Minimal and cmc surfaces in $\mathbb{S}^3$ foliated by circles

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Circle-foliated surfaces in $\mathbb{R}^3$

A circle foliated surface in $\mathbb{R}^3$ is a surface parametrized by

$$X(t,\theta) = c(t) + r(t)(\cos\theta e_1 + \sin\theta e_2),$$

where e_1 and e_2 are smooth ON vectors, and $c(t)$ and $r(t)$ are smooth.

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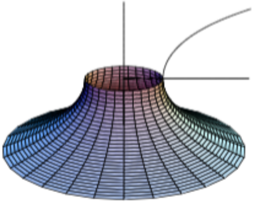
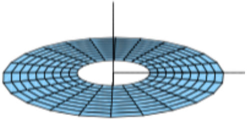
An isometric immersion $\psi : M \to \mathbb{R}^3$ is

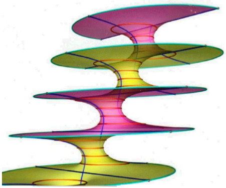
- **minimal** if $H = 0$
- a **cmc surface** if $H = const$

Circle-foliated minimal surfaces in $\mathbb{R}^3$:

Plane, Catenoid, Riemann's minimal surface, (Helicoid)







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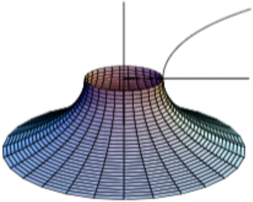
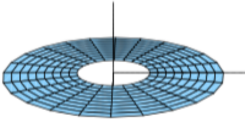
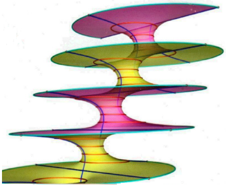
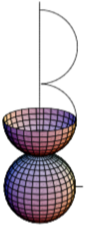
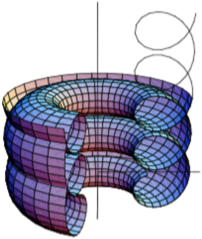
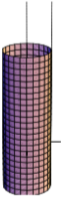




Circle-foliated cmc surfaces in $\mathbb{R}^3$:
Delaunay surfaces (cmc surfaces of rotation), sphere (Nitsche '89)



Circle foliated surfaces in $\mathbb{S}^3$

Let $\mathbb{S}^3 \subset \mathbb{R}^4$ be the unit sphere centered at the origin.

A smooth complete surface $\Sigma \subset \mathbb{S}^3$ is *foliated by circles (circle-foliated)* if, for a smooth orthonormal frames $\{e_1(t), e_2(t)\}$ of $\mathbb{R}^4$ and smooth $c(t)$ and $r(t)$,

$$X(t,\theta) = c(t) + r(t)(\cos\theta e_1(t) + \sin\theta e_2(t)) \tag{1}$$

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Let $\psi : \Sigma \to \mathbb{S}^3 \subset \mathbb{R}^4$ be an isometric immersion

- ψ is **minimal** iff $\triangle_\Sigma \psi = -2\psi$
- ψ has **cmc** H iff $\triangle_\Sigma \psi = 2H\nu - 2\psi$, where ν is a unit normal of $\psi(\Sigma) \subset \mathbb{S}^3$.

Examples of circle-foliated minimal surfaces in $\mathbb{S}^3$:

- (Ruled minimal surfaces in $\mathbb{S}^3$, B. Lawson) Every ruled minimal surface in $\mathbb{S}^3$ is an open submanifold of $\mathcal{M}_\alpha$ given by

$$T(x, y) = (\cos x \cos y, \sin x \cos y, \cos \alpha x \sin y, \sin \alpha x \sin y)$$

  for some $\alpha \geq 0$. ($\mathcal{M}_0$: great sphere, $\mathcal{M}_1$: Clifford torus)

- Rotationally symmetric minimal surfaces

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Theorem (Kutev-Milousheva, 2010)

There are two types of circle-foliated minimal surfaces in $\mathbb{S}^3$:

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Theorem (P)

1. Circle-foliated minimal surface in $\mathbb{S}^3$ is either ruled or rotationally symmetric

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1. Circle-foliated minimal surface in $\mathbb{S}^3$ is either ruled or rotationally symmetric

2. Circle-foliated cmc surface in $\mathbb{S}^3$ is either a sphere, or ruled or rotationally symmetric

Rotationally symmetric minimal surfaces in $\mathbb{S}^3$ (Hynd-McCuan-P)

Stereographic 3-sphere $\mathbb{S}^3 = \left(\mathbb{R}^3, \frac{4ds_0^2}{(1+|X|^2)^2}\right)$.

