∇ is the gradient on Σ.

General setting, stability

The index form $\mathcal{I}$: symmetric bilinear form on $H^1(\Sigma)$

$$\mathcal{I}(f,g) = \int_\Sigma \langle \nabla f, \nabla g\rangle - (|\sigma|^2 + \mathrm{Ric}(N))fg - \int_{\partial\Sigma} q\, fg,$$

where ∇ is the gradient on Σ.

Σ is stable iff

$$\mathcal{I}(f,f) \geq 0, \quad \forall f \in H^1(\Sigma)$$

General setting, stability

The index form $\mathcal{I}$: symmetric bilinear form on $H^1(\Sigma)$

$$\mathcal{I}(f,g) = \int_\Sigma \langle \nabla f, \nabla g \rangle - (|\sigma|^2 + \mathrm{Ric}(N)) fg - \int_{\partial\Sigma} q\, fg,$$

where ∇ is the gradient on Σ.

Σ is stable iff

$$\mathcal{I}(f,f) \geq 0, \quad \forall f \in H^1(\Sigma)$$

Remark

More generally, if $\partial\Sigma$ has several components $\Gamma_1, \ldots, \Gamma_k$ and has constant angle of contact θ_i with $\partial\mathcal{B}$ along Γ_i, for each i, then it is a critical point for the energy:

$$E(t) = |\psi_t(\Sigma)| - \sum_{i=1}^{k} \cos\theta_i\, W_i(t)$$

Capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

Examples: spherical caps.

Capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

Examples: spherical caps.

Question

Are there any other capillary immersions in half-spaces ?

Capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

Examples: spherical caps.

Question

Are there any other capillary immersions in half-spaces ?

[Wente, 1980] An embedded capillary hypersurface in a half-space is a spherical cap (Alexandrov's reflection technique).

Capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

Examples: spherical caps.

Question

Are there any other capillary immersions in half-spaces ?

[Wente, 1980] An embedded capillary hypersurface in a half-space is a spherical cap (Alexandrov's reflection technique).

By the argument of Nitsche, a disk type capillary surface in a half-space in $\mathbb{R}^3$ is a spherical cap.

Capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

Examples: spherical caps.

Question

Are there any other capillary immersions in half-spaces ?

[Wente, 1980] An embedded capillary hypersurface in a half-space is a spherical cap (Alexandrov's reflection technique).

By the argument of Nitsche, a disk type capillary surface in a half-space in $\mathbb{R}^3$ is a spherical cap.

[Marinov, 2012] Stable capillary surface in a half-space in $\mathbb{R}^3$ with embedded boundary $\Rightarrow$ spherical cap.

Capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

Examples: spherical caps.

Question

Are there any other capillary immersions in half-spaces ?

[Wente, 1980] An embedded capillary hypersurface in a half-space is a spherical cap (Alexandrov's reflection technique).

By the argument of Nitsche, a disk type capillary surface in a half-space in $\mathbb{R}^3$ is a spherical cap.

[Marinov, 2012] Stable capillary surface in a half-space in $\mathbb{R}^3$ with embedded boundary $\Rightarrow$ spherical cap.

[Choe-Koiso, 2014] Same result in $\mathbb{R}^3$ and for $n \geq 3$, a stable capillary hypersurface with contact angle $\theta \geq \pi/2$ and convex boundary in a half-space in $\mathbb{R}^{n+1}$ is a spherical cap.

Stable capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

Theorem

Let $\psi : \Sigma \to \mathbb{R}^{n+1}$, $n \geq 2$, *be a stable immersed capillary hypersurface in a half-space in* $\mathbb{R}^{n+1}$ *with contact angle* $0 < \theta \leq \pi/2$.

(i) *If* $\theta = \pi/2$, *then* $\psi(\Sigma)$ *is a hemisphere.*

(ii) *If* $\theta < \pi/2$ *and the restriction of* ψ *to each component of* $\partial\Sigma$ *is an embedding, then* $\psi(\Sigma)$ *is a spherical cap.*

Stable capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

Theorem

Let $\psi : \Sigma \to \mathbb{R}^{n+1}$, $n \geq 2$, be a stable immersed capillary hypersurface in a half-space in $\mathbb{R}^{n+1}$ with contact angle $0 < \theta \leq \pi/2$.

(i) *If $\theta = \pi/2$, then $\psi(\Sigma)$ is a hemisphere.*

(ii) *If $\theta < \pi/2$ and the restriction of ψ to each component of $\partial\Sigma$ is an embedding, then $\psi(\Sigma)$ is a spherical cap.*

We may assume the half-space is $\{x_{n+1} \geq 0\}$. Let $e_{n+1} = (0, \ldots, 0, 1)$.

Lemma

$\int_\Sigma 1 + H\langle \psi, N\rangle + \cos\theta \langle N, e_{n+1}\rangle = 0$

Stable capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

Theorem

Let $\psi : \Sigma \to \mathbb{R}^{n+1}$, $n \geq 2$, be a stable immersed capillary hypersurface in a half-space in $\mathbb{R}^{n+1}$ with contact angle $0 < \theta \leq \pi/2$.

(i) *If $\theta = \pi/2$, then $\psi(\Sigma)$ is a hemisphere.*

(ii) *If $\theta < \pi/2$ and the restriction of ψ to each component of $\partial\Sigma$ is an embedding, then $\psi(\Sigma)$ is a spherical cap.*

We may assume the half-space is $\{x_{n+1} \geq 0\}$. Let $e_{n+1} = (0, \ldots, 0, 1)$.

Lemma

$\int_\Sigma 1 + H\langle \psi, N\rangle + \cos\theta \langle N, e_{n+1}\rangle = 0$

The proof of the lemma combines 2 formulas

Stable capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

Theorem

Let $\psi : \Sigma \to \mathbb{R}^{n+1}$, $n \geq 2$, be a stable immersed capillary hypersurface in a half-space in $\mathbb{R}^{n+1}$ with contact angle $0 < \theta \leq \pi/2$.

(i) *If $\theta = \pi/2$, then $\psi(\Sigma)$ is a hemisphere.*

(ii) *If $\theta < \pi/2$ and the restriction of ψ to each component of $\partial\Sigma$ is an embedding, then $\psi(\Sigma)$ is a spherical cap.*

We may assume the half-space is $\{x_{n+1} \geq 0\}$. Let $e_{n+1} = (0, \ldots, 0, 1)$.

Lemma

$\int_\Sigma 1 + H\langle \psi, N\rangle + \cos\theta \langle N, e_{n+1}\rangle = 0$

The proof of the lemma combines 2 formulas

- A Minkowski formula for hypersurfaces with boundary:

Stable capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

Theorem

Let $\psi : \Sigma \to \mathbb{R}^{n+1}$, $n \geq 2$, be a stable immersed capillary hypersurface in a half-space in $\mathbb{R}^{n+1}$ with contact angle $0 < \theta \leq \pi/2$.

(i) *If $\theta = \pi/2$, then $\psi(\Sigma)$ is a hemisphere.*

(ii) *If $\theta < \pi/2$ and the restriction of ψ to each component of $\partial\Sigma$ is an embedding, then $\psi(\Sigma)$ is a spherical cap.*

We may assume the half-space is $\{x_{n+1} \geq 0\}$. Let $e_{n+1} = (0, \ldots, 0, 1)$.

