SURFACES WITH PRESCRIBED MEAN CURVATURE

EXISTENCE OF STABLE SURFACES AND THEIR CHARACTERIZATION AS RADIAL GRAPHS

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The problem

Fix:

a cone $C_\beta = \{ p = (x, y, z) \in \mathbb{R}^3 \ / \ z > |p| \cos\beta \}$

a Jordan curve $\Gamma \subset \overline{C_\beta} \setminus \{0\}$

a mapping $H : \overline{C_\beta} \longrightarrow \mathbb{R}$

Find conditions on H such that:

① $\forall \lambda > 0$ one can find an H-surface spanning $\lambda\Gamma$ and contained in $\overline{C_\beta} \setminus \{0\}$

H-surface = surface with mean curvature $H(p)$ at every point p.

② if Γ admits a radial representation then any stable H-surface spanning Γ can be represented as a radial graph.

SOME KNOWN RESULTS

Minimal surfaces

Radó (1933)

- $\Gamma = \{ (x,y,g(x,y)) \mid (x,y) \in \partial D \}$
  with $g : \partial D \to \mathbb{R}$ continuous
- $D \subset \mathbb{R}^2$ convex
- S minimal surface spanning Γ

Then S is a cartesian graph :

$$S = \{ (x,y,\phi(x,y)) \ : \ (x,y) \in \overline{D} \}$$

SOME KNOWN RESULTS

Surfaces with constant mean curvature $H \geq 0$

Serrin (1969)

- $\Gamma = \{ (x, y, g(x,y)) \mid (x,y) \in \partial D \}$ with $g : \partial D \to \mathbb{R}$ continuous and $\partial D \in C^2$
- curvature of $\partial D \geq 2H$
- S embedded H-surface spanning Γ contained in a slab $\left[-\frac{1}{2H}, \frac{1}{2H}\right] \times \mathbb{R}^2$

Then S is a cartesian graph.

Gulliver & Spruck (1972)

- Γ as above, with $g \in C^2$
- curvature of $\partial D \geq 2H$
- S H-surface spanning Γ, contained in a slab $\left[-\frac{1}{2H}, \frac{1}{2H}\right] \times \mathbb{R}^2$

Then S is a cartesian graph.

Sauvigny (1982)

surfaces with <u>prescribed mean curvature</u>

$Z_r = \{(x,y,z) : \quad x^2+y^2 \le r^2\}$

$H : Z_r \longrightarrow \mathbb{R}$ regular

with $\quad 2r\|H\|_\infty \le 1$.

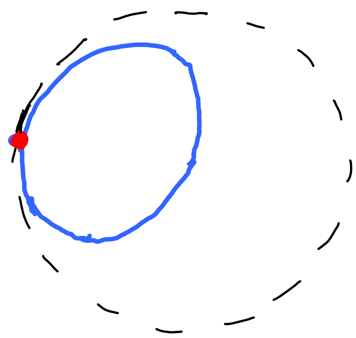
$\Gamma \subset Z_r$ Jordan curve.

Assume $\Gamma = \{(x,y,g(x,y)) \mid (x,y)\in\partial D\}$

with $D \subset \mathbb{R}^2$ r-convex, namely

$\forall p \in \partial D \ \exists$ r-disk B

s.t. $\overline{B} \supset \overline{D}$, $\quad p \in \partial B$



Moreover assume $\dfrac{\partial H}{\partial z} \geq 0$

Then <u>stable</u> H-surfaces spanning Γ are cartesian graphs, too.

Consider the analogous problem with respect to RADIAL PROJECTION (and cones instead of cylinders).

$$C_\beta = \{ p = (x,y,z) \in \mathbb{R}^3 \mid z > |p| \cos\beta \}$$

with $0 < \beta < \frac{\pi}{2}$

THEOREM 1 $H : \overline{C_\beta} \longrightarrow \mathbb{R}$ of class C^1

$$(1) \qquad |H(p)|\,|p| \le \frac{\cos\beta}{2(1+\cos\beta)} \qquad \forall p \in \overline{C_\beta}$$

Then $\forall$ rectifiable Jordan curve $\Gamma \subset \overline{C_\beta} \setminus \{0\}$ $\exists$ H-surface spanning Γ contained in $\overline{C_\beta} \setminus \{0\}$

COROLLARY $H : \mathbb{R}^3 \longrightarrow \mathbb{R}$ of class C^1

Let Γ be a rectifiable Jordan curve in $\mathbb{R}^3_+ = \{ (x,y,z) \in \mathbb{R}^3 : z > 0 \}$.