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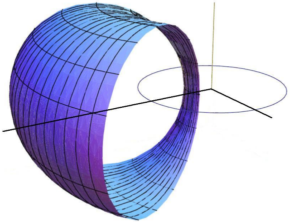
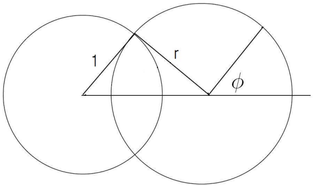
Parametrization of rotationally symmetric surface:

$$X(\theta,\phi) = (\sqrt{1+r^2} + r\cos\phi)(\cos\theta, \sin\theta, 0) + (r\sin\phi)e3.$$





$$H_s = \frac{\sqrt{r^2+1}\left(r(r^2+1)r'' - r'^2 + r^4 - 1\right)}{2r(r'^2+r^2+1)^{3/2}}$$

$$c = \frac{r}{\sqrt{(r^2+1)(r'^2+r^2+1)}} + \frac{H_s}{r^2+1}$$

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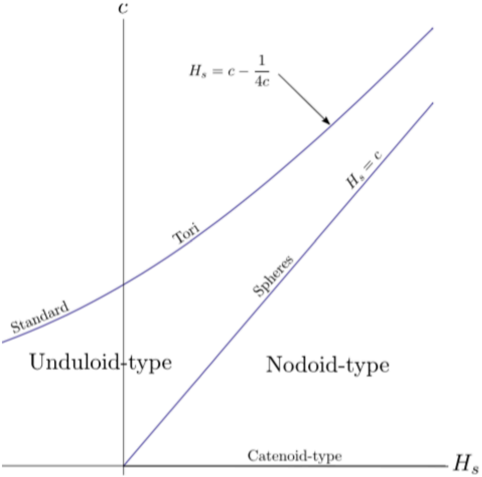
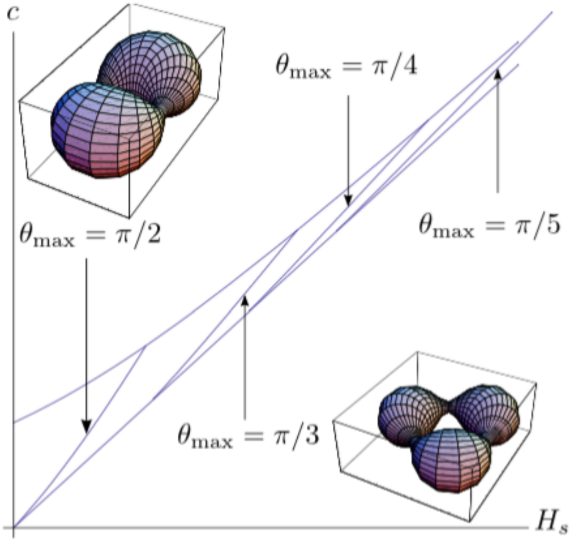
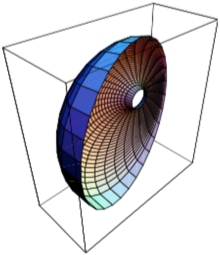



c
$\theta_{\max} = \pi/4$
$\theta_{\max} = \pi/2$
$\theta_{\max} = \pi/5$
$\theta_{\max} = \pi/3$
H_s



Remark

1. S. Brendle proved that Clifford torus is the only embedded minimal torus in $\mathbb{S}^3$ (Lawson's conjecture).

2. B.Andrews and H. Li showed that every embedded cmc torus in $\mathbb{S}^3$ is rotationally symmetric.

Frenet type formula by Frank-Giering:

Let $\{P_t\}$ be a smooth one-parameter family of planes in $\mathbb{R}^4$ passing through the origin. There is a one-parameter family of orthonormal frames $\{e_1(t), e_2(t), e_3(t), e_4(t)\}$ of $\mathbb{R}^4$ such that $e_1(t)$ and $e_2(t)$ span P_t, and the following equations hold

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}' = \begin{pmatrix} 0 & \beta & \kappa & 0 \\ -\beta & 0 & 0 & \tau \\ -\kappa & 0 & 0 & \eta \\ 0 & -\tau & -\eta & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix},$$

where $' = \partial/\partial t$ and $\kappa^2 \geq \tau^2$.