Lemma

$\int_\Sigma 1 + H\langle \psi, N\rangle + \cos\theta \langle N, e_{n+1}\rangle = 0$

The proof of the lemma combines 2 formulas

- A Minkowski formula for hypersurfaces with boundary:

$\mathrm{div}(\psi - \langle \psi, N\rangle N) = n(1 + H\langle \psi, N\rangle)$

Stable capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

Integrating

$$\int_\Sigma \langle \psi, \nu \rangle = n \int_\Sigma 1 + H \langle \psi, N \rangle$$

Stable capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

Integrating

$$\int_\Sigma \langle \psi, \nu \rangle = n \int_\Sigma 1 + H\langle \psi, N \rangle$$

- A formula for the integral of the unit normal

Integrating

$$\int_\Sigma \langle \psi, \nu \rangle = n \int_\Sigma 1 + H\langle \psi, N \rangle$$

- A formula for the integral of the unit normal

Proposition

Let $\psi : \Sigma \to \mathbb{R}^{n+1}$ be an immersion, Σ compact orientable. Then,

$$n \int_\Sigma N = \int_{\partial\Sigma} \langle \psi, \nu \rangle N - \langle \psi, N \rangle \nu$$

where ν is the outward unit normal to $\partial\Sigma$ in Σ.

Proof of the Proposition: Let $\vec{a}$ be a constant vector field on $\mathbb{R}^{n+1}$. Set

$$X = \langle \vec{a}, N \rangle \psi^T - \langle \psi, N \rangle \vec{a}^T,$$

then

$$\operatorname{div} X = n \langle \vec{a}, N \rangle$$

Integrating gives the result.

Stable capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

Proof of the Proposition: Let $\vec{a}$ be a constant vector field on $\mathbb{R}^{n+1}$. Set

$$X = \langle \vec{a}, N\rangle \psi^T - \langle \psi, N\rangle \vec{a}^T,$$

then

$$\operatorname{div} X = n\langle \vec{a}, N\rangle$$

Integrating gives the result.

On $\partial\Sigma : \cos\theta\, N + \sin\theta\, \nu = -e_{n+1}$, where $e_{n+1} = (0, \ldots, 0, 1)$. So,

$$n \int_\Sigma N = -\frac{1}{\cos\theta} \left(\int_{\partial\Sigma} \langle \psi, \nu\rangle \right) e_{n+1}.$$

Stable capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

We can use $\phi = 1 + H\langle \psi, N\rangle + \cos\theta \langle N, e_{n+1}\rangle$ as a test function. We have

$$\mathcal{I}(\phi,\phi) = -\int_\Sigma \left[|\sigma|^2 - nH^2\right] + (n-1)\sin\theta\cos\theta\left[H\,|\partial\Sigma| + \sin\theta\int_{\partial\Sigma} H_{\partial\Sigma}\right].$$

where $H_{\partial\Sigma}$ is the mean curvature of $\partial\Sigma$ in $\mathbb{R}^n \times \{0\}$ computed with respect to the unit normal $\overline{\nu}$ for which $\{N, \nu\}$ and $\{\overline{N}, \bar{\nu}\}$ have the same orientation (in $(T\partial\Sigma)^\perp$).

Stable capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

We can use $\phi = 1 + H\langle \psi, N\rangle + \cos\theta \langle N, e_{n+1}\rangle$ as a test function. We have

$$\mathcal{I}(\phi,\phi) = -\int_\Sigma \left[|\sigma|^2 - nH^2\right] + (n-1)\sin\theta\cos\theta\left[H\,|\partial\Sigma| + \sin\theta\int_{\partial\Sigma} H_{\partial\Sigma}\right].$$

where $H_{\partial\Sigma}$ is the mean curvature of $\partial\Sigma$ in $\mathbb{R}^n \times \{0\}$ computed with respect to the unit normal $\bar{\nu}$ for which $\{N, \nu\}$ and $\{\overline{N}, \bar{\nu}\}$ have the same orientation (in $(T\partial\Sigma)^\perp$).

By stability $\mathcal{I}(\phi,\phi) \geq 0$.

Stable capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

We can use $\phi = 1 + H\langle \psi, N\rangle + \cos\theta\langle N, e_{n+1}\rangle$ as a test function. We have

$$\mathcal{I}(\phi,\phi) = -\int_\Sigma \left[|\sigma|^2 - nH^2\right] \\ + (n-1)\sin\theta\cos\theta\left[H\,|\partial\Sigma| + \sin\theta\int_{\partial\Sigma} H_{\partial\Sigma}\right].$$

where $H_{\partial\Sigma}$ is the mean curvature of $\partial\Sigma$ in $\mathbb{R}^n \times \{0\}$ computed with respect to the unit normal $\bar{\nu}$ for which $\{N, \nu\}$ and $\{\overline{N}, \bar{\nu}\}$ have the same orientation (in $(T\partial\Sigma)^\perp$).

By stability $\mathcal{I}(\phi,\phi) \geq 0$.
In particular, as $|\sigma|^2 \geq nH^2$, when $\theta = \pi/2$, we get $|\sigma|^2 \equiv nH^2$

Stable capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

We can use $\phi = 1 + H\langle\psi, N\rangle + \cos\theta\langle N, e_{n+1}\rangle$ as a test function. We have

$$\mathcal{I}(\phi,\phi) = -\int_\Sigma \left[|\sigma|^2 - nH^2\right] + (n-1)\sin\theta\cos\theta\left[H\,|\partial\Sigma| + \sin\theta\int_{\partial\Sigma} H_{\partial\Sigma}\right].$$

where $H_{\partial\Sigma}$ is the mean curvature of $\partial\Sigma$ in $\mathbb{R}^n \times \{0\}$ computed with respect to the unit normal $\overline{\nu}$ for which $\{N,\nu\}$ and $\{\overline{N},\bar{\nu}\}$ have the same orientation (in $(T\partial\Sigma)^\perp$).

By stability $\mathcal{I}(\phi,\phi) \geq 0$.
In particular, as $|\sigma|^2 \geq nH^2$, when $\theta = \pi/2$, we get $|\sigma|^2 \equiv nH^2$
$\Rightarrow \psi(\Sigma)$ is spherical.

Stable capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

We can use $\phi = 1 + H\langle\psi, N\rangle + \cos\theta\langle N, e_{n+1}\rangle$ as a test function. We have

$$\mathcal{I}(\phi,\phi) = -\int_\Sigma \left[|\sigma|^2 - nH^2\right] + (n-1)\sin\theta\cos\theta\left[H\,|\partial\Sigma| + \sin\theta\int_{\partial\Sigma} H_{\partial\Sigma}\right].$$

where $H_{\partial\Sigma}$ is the mean curvature of $\partial\Sigma$ in $\mathbb{R}^n \times \{0\}$ computed with respect to the unit normal $\bar{\nu}$ for which $\{N, \nu\}$ and $\{\overline{N}, \bar{\nu}\}$ have the same orientation (in $(T\partial\Sigma)^\perp$).