Basic computations

Let Σ be circle-foliated in $\mathbb{S}^3$, parametrized by,

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- Let $\tilde{P}_t$ be the plane containing circle.

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Then $c(t) \perp \tilde{P}_t$ and $c_1 = 0 = c_2$ for $c_i = c \cdot e_i$.
Let

$$\begin{aligned} c'(t) &= \sum_{i=1}^{4} \alpha_i e_i = (c_3 e_3 + c_4 e_4)' \\ &= -\kappa c_3 e_1 - \tau c_4 e_2 + (c_3' - \eta c_4) e_3 + (c_4' + \eta c_3) e_4 \end{aligned}$$

$$\begin{aligned} \Rightarrow X_t =& (\alpha_1 + r' \cos\theta - r\beta \sin\theta) e_1 + (\alpha_2 + r' \sin\theta + r\beta\cos\theta) e_2 \\ &+ (\alpha_3 + r\kappa\cos\theta) e_3 + (\alpha_4 + r\tau \sin\theta) e_4, \\ X_\theta =& -r\sin\theta e_1 + r\cos\theta e_2. \end{aligned}$$

Let N be the normal of Σ. Then

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Since

$$\begin{aligned} *(X_t \wedge X_\theta \wedge X) &= r\cos\theta\left[c_3(\alpha_4 + r\tau\sin\theta) - c_4(\alpha_3 + r\kappa\cos\theta)\right] e_1 \\ &\quad + r\sin\theta\left[c_3(\alpha_4 + r\tau\sin\theta) - c_4(\alpha_3 + r\kappa\cos\theta)\right] e_2 \\ &\quad + \left[rc_4(r' + \alpha_1\cos\theta + \alpha_2\sin\theta) - r^2(\alpha_4 + r\tau\sin\theta)\right] e_3 \\ &\quad - \left[rc_3(r' + \alpha_1\cos\theta + \alpha_2\sin\theta) - r^2(\alpha_3 + r\kappa\cos\theta)\right] e_4, \end{aligned}$$

let

$$\begin{aligned} N &= \frac{*(X_t \wedge X_\theta \wedge X)}{r\left[c_3(\alpha_4 + r\tau\sin\theta) - c_4(\alpha_3 + r\kappa\cos\theta)\right]} \\ &= \epsilon\cos\theta e_1 + \epsilon\sin\theta e_2 + \gamma e_3 + \delta e_4 \end{aligned}$$

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Let $E = X_t \cdot X_t, \cdots,\ l = X_{tt} \cdot N, \cdots$, and

$$\mathcal{H} := lG + nE - 2mF = 2H(EG - F^2)\|N\| \tag{2}$$

Ruled minimal surfaces in $\mathbb{S}^3$

If X is **ruled**, then $c(t) \equiv 0$ and $r(t) \equiv 1$
$\Rightarrow c_i = 0$, $\alpha_i = 0$, $r' = 0$, and

$$\mathcal{H} = \eta\kappa^2 \cos^2\theta + \eta\tau^2 \sin^2\theta - (\kappa'\tau - \kappa\tau')\cos\theta\sin\theta - \beta\kappa\tau$$

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In the Frenet equation, we may let

$$e_1 = (\cos t, \sin t, 0, 0), \;\; e_2 = (0, 0, \cos \tau t, \sin \tau t).$$

Then

$$X(t, \theta) = \cos\theta e_1 + \sin\theta e_2$$

is $\mathcal{M}_\tau$.

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If ii holds, then we may assume that $\kappa = \tau$. Then $\eta = \beta$ from $\mathcal{H} = 0$. For φ with $\varphi' = \beta$, the new orthonormal frame fields

$$\tilde{e}_1 = \cos\phi\, e_1 - \sin\phi\, e_2,\ \ \tilde{e}_2 = \sin\phi\, e_1 + \cos\phi\, e_2$$
$$\tilde{e}_3 = \cos\phi\, e_3 - \sin\phi\, e_4,\ \ \tilde{e}_4 = \sin\phi\, e_3 + \cos\phi\, e_4$$

satisfies

$$\tilde{e}_1' = \kappa\, \tilde{e}_3,\ \ \tilde{e}_2' = \kappa\, \tilde{e}_4$$
$$\tilde{e}_3' = -\kappa \tilde{e}_1,\ \ \tilde{e}_4' = -\kappa \tilde{e}_2,$$

and X is the Clifford torus.