By stability $\mathcal{I}(\phi,\phi) \geq 0$.
In particular, as $|\sigma|^2 \geq nH^2$, when $\theta = \pi/2$, we get $|\sigma|^2 \equiv nH^2$ $\Rightarrow \psi(\Sigma)$ is spherical.

For $\theta < \pi/2$, we will show $v := \langle N, e_{n+1}\rangle$ does not change sign on Σ.

Stable capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

We can use $\phi = 1 + H\langle \psi, N\rangle + \cos\theta\langle N, e_{n+1}\rangle$ as a test function.
We have

$$\mathcal{I}(\phi,\phi) = -\int_\Sigma \left[|\sigma|^2 - nH^2\right] + (n-1)\sin\theta\cos\theta\left[H\,|\partial\Sigma| + \sin\theta\int_{\partial\Sigma} H_{\partial\Sigma}\right].$$

where $H_{\partial\Sigma}$ is the mean curvature of $\partial\Sigma$ in $\mathbb{R}^n \times \{0\}$ computed with respect to the unit normal $\bar{\nu}$ for which $\{N, \nu\}$ and $\{\overline{N}, \bar{\nu}\}$ have the same orientation (in $(T\partial\Sigma)^\perp$).

By stability $\mathcal{I}(\phi,\phi) \geq 0$.
In particular, as $|\sigma|^2 \geq nH^2$, when $\theta = \pi/2$, we get $|\sigma|^2 \equiv nH^2$
$\Rightarrow \psi(\Sigma)$ is spherical.

For $\theta < \pi/2$, we will show $v := \langle N, e_{n+1}\rangle$ does not change sign on Σ.

By contradiction, if not, there exists a real α so that $w := v_- + \alpha v_+$ satisfies $\int_\Sigma w = 0$.

Use w as a test function.

Stable capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

Use w as a test function. We have

$$\mathcal{I}(w, w) = -nH \cot\theta \, |\partial\Sigma| - (n-1)\cos\theta \int_{\partial\Sigma} H_{\partial\Sigma}.$$

Stable capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

Use w as a test function. We have

$$\mathcal{I}(w,w) = -nH\,\cot\theta\,|\partial\Sigma| - (n-1)\cos\theta\int_{\partial\Sigma} H_{\partial\Sigma}.$$

$$\begin{aligned}\mathcal{I}(\phi,\phi) = &-\int_{\Sigma}\left[|\sigma|^2 - nH^2\right] \\ &+ (n-1)\sin\theta\cos\theta\left[H\,|\partial\Sigma| + \sin\theta\int_{\partial\Sigma} H_{\partial\Sigma}\right].\end{aligned}$$

Stable capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

Use w as a test function. We have

$$\mathcal{I}(w, w) = -nH \cot\theta \, |\partial\Sigma| - (n-1)\cos\theta \int_{\partial\Sigma} H_{\partial\Sigma}.$$

$$\begin{aligned}\mathcal{I}(\phi, \phi) = &- \int_{\Sigma} \left[|\sigma|^2 - nH^2\right] \\ &+ (n-1)\sin\theta\cos\theta \left[H\,|\partial\Sigma| + \sin\theta \int_{\partial\Sigma} H_{\partial\Sigma}\right].\end{aligned}$$

Stability $\Longrightarrow \mathcal{I}(\phi, \phi) + \sin^2\theta \, \mathcal{I}(w, w) \geq 0.$

So

$$-\int_\Sigma \left[|\sigma|^2 - nH^2\right] - \sin\theta\cos\theta\, H|\partial\Sigma| \geq 0.$$

So

$$-\int_{\Sigma}\left[|\sigma|^2 - nH^2\right] - \sin\theta\cos\theta\, H|\partial\Sigma| \geq 0.$$

But $|\sigma|^2 \geq nH^2$, $\theta < \pi/2$ and $H > 0$, a contradiction.

Stable capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

So

$$-\int_\Sigma \left[|\sigma|^2 - nH^2\right] - \sin\theta\cos\theta\, H|\partial\Sigma| \geq 0.$$

But $|\sigma|^2 \geq nH^2$, $\theta < \pi/2$ and $H > 0$, a contradiction.

Conclusion: v does not change sign on Σ.

Stable capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

So

$$-\int_\Sigma \left[|\sigma|^2 - nH^2\right] - \sin\theta\cos\theta\, H|\partial\Sigma| \geq 0.$$

But $|\sigma|^2 \geq nH^2$, $\theta < \pi/2$ and $H > 0$, a contradiction.

Conclusion: v does not change sign on Σ.

On $\partial\Sigma$, $v = -\cos\theta < 0$. So $v \leq 0$ on Σ.

Stable capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

So

$$-\int_\Sigma \left[|\sigma|^2 - nH^2\right] - \sin\theta\cos\theta\, H|\partial\Sigma| \geq 0.$$

But $|\sigma|^2 \geq nH^2$, $\theta < \pi/2$ and $H > 0$, a contradiction.

Conclusion: v does not change sign on Σ.

On $\partial\Sigma$, $v = -\cos\theta < 0$. So $v \leq 0$ on Σ.

We have $\Delta v = -|\sigma|^2 v \geq 0$.

Stable capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

So

$$-\int_\Sigma \left[|\sigma|^2 - nH^2\right] - \sin\theta\cos\theta\, H|\partial\Sigma| \geq 0.$$

But $|\sigma|^2 \geq nH^2$, $\theta < \pi/2$ and $H > 0$, a contradiction.

Conclusion: v does not change sign on Σ.

On $\partial\Sigma$, $v = -\cos\theta < 0$. So $v \leq 0$ on Σ.

We have $\Delta v = -|\sigma|^2 v \geq 0$. Maximum principle $\Rightarrow$ $v < 0$ on Σ.

Stable capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

So

$$-\int_\Sigma \left[|\sigma|^2 - nH^2\right] - \sin\theta\cos\theta\, H|\partial\Sigma| \geq 0.$$

But $|\sigma|^2 \geq nH^2$, $\theta < \pi/2$ and $H > 0$, a contradiction.

Conclusion: v does not change sign on Σ.

On $\partial\Sigma$, $v = -\cos\theta < 0$. So $v \leq 0$ on Σ.

We have $\Delta v = -|\sigma|^2 v \geq 0$. Maximum principle $\Rightarrow$ $v < 0$ on Σ.

$\Longrightarrow \psi(\Sigma)$ is a local vertical graph at each point.

Stable capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

So

$$-\int_\Sigma \left[|\sigma|^2 - nH^2\right] - \sin\theta\cos\theta\, H|\partial\Sigma| \geq 0.$$

But $|\sigma|^2 \geq nH^2$, $\theta < \pi/2$ and $H > 0$, a contradiction.

Conclusion: v does not change sign on Σ.

On $\partial\Sigma$, $v = -\cos\theta < 0$. So $v \leq 0$ on Σ.

We have $\Delta v = -|\sigma|^2 v \geq 0$. Maximum principle $\Rightarrow v < 0$ on Σ.

$\Longrightarrow \psi(\Sigma)$ is a local vertical graph at each point.

Set $\partial\Sigma = \Gamma_1 \cup \ldots \cup \Gamma_k$, each Γ_i connected.

Stable capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

So

$$-\int_{\Sigma}\left[|\sigma|^2 - nH^2\right] - \sin\theta\cos\theta\, H|\partial\Sigma| \geq 0.$$

But $|\sigma|^2 \geq nH^2$, $\theta < \pi/2$ and $H > 0$, a contradiction.

Conclusion: v does not change sign on Σ.

On $\partial\Sigma$, $v = -\cos\theta < 0$. So $v \leq 0$ on Σ.

We have $\Delta v = -|\sigma|^2 v \geq 0$. Maximum principle $\Rightarrow$ $v < 0$ on Σ.

$\Longrightarrow \psi(\Sigma)$ is a local vertical graph at each point.

Set $\partial\Sigma = \Gamma_1 \cup \ldots \cup \Gamma_k$, each Γ_i connected.

As ψ restricted to Γ_I is an embedding, $\psi(\Gamma_i)$ separates $\mathbb{R}^n \times \{0\}$ into 2 components.

Stable capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

So

$$-\int_\Sigma \left[|\sigma|^2 - nH^2\right] - \sin\theta\cos\theta\, H|\partial\Sigma| \geq 0.$$

But $|\sigma|^2 \geq nH^2$, $\theta < \pi/2$ and $H > 0$, a contradiction.

Conclusion: v does not change sign on Σ.

On $\partial\Sigma$, $v = -\cos\theta < 0$. So $v \leq 0$ on Σ.

We have $\Delta v = -|\sigma|^2 v \geq 0$. Maximum principle $\Rightarrow v < 0$ on Σ.

$\Longrightarrow \psi(\Sigma)$ is a local vertical graph at each point.

Set $\partial\Sigma = \Gamma_1 \cup \ldots \cup \Gamma_k$, each Γ_i connected.

As ψ restricted to Γ_I is an embedding, $\psi(\Gamma_i)$ separates $\mathbb{R}^n \times \{0\}$ into 2 components. Call D_i the component onto which $\psi(\Sigma)$ does not project near Γ_i.

Stable capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

Consider $\widetilde{\Sigma}$ obtained by gluing D_i to Σ along Γ_i, for $i = 1, \dots, k$.

Let P be the orthogonal projection onto $\mathbb{R}^n \times \{0\}$.

Set $F = P \circ \psi$.

Stable capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

Consider $\widetilde{\Sigma}$ obtained by gluing D_i to Σ along Γ_i, for $i = 1, \ldots, k$.

Let P be the orthogonal projection onto $\mathbb{R}^n \times \{0\}$.

Set $F = P \circ \psi$.

Define $\widetilde{F} : \widetilde{\Sigma} \to \mathbb{R}^n \times \{0\}$ by

$$\widetilde{F} = \begin{cases} F & \text{on} \quad \Sigma \\ \text{identity} & \text{on} \quad D_i, i = 1, \ldots, k. \end{cases}$$

Stable capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

Consider $\widetilde{\Sigma}$ obtained by gluing D_i to Σ along Γ_i, for $i = 1, \ldots, k$.
Let P be the orthogonal projection onto $\mathbb{R}^n \times \{0\}$.
Set $F = P \circ \psi$.
Define $\widetilde{F} : \widetilde{\Sigma} \to \mathbb{R}^n \times \{0\}$ by

$$\widetilde{F} = \begin{cases} F & \text{on} \quad \Sigma \\ \text{identity} & \text{on} \quad D_i, i = 1, \ldots, k. \end{cases}$$

$\widetilde{F}$ is a proper map and is local homeomorphism $\Rightarrow \widetilde{F}$ is a covering $\Rightarrow \widetilde{F}$ is a global homeomorphism.

Stable capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

Consider $\widetilde{\Sigma}$ obtained by gluing D_i to Σ along Γ_i, for $i = 1, \ldots, k$.

Let P be the orthogonal projection onto $\mathbb{R}^n \times \{0\}$.

Set $F = P \circ \psi$.

Define $\widetilde{F} : \widetilde{\Sigma} \to \mathbb{R}^n \times \{0\}$ by

$$\widetilde{F} = \begin{cases} F & \text{on} \quad \Sigma \\ \text{identity} & \text{on} \quad D_i, i = 1, \ldots, k. \end{cases}$$

$\widetilde{F}$ is a proper map and is local homeomorphism $\Rightarrow$ $\widetilde{F}$ is a covering $\Rightarrow$ $\widetilde{F}$ is a global homeomorphism.

$\Rightarrow$ $\psi(\Sigma)$ is a global graph

Stable capillary hypersurfaces in a half-space in $\mathbb{R}^{n+1}$

Consider $\widetilde{\Sigma}$ obtained by gluing D_i to Σ along Γ_i, for $i = 1, \ldots, k$.

Let P be the orthogonal projection onto $\mathbb{R}^n \times \{0\}$.

Set $F = P \circ \psi$.

Define $\widetilde{F} : \widetilde{\Sigma} \to \mathbb{R}^n \times \{0\}$ by

$$\widetilde{F} = \begin{cases} F & \text{on} \quad \Sigma \\ \text{identity} & \text{on} \quad D_i, i = 1, \ldots, k. \end{cases}$$

$\widetilde{F}$ is a proper map and is local homeomorphism $\Rightarrow$ $\widetilde{F}$ is a covering
$\Rightarrow$ $\widetilde{F}$ is a global homeomorphism.

$\Rightarrow$ $\psi(\Sigma)$ is a global graph

$\Rightarrow$ $\psi(\Sigma)$ is a spherical cap (Wente).

Capillary hypersurfaces in a slab in $\mathbb{R}^3$

Examples: cylinders, unduloids, nodoids, catenoids

Capillary hypersurfaces in a slab in $\mathbb{R}^3$

Examples: cylinders, unduloids, nodoids, catenoids

[Wente, 1980] An embedded capillary hypersurface in in a slab in $\mathbb{R}^{n+1}$ is rotationally invariant, i.e a spherical cap or a slice of a Delaunay hypersurface (Alexandrov's reflection technique).

Capillary hypersurfaces in a slab in $\mathbb{R}^3$

Examples: cylinders, unduloids, nodoids, catenoids

[Wente, 1980] An embedded capillary hypersurface in in a slab in $\mathbb{R}^{n+1}$ is rotationally invariant, i.e a spherical cap or a slice of a Delaunay hypersurface (Alexandrov's reflection technique).

[Wente, 1993] There exist capillary annuli in a slab in $\mathbb{R}^3$ which are immersed and non-rotational.

Capillary hypersurfaces in a slab in $\mathbb{R}^3$

Examples: cylinders, unduloids, nodoids, catenoids

[Wente, 1980] An embedded capillary hypersurface in in a slab in $\mathbb{R}^{n+1}$ is rotationally invariant, i.e a spherical cap or a slice of a Delaunay hypersurface (Alexandrov's reflection technique).

[Wente, 1993] There exist capillary annuli in a slab in $\mathbb{R}^3$ which are immersed and non-rotational.