Not-ruled minimal surfaces in $\mathbb{S}^3$

If X **is not ruled**, then we let

$$N = \cos\theta e_1 + \sin\theta e_2 + \gamma e_3 + \delta e_4,$$

$$\gamma = \frac{c_4(r' + \alpha_1\cos\theta + \alpha_2\sin\theta) - r(\alpha_4 + r\tau\sin\theta)}{c_3(\alpha_4 + r\tau\sin\theta) - c_4(\alpha_3 + r\kappa\cos\theta)},$$

$$\delta = \frac{c_3(r' + \alpha_1\cos\theta + \alpha_2\sin\theta) - r(\alpha_3 + r\kappa\cos\theta)}{c_3(\alpha_4 + r\tau\sin\theta) - c_4(\alpha_3 + r\kappa\cos\theta)}$$

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$\tilde{\mathcal{H}} = [c_3(\alpha_4 + r\tau\sin\theta) - c_4(\alpha_3 + r\kappa\cos\theta)]\,\mathcal{H}$ is a trigonometric polynomial of degree 3.

We have $\kappa = 0$ and $\tau = 0$ from the coefficients of the Fourier series expansion of $\tilde{\mathcal{H}}$.

Then the plane $span\{e_1, e_2\}$ is fixed. Hence we may let $\beta = 0$, $\eta = 0$.

Then $\alpha_1 = \alpha_2 = 0$, $\alpha_3 = c_3'$, $\alpha_4 = c_4'$, and

$$E = \alpha_3^2 + \alpha_4^2 + r'^2, \ F = 0, \ G = r^2$$

Moreover $c(t)$ lies in the plane $span\{e_3, e_4\}$.

$\Rightarrow X$ is rotationally symmetric.

Ruled cmc surfaces in $\mathbb{S}^3$

Theorem (P.) *Ruled surface of cmc H in $\mathbb{S}^3$ is given by*

$$X(t,\theta) = \cos\theta e_1 + \sin\theta e_2,$$

where e_1 and e_2 are part of an orthonormal frame $e_1, \ldots, e_4$ of $\mathbb{R}^4$ satisfying

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 2H \\ 0 & -1 & -2H & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}.$$

Sketch of proof

We may let $N = -\tau sin\theta e_3 + \kappa \cos\theta e_4$.

From
$$\mathcal{H} = lG + nE - 2mF = 2H(EG - F^2)\|N\|,$$

$$2H(EG - F^2)\|N\| - \mathcal{H} = 2H\left(\kappa^2\cos^2\theta + \tau^2\sin^2\theta\right)^{\frac{3}{2}}$$
$$-\left(\eta\kappa^2\cos^2\theta + \eta\tau^2\sin^2\theta + (-\kappa'\tau + \kappa\tau')\cos\theta\sin\theta - \beta\kappa\tau\right) = 0.$$

Then $\kappa^2 = \tau^2 > 0$ and $\kappa^2(2H|\kappa| - \eta + \beta) = 0$.

From
$$\Delta X = \frac{1}{\kappa}\left\{X_{tt} - \left(\frac{\beta}{\kappa}\right)_t X_\theta - 2\frac{\beta}{\kappa}X_{t\theta} + \frac{\kappa^2 + \beta^2}{\kappa}X_{\theta\theta}\right\}$$
$$\nu = N/\|N\| = -\sin\theta e_3 + \cos\theta e_4.$$

Then $(\Delta X - 2H\nu + 2X)\cdot e_1$ is

$$-\beta^2\cos\theta - \kappa^2\cos\theta - \beta'\sin\theta + \left(\frac{\beta}{\kappa}\right)'\sin\theta + \frac{\beta^2}{\kappa}\cos\theta + \kappa\cos\theta = 0.$$

Hence $\kappa = 1$ and $\eta = 2H + \beta$.

For ϕ with $\phi' = \beta$, let

$$\tilde{e}_1 = \cos\phi\, e_1 - \sin\phi\, e_2,\ \ \tilde{e}_2 = \sin\phi\, e_1 + \cos\phi\, e_2,$$
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Then

$$\begin{pmatrix} \tilde{e}_1 \\ \tilde{e}_2 \\ \tilde{e}_3 \\ \tilde{e}_4 \end{pmatrix}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 2H \\ 0 & -1 & -2H & 0 \end{pmatrix} \begin{pmatrix} \tilde{e}_1 \\ \tilde{e}_2 \\ \tilde{e}_3 \\ \tilde{e}_4 \end{pmatrix}.$$

Thank you!