[Vogel 1989, Finn-Vogel 1993, Zhou 1997] studied stability of rotationally invariant capillary surfaces between 2 parallel planes in $\mathbb{R}^3$

Capillary hypersurfaces in a slab in $\mathbb{R}^3$

Examples: cylinders, unduloids, nodoids, catenoids

[Wente, 1980] An embedded capillary hypersurface in in a slab in $\mathbb{R}^{n+1}$ is rotationally invariant, i.e a spherical cap or a slice of a Delaunay hypersurface (Alexandrov's reflection technique).

[Wente, 1993] There exist capillary annuli in a slab in $\mathbb{R}^3$ which are immersed and non-rotational.

[Vogel 1989, Finn-Vogel 1993, Zhou 1997] studied stability of rotationally invariant capillary surfaces between 2 parallel planes in $\mathbb{R}^3+$ [Fel-Rubinstein, 2015]

Capillary hypersurfaces in a slab in $\mathbb{R}^3$

Examples: cylinders, unduloids, nodoids, catenoids

[Wente, 1980] An embedded capillary hypersurface in in a slab in $\mathbb{R}^{n+1}$ is rotationally invariant, i.e a spherical cap or a slice of a Delaunay hypersurface (Alexandrov's reflection technique).

[Wente, 1993] There exist capillary annuli in a slab in $\mathbb{R}^3$ which are immersed and non-rotational.

[Vogel 1989, Finn-Vogel 1993, Zhou 1997] studied stability of rotationally invariant capillary surfaces between 2 parallel planes in $\mathbb{R}^3+$ [Fel-Rubinstein, 2015]

[Ros, 2007] stable capillary in a slab $\subset \mathbb{R}^3$, contact angle $\theta = \pi/2 \Rightarrow$ right circular cylinder.

Stable capillary surfaces in a slab in $\mathbb{R}^3$

Consider the slab $\mathcal{B} = \{0 \leq x_3 \leq 1\} \subset \mathbb{R}^3$, $\partial\mathcal{B} = \Pi_0 \cup \Pi_1$.

Theorem

Let $\psi : \Sigma \to \mathcal{B}$, a capillary immersion of a surface Σ of genus 0 making contact angles θ_0 and θ_1 with Π_0 and Π_1, respectively. If ψ is stable, then $\psi(\Sigma)$ is a surface of revolution.

Stable capillary surfaces in a slab in $\mathbb{R}^3$

Proof: Let γ be a component of $\partial\Sigma$, with $\psi(\gamma) \subset \Pi_0$.

Stable capillary surfaces in a slab in $\mathbb{R}^3$

Proof: Let γ be a component of $\partial\Sigma$, with $\psi(\gamma) \subset \Pi_0$.

Consider $\mathcal{C}$: circumscribed circle in Π_0 about $\psi(\gamma)$, can be assumed to have center at the origin.

Stable capillary surfaces in a slab in $\mathbb{R}^3$

Proof: Let γ be a component of $\partial\Sigma$, with $\psi(\gamma) \subset \Pi_0$.

Consider $\mathcal{C}$: circumscribed circle in Π_0 about $\psi(\gamma)$, can be assumed to have center at the origin.

u : Jacobi function associated to rotations around the x_3−axis, that is,

$$p \in \Sigma, \quad u(p) = \langle \psi(p) \wedge e_3, N(p) \rangle$$

Stable capillary surfaces in a slab in $\mathbb{R}^3$

Proof: Let γ be a component of $\partial\Sigma$, with $\psi(\gamma) \subset \Pi_0$.

Consider $\mathcal{C}$: circumscribed circle in Π_0 about $\psi(\gamma)$, can be assumed to have center at the origin.

u : Jacobi function associated to rotations around the x_3-axis, that is,

$$p \in \Sigma, \quad u(p) = \langle \psi(p) \wedge e_3, N(p) \rangle$$

u verifies

$$\Delta u + |\sigma|^2 u = 0 \qquad on \ \Sigma$$
$$\frac{\partial u}{\partial \nu} = q\, u \qquad \text{on } \partial\Sigma$$

Stable capillary surfaces in a slab in $\mathbb{R}^3$

Proof: Let γ be a component of $\partial\Sigma$, with $\psi(\gamma) \subset \Pi_0$.

Consider $\mathcal{C}$: circumscribed circle in Π_0 about $\psi(\gamma)$, can be assumed to have center at the origin.

u : Jacobi function associated to rotations around the x_3-axis, that is,

$$p \in \Sigma, \quad u(p) = \langle \psi(p) \wedge e_3, N(p) \rangle$$

u verifies

$$\Delta u + |\sigma|^2 u = 0 \qquad on \;\; \Sigma$$
$$\frac{\partial u}{\partial \nu} = q\, u \qquad \text{on} \;\; \partial\Sigma$$

Aim: prove that $u \equiv 0$

Stable capillary surfaces in a slab in $\mathbb{R}^3$

A known fact: $\mathcal{C} \cap \psi(\gamma)$ contains at least 2 points.

Stable capillary surfaces in a slab in $\mathbb{R}^3$

A known fact: $\mathcal{C} \cap \psi(\gamma)$ contains at least 2 points.

At each of these points p, we have : $u(p) = \frac{\partial u}{\partial \nu}(p) = 0$.

Stable capillary surfaces in a slab in $\mathbb{R}^3$

A known fact: $\mathcal{C} \cap \psi(\gamma)$ contains at least 2 points.

At each of these points p, we have : $u(p) = \frac{\partial u}{\partial \nu}(p) = 0$.

A closest point on $\psi(\gamma)$ to the origin gives another such point p.

Stable capillary surfaces in a slab in $\mathbb{R}^3$

A known fact: $\mathcal{C} \cap \psi(\gamma)$ contains at least 2 points.

At each of these points p, we have : $u(p) = \frac{\partial u}{\partial \nu}(p) = 0$.

A closest point on $\psi(\gamma)$ to the origin gives another such point p.

Maximum principle $\Rightarrow$ u changes sign in any neighborhood of p unless $u \equiv 0$ in a neighborhood of p and so $u \equiv 0$ on Σ (unique continuation principle),

Stable capillary surfaces in a slab in $\mathbb{R}^3$

A known fact: $\mathcal{C} \cap \psi(\gamma)$ contains at least 2 points.

At each of these points p, we have : $u(p) = \frac{\partial u}{\partial \nu}(p) = 0$.

A closest point on $\psi(\gamma)$ to the origin gives another such point p.

Maximum principle $\Rightarrow$ u changes sign in any neighborhood of p unless $u \equiv 0$ in a neighborhood of p and so $u \equiv 0$ on Σ (unique continuation principle),

So, assume $u \not\equiv 0$, then $p \in$ the boundary of at least 2 nodal domains of u,

Stable capillary surfaces in a slab in $\mathbb{R}^3$

A known fact: $\mathcal{C} \cap \psi(\gamma)$ contains at least 2 points.

At each of these points p, we have : $u(p) = \frac{\partial u}{\partial \nu}(p) = 0$.

A closest point on $\psi(\gamma)$ to the origin gives another such point p.

Maximum principle $\Rightarrow$ u changes sign in any neighborhood of p unless $u \equiv 0$ in a neighborhood of p and so $u \equiv 0$ on Σ (unique continuation principle),

So, assume $u \not\equiv 0$, then $p \in$ the boundary of at least 2 nodal domains of u,

Σ has genus 0 $\Longrightarrow$ u has at least 3 nodal domains $\Sigma_1, \Sigma_2, \ldots$.

Stable capillary surfaces in a slab in $\mathbb{R}^3$

A known fact: $\mathcal{C} \cap \psi(\gamma)$ contains at least 2 points.

At each of these points p, we have : $u(p) = \frac{\partial u}{\partial \nu}(p) = 0$.

A closest point on $\psi(\gamma)$ to the origin gives another such point p.

Maximum principle $\Rightarrow$ u changes sign in any neighborhood of p unless $u \equiv 0$ in a neighborhood of p and so $u \equiv 0$ on Σ (unique continuation principle),

So, assume $u \not\equiv 0$, then $p \in$ the boundary of at least 2 nodal domains of u,

Σ has genus 0 $\Longrightarrow$ u has at least 3 nodal domains $\Sigma_1, \Sigma_2, \ldots$.

Define $\widetilde{u} \in H^1(\Sigma)$ by

$$\widetilde{u} = \begin{cases} u & \text{on} \quad \Sigma_1 \\ \alpha\, u & \text{on} \quad \Sigma_2 \\ 0 & \text{on} \quad \Sigma \setminus (\Sigma_1 \cup \Sigma_2) \end{cases}$$

α chosen so that $\int_\Sigma \widetilde{u} = 0$.

Stable capillary surfaces in a slab in $\mathbb{R}^3$

We have: $\mathcal{I}(\widetilde{u}, \widetilde{u}) = 0$.

Stable capillary surfaces in a slab in $\mathbb{R}^3$

We have: $\mathcal{I}(\widetilde{u}, \widetilde{u}) = 0$.

ψ is stable $\Longrightarrow$ $\widetilde{u}$ lies in the kernel of $\mathcal{I}$, i.e $\widetilde{u}$ is a Jacobi function

Stable capillary surfaces in a slab in $\mathbb{R}^3$

We have: $\mathcal{I}(\widetilde{u}, \widetilde{u}) = 0$.

ψ is stable $\Longrightarrow$ $\widetilde{u}$ lies in the kernel of $\mathcal{I}$, i.e $\widetilde{u}$ is a Jacobi function

but, $\widetilde{u}$ vanishes on a non empty open set , the unique continuation principle $\Longrightarrow \widetilde{u} \equiv 0 \Longrightarrow u \equiv 0$, a contradiction.

Stable capillary surfaces in a slab in $\mathbb{R}^3$

We have: $\mathcal{I}(\widetilde{u}, \widetilde{u}) = 0$.

ψ is stable $\Longrightarrow$ $\widetilde{u}$ lies in the kernel of $\mathcal{I}$, i.e $\widetilde{u}$ is a Jacobi function

but, $\widetilde{u}$ vanishes on a non empty open set , the unique continuation principle $\Longrightarrow \widetilde{u} \equiv 0 \Longrightarrow u \equiv 0$, a contradiction.

Conclusion: $u \equiv 0$, that is, $\psi(\Sigma)$ is invariant under rotations around x_3-axis.

Case of a slab in $\mathbb{R}^{n+1}$ and $\theta = \pi/2$

[Ros, 2007] stable capillary in a slab $\subset \mathbb{R}^3$, contact angle $\theta = \pi/2 \Rightarrow$ right circular cylinder.

Case of a slab in $\mathbb{R}^{n+1}$ and $\theta = \pi/2$

[Ros, 2007] stable capillary in a slab $\subset \mathbb{R}^3$, contact angle $\theta = \pi/2 \Rightarrow$ right circular cylinder.

Stability of embedded rotationally invariant capillary hypersurfaces in slices in $\mathbb{R}^{n+1}$with contact angle $\theta = \pi/2$ was studied by [Athanassenas, Vogel ,1987] for $n = 2$ and by [Pedrosa-Ritoré, 1999] for any n.

Case of a slab in $\mathbb{R}^{n+1}$ and $\theta = \pi/2$

[Ros, 2007] stable capillary in a slab $\subset \mathbb{R}^3$, contact angle $\theta = \pi/2 \Rightarrow$ right circular cylinder.

Stability of embedded rotationally invariant capillary hypersurfaces in slices in $\mathbb{R}^{n+1}$with contact angle $\theta = \pi/2$ was studied by [Athanassenas, Vogel ,1987] for $n = 2$ and by [Pedrosa-Ritoré, 1999] for any n.

For $2 \leq n \leq 7$ only circular cylinders can be stable.

Case of a slab in $\mathbb{R}^{n+1}$ and $\theta = \pi/2$

[Ros, 2007] stable capillary in a slab $\subset \mathbb{R}^3$, contact angle $\theta = \pi/2 \Rightarrow$ right circular cylinder.

Stability of embedded rotationally invariant capillary hypersurfaces in slices in $\mathbb{R}^{n+1}$with contact angle $\theta = \pi/2$ was studied by [Athanassenas, Vogel ,1987] for $n = 2$ and by [Pedrosa-Ritoré, 1999] for any n.

For $2 \leq n \leq 7$ only circular cylinders can be stable.

For $n \geq 9$ there exist unduloids which are stable.

Case of a slab in $\mathbb{R}^{n+1}$ and $\theta = \pi/2$

[Ros, 2007] stable capillary in a slab $\subset \mathbb{R}^3$, contact angle $\theta = \pi/2 \Rightarrow$ right circular cylinder.

Stability of embedded rotationally invariant capillary hypersurfaces in slices in $\mathbb{R}^{n+1}$with contact angle $\theta = \pi/2$ was studied by [Athanassenas, Vogel ,1987] for $n = 2$ and by [Pedrosa-Ritoré, 1999] for any n.

For $2 \leq n \leq 7$ only circular cylinders can be stable.

For $n \geq 9$ there exist unduloids which are stable.

The case $n = 8$ is open.

Case of a slab in $\mathbb{R}^{n+1}$ and $\theta = \pi/2$

[Ros, 2007] stable capillary in a slab $\subset \mathbb{R}^3$, contact angle $\theta = \pi/2 \Rightarrow$ right circular cylinder.

Stability of embedded rotationally invariant capillary hypersurfaces in slices in $\mathbb{R}^{n+1}$with contact angle $\theta = \pi/2$ was studied by [Athanassenas, Vogel ,1987] for $n = 2$ and by [Pedrosa-Ritoré, 1999] for any n.

For $2 \leq n \leq 7$ only circular cylinders can be stable.

For $n \geq 9$ there exist unduloids which are stable.

The case $n = 8$ is open.

Theorem

Let $\psi : \Sigma \to \mathbb{R}^{n+1}$, $n \geq 2$, be an immersed capillary hypersurface connecting two horizontal hyperplanes in $\mathbb{R}^{n+1}$ with contact angle $\theta = \pi/2$. Suppose that the restriction of ψ to each component of $\partial\Sigma$ is an embedding.
If ψ is stable then $\psi(\Sigma)$ is either a circular vertical cylinder or a vertical graph which is rotationally invariant around a vertical axis.

Case of a slab in $\mathbb{R}^{n+1}$ and $\theta = \pi/2$

Proof: Set $v := \langle N, e_{n+1} \rangle$, where $e_{n+1} = (0, \ldots, 1)$. Then, $v \equiv 0$ on $\partial\Sigma$.

Case of a slab in $\mathbb{R}^{n+1}$ and $\theta = \pi/2$

Proof: Set $v := \langle N, e_{n+1} \rangle$, where $e_{n+1} = (0, \ldots, 1)$. Then, $v \equiv 0$ on $\partial\Sigma$.

If $v \equiv 0$ then Σ is a circular vertical cylinder.

Case of a slab in $\mathbb{R}^{n+1}$ and $\theta = \pi/2$

Proof: Set $v := \langle N, e_{n+1} \rangle$, where $e_{n+1} = (0, \dots, 1)$. Then, $v \equiv 0$ on $\partial\Sigma$.

If $v \equiv 0$ then Σ is a circular vertical cylinder.

Suppose $v \not\equiv 0$, we will show it has a sign inside Σ.

Case of a slab in $\mathbb{R}^{n+1}$ and $\theta = \pi/2$

Proof: Set $v := \langle N, e_{n+1} \rangle$, where $e_{n+1} = (0, \ldots, 1)$. Then, $v \equiv 0$ on $\partial\Sigma$.

If $v \equiv 0$ then Σ is a circular vertical cylinder.

Suppose $v \not\equiv 0$, we will show it has a sign inside Σ.

By contradiction, if v changes sign, take $\alpha \in \mathbb{R}$ so that $\widetilde{v} := v_+ + \alpha v_-$ satisfies $\int_\Sigma \widetilde{v} = 0$.

Case of a slab in $\mathbb{R}^{n+1}$ and $\theta = \pi/2$

Proof: Set $v := \langle N, e_{n+1} \rangle$, where $e_{n+1} = (0, \ldots, 1)$. Then, $v \equiv 0$ on $\partial\Sigma$.

If $v \equiv 0$ then Σ is a circular vertical cylinder.

Suppose $v \not\equiv 0$, we will show it has a sign inside Σ.

By contradiction, if v changes sign, take $\alpha \in \mathbb{R}$ so that $\widetilde{v} := v_+ + \alpha v_-$ satisfies $\int_\Sigma \widetilde{v} = 0$.

Then (using that $v \equiv 0$ on $\partial\Sigma$) we have: $\mathcal{I}(\widetilde{v}, \widetilde{v}) = 0$.

Case of a slab in $\mathbb{R}^{n+1}$ and $\theta = \pi/2$

Proof: Set $v := \langle N, e_{n+1} \rangle$, where $e_{n+1} = (0, \ldots, 1)$. Then, $v \equiv 0$ on $\partial\Sigma$.

If $v \equiv 0$ then Σ is a circular vertical cylinder.

Suppose $v \not\equiv 0$, we will show it has a sign inside Σ.

By contradiction, if v changes sign, take $\alpha \in \mathbb{R}$ so that $\widetilde{v} := v_+ + \alpha v_-$ satisfies $\int_\Sigma \widetilde{v} = 0$.

Then (using that $v \equiv 0$ on $\partial\Sigma$) we have: $\mathcal{I}(\widetilde{v}, \widetilde{v}) = 0$. Stability

$\Longrightarrow$ $\widetilde{v}$ is a Jacobi function.

Case of a slab in $\mathbb{R}^{n+1}$ and $\theta = \pi/2$

Proof: Set $v := \langle N, e_{n+1} \rangle$, where $e_{n+1} = (0, \ldots, 1)$. Then, $v \equiv 0$ on $\partial\Sigma$.

If $v \equiv 0$ then Σ is a circular vertical cylinder.

Suppose $v \not\equiv 0$, we will show it has a sign inside Σ.

By contradiction, if v changes sign, take $\alpha \in \mathbb{R}$ so that $\widetilde{v} := v_+ + \alpha v_-$ satisfies $\int_\Sigma \widetilde{v} = 0$.

Then (using that $v \equiv 0$ on $\partial\Sigma$) we have: $\mathcal{I}(\widetilde{v}, \widetilde{v}) = 0$. Stability $\Longrightarrow$ $\widetilde{v}$ is a Jacobi function. So $\frac{\partial \widetilde{v}}{\partial \nu} = q\widetilde{v} = 0$ on $\partial\Sigma$.

Case of a slab in $\mathbb{R}^{n+1}$ and $\theta = \pi/2$

Proof: Set $v := \langle N, e_{n+1} \rangle$, where $e_{n+1} = (0, \ldots, 1)$. Then, $v \equiv 0$ on $\partial\Sigma$.

If $v \equiv 0$ then Σ is a circular vertical cylinder.

Suppose $v \not\equiv 0$, we will show it has a sign inside Σ.

By contradiction, if v changes sign, take $\alpha \in \mathbb{R}$ so that $\widetilde{v} := v_+ + \alpha v_-$ satisfies $\int_\Sigma \widetilde{v} = 0$.

Then (using that $v \equiv 0$ on $\partial\Sigma$) we have: $\mathcal{I}(\widetilde{v}, \widetilde{v}) = 0$. Stability

$\Longrightarrow$ $\widetilde{v}$ is a Jacobi function. So $\frac{\partial \widetilde{v}}{\partial \nu} = q\widetilde{v} = 0$ on $\partial\Sigma$.

$\psi(\Sigma)$ extends analytically by reflection across Π_1 and Π_2. Uniqueness in Cauchy-Kowalevski's theorem $\Rightarrow \widetilde{v} \equiv 0$, i.e $v \equiv 0$ in a neighborhood of $\partial\Sigma$

Case of a slab in $\mathbb{R}^{n+1}$ and $\theta = \pi/2$

Proof: Set $v := \langle N, e_{n+1} \rangle$, where $e_{n+1} = (0, \ldots, 1)$. Then, $v \equiv 0$ on $\partial\Sigma$.

If $v \equiv 0$ then Σ is a circular vertical cylinder.

Suppose $v \not\equiv 0$, we will show it has a sign inside Σ.

By contradiction, if v changes sign, take $\alpha \in \mathbb{R}$ so that $\widetilde{v} := v_+ + \alpha v_-$ satisfies $\int_\Sigma \widetilde{v} = 0$.

Then (using that $v \equiv 0$ on $\partial\Sigma$) we have: $\mathcal{I}(\widetilde{v}, \widetilde{v}) = 0$. Stability

$\Longrightarrow$ $\widetilde{v}$ is a Jacobi function. So $\frac{\partial \widetilde{v}}{\partial \nu} = q\widetilde{v} = 0$ on $\partial\Sigma$.

$\psi(\Sigma)$ extends analytically by reflection across Π_1 and Π_2. Uniqueness in Cauchy-Kowalevski's theorem $\Rightarrow \widetilde{v} \equiv 0$, i.e $v \equiv 0$ in a neighborhood of $\partial\Sigma \Rightarrow v \equiv 0$ on Σ, contradiction.

Case of a slab in $\mathbb{R}^{n+1}$ and $\theta = \pi/2$

Proof: Set $v := \langle N, e_{n+1} \rangle$, where $e_{n+1} = (0, \ldots, 1)$. Then, $v \equiv 0$ on $\partial\Sigma$.

If $v \equiv 0$ then Σ is a circular vertical cylinder.

Suppose $v \not\equiv 0$, we will show it has a sign inside Σ.

By contradiction, if v changes sign, take $\alpha \in \mathbb{R}$ so that $\widetilde{v} := v_+ + \alpha v_-$ satisfies $\int_\Sigma \widetilde{v} = 0$.

Then (using that $v \equiv 0$ on $\partial\Sigma$) we have: $\mathcal{I}(\widetilde{v}, \widetilde{v}) = 0$. Stability

$\implies \widetilde{v}$ is a Jacobi function. So $\frac{\partial \widetilde{v}}{\partial \nu} = q\widetilde{v} = 0$ on $\partial\Sigma$.

$\psi(\Sigma)$ extends analytically by reflection across Π_1 and Π_2. Uniqueness in Cauchy-Kowalevski's theorem $\Rightarrow \widetilde{v} \equiv 0$, i.e $v \equiv 0$ in a neighborhood of $\partial\Sigma \Rightarrow v \equiv 0$ on Σ, contradiction. So v doesn't change sign inside Σ, say $v \geq 0$.

Case of a slab in $\mathbb{R}^{n+1}$ and $\theta = \pi/2$

$$\begin{aligned} v &\geq 0, \\ \Delta v &= -|\sigma|^2 v \leq 0, \\ v &= 0 \quad \text{on} \quad \partial\Sigma. \end{aligned}$$

Maximum principle $\Rightarrow v > 0$ on $\text{int}(\Sigma) \Rightarrow \psi(\text{int}(\Sigma))$ is a local vertical graph.

Case of a slab in $\mathbb{R}^{n+1}$ and $\theta = \pi/2$

$$
\begin{aligned}
v &\geq 0, \\
\Delta v &= -|\sigma|^2 v \leq 0, \\
v &= 0 \quad \text{on} \quad \partial\Sigma.
\end{aligned}
$$

Maximum principle $\Rightarrow v > 0$ on $\mathrm{int}(\Sigma) \Rightarrow \psi(\mathrm{int}(\Sigma))$ is a local vertical graph.

Set $\partial\Sigma = \Gamma_1 \cup \ldots \cup \Gamma_k$. For each i, $\psi(\Gamma_i)$ separates Π_1 or Π_2 into 2 components. Consider the orthogonal $P : \mathbb{R}^{n+1} \to \Pi_1$ and set $F = P \circ \psi$.

Case of a slab in $\mathbb{R}^{n+1}$ and $\theta = \pi/2$

$$
\begin{aligned}
v &\geq 0, \\
\Delta v &= -|\sigma|^2 v \leq 0, \\
v &= 0 \quad \text{on} \quad \partial\Sigma.
\end{aligned}
$$

Maximum principle $\Rightarrow v > 0$ on $\text{int}(\Sigma) \Rightarrow \psi(\text{int}(\Sigma))$ is a local vertical graph.

Set $\partial\Sigma = \Gamma_1 \cup \ldots \cup \Gamma_k$. For each i, $\psi(\Gamma_i)$ separates Π_1 or Π_2 into 2 components. Consider the orthogonal $P : \mathbb{R}^{n+1} \to \Pi_1$ and set $F = P \circ \psi$.

Case of a slab in $\mathbb{R}^{n+1}$ and $\theta = \pi/2$

Let $p \in \Gamma_i$, $\gamma : (-\epsilon, 0] \to \Sigma$ a curve parametrized by arclength so that $\gamma(0) = p$ and $\dot{\gamma}(0) = \nu(p)$.

$$
\begin{aligned}
\frac{d}{dt}\langle F(\gamma(t)) - F(p), N(p)\rangle|_0 &= 0 \\
\frac{d^2}{dt^2}\langle F(\gamma(t) - F(p), N(p)\rangle|_0 &= \langle \frac{D}{dt}\psi(\gamma')|_0, N(p)\rangle \\
&= \sigma(\nu, \nu)
\end{aligned}
$$

Case of a slab in $\mathbb{R}^{n+1}$ and $\theta = \pi/2$

Let $p \in \Gamma_i$, $\gamma : (-\epsilon, 0] \to \Sigma$ a curve parametrized by arclength so that $\gamma(0) = p$ and $\dot\gamma(0) = \nu(p)$.

$$\begin{aligned}
\frac{d}{dt}\langle F(\gamma(t)) - F(p), N(p)\rangle|_0 &= 0 \\
\frac{d^2}{dt^2}\langle F(\gamma(t) - F(p), N(p)\rangle|_0 &= \langle \frac{D}{dt}\psi(\gamma')|_0, N(p)\rangle \\
&= \sigma(\nu, \nu)
\end{aligned}$$

Note that

$$\frac{\partial v}{\partial \nu} = -\sigma(\nu, \nu)\langle \nu, e_{n+1}\rangle = \begin{cases} +\sigma(\nu, \nu) & \text{if} \quad \psi(\Gamma_i) \subset \Pi_1 \\ -\sigma(\nu, \nu) & \text{if} \quad \psi(\Gamma_i) \subset \Pi_2 \end{cases}$$

Case of a slab in $\mathbb{R}^{n+1}$ and $\theta = \pi/2$

Strong maximum principle $\Rightarrow \frac{\partial v}{\partial \nu} < 0$ on $\partial\Sigma$

Case of a slab in $\mathbb{R}^{n+1}$ and $\theta = \pi/2$

Strong maximum principle $\Rightarrow \frac{\partial v}{\partial \nu} < 0$ on $\partial\Sigma$

$\Longrightarrow$ for a small neighborhood U_i of Γ_i in Σ, $F(U_i \setminus \Gamma_i)$ is contained in the component of $\Pi_1 \setminus F(\Gamma_i)$ having $N(p)$ as outward (resp. inward) normal at $F(p)$ if $\psi(\Gamma_i) \subset \Pi_1$ (resp. if $\Psi(\Gamma_i) \subset \Pi_2$).

Case of a slab in $\mathbb{R}^{n+1}$ and $\theta = \pi/2$

Strong maximum principle $\Rightarrow \frac{\partial v}{\partial \nu} < 0$ on $\partial\Sigma$
$\Longrightarrow$ for a small neighborhood U_i of Γ_i in Σ, $F(U_i \setminus \Gamma_i)$ is contained in the component of $\Pi_1 \setminus F(\Gamma_i)$ having $N(p)$ as outward (resp. inward) normal at $F(p)$ if $\psi(\Gamma_i) \subset \Pi_1$ (resp. if $\Psi(\Gamma_i) \subset \Pi_2$).

Using a topological argument, as before, we conclude that $\psi(\Sigma)$ is globally a graph over a domain in Π_1 and thus $\psi(\Sigma)$ is rotationally invariant (Wente).

An open question

Question

Is a stable capillary hypersurface in a half-space or a slab in $\mathbb{R}^{n+1}$ *necessarily rotationally invariant ?*