Let $\beta \in (0, \frac{\pi}{2})$ be the angular radius of Γ, i.e. $\beta = \inf\{ \theta : \Gamma \subset \text{cone of angular radius } \theta \}$

If (1) holds, then $\forall \lambda > 0$ $\exists$ H-surface spanning $\lambda\Gamma$

Some ideas of the proof

- Analytical formulation of the problem

$$B = \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 < 1\}$$

Look for solutions $X : \overline{B} \to \mathbb{R}^3$ of

$$(P) \begin{cases} \Delta X = 2H(X)\, X_u \wedge X_v & \text{in } B \\ |X_u|^2 - |X_v|^2 = 0 = X_u \cdot X_v & \text{in } B \\ X|_{\partial B} : \partial B \to \Gamma & \text{oriented parametrization of } \Gamma \end{cases}$$

- Variational approach

Solutions of (P) are formally critical points of

$$E(X) = \frac{1}{2}\int_B |\nabla X|^2 + 2\int_B Q(X) \cdot X_u \wedge X_v$$

$\left(Q : \mathbb{R}^3 \to \mathbb{R}^3 \quad \text{s.t. } \operatorname{div} Q = H \right)$

in the class of admissible functions

$W^{1,2}(B, \mathbb{R}^3) \cap C^0(\partial B, \Gamma)$ + three pt condition

· Minimization problem

Choose $Q(p) = m(p)\, p$ $\quad m(p) = \int_0^1 H(sp)\, s^2 ds$

Using (1) $\quad |H(p)|\,|p| \le \dfrac{\cos\beta}{2(1+\cos\beta)}$

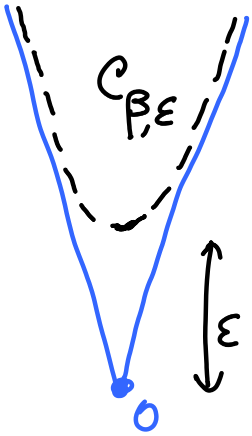
one finds

$$C_1 \int_B |\nabla X|^2 \le E(X) \le C_2 \int_B |\nabla X|^2$$

as $X \in \{$ admissible functions with image in $\overline{C_\beta}\}$

· The minimizer does not touch the obstacle

Smooth the obstacle



Minimize in the class of admissible functions with image in $\overline{C_{\beta,\varepsilon}}$

Use the geometric maximum principle (Gulliver & Spruck, 1972)

$|H(p)| <$ mean curvature of $\partial C_{\beta,\varepsilon}$ $\quad \forall p \in \partial C_{\beta,\varepsilon}$

II PART : Characterization of the H-surface as a radial graph

1. STABILITY PROPERTY OF THE LEAST-ENERGY H-SURFACE

Consider the H-surface $X : \overline{B} \to \mathbb{R}^3$ found as a minimizer of

$$E(Y) = \frac{1}{2}\int |\nabla Y|^2 + 2\int Q(Y)\cdot Y_u \wedge Y_v$$

in the class of admissible functions

Let $F(Y) = \int |Y_u \wedge Y_v| + 2\int Q(Y)\cdot Y_u \wedge Y_v$

Then $\frac{d}{dt} F(X+t\varphi N)\Big|_{t=0} = 0$ and

$$\frac{d^2}{dt^2} F(X+t\varphi N)\Big|_{t=0} \geq 0$$

$\forall \varphi \in C_c^\infty(B,\mathbb{R})$. Here $N = \dfrac{X_u \wedge X_v}{|X_u \wedge X_v|}$.

$$\frac{d^2}{dt^2} F(X+t\varphi N)\Big|_{t=0} = \int_B \left(|\nabla\varphi|^2 - 2q(u,v)\varphi^2\right)$$

where $q = |X_u|^2\left[2H(X)^2 - K(X) - \nabla H(X)\cdot X\right]$

Definition An H-surface $X : \overline{B} \longrightarrow \mathbb{R}^3$ is said to be stable if

$$\int (|\nabla\varphi|^2 - 2\, q(u,v)\, \varphi^2) \geq 0 \quad \forall \varphi \in C_c^\infty(B, \mathbb{R})$$

where $q = |X_u|^2 \left[2H(X)^2 - K(X) - \nabla H(X) \cdot X \right]$

Remarks

① The H-surface given by Thm 1 as a least-energy solution of (P) is stable.

② If X is a stable H-surface then

$$\lambda_1(-\Delta - 2q) \geq 0$$

i.e. the maximum principle holds for the Schrödinger operator $-\Delta - 2q$

② β-CONVEX SUBSETS OF S^2

<u>Notation</u> Given a domain Ω in S^2 let $C_\Omega = \{ \lambda p \; : \; \lambda > 0, \; p \in \Omega \} \subset \mathbb{R}^3$.

(cone spanned by Ω).

Given $\hat{p} \in S^2$ and $\beta \in (0, \frac{\pi}{2})$ set

$C_\beta(\hat{p}) = \{ p \in \mathbb{R}^3 \; : \; p \cdot \hat{p} > |p| \cos\beta \}$

(cone of angular radius β, in the direction of $\hat{p}$)

<u>Definition</u> Fix $\beta \in (0, \frac{\pi}{2})$.

A domain Ω in S^2 is β-convex if

$\forall p \in \partial\Omega \quad \exists \; \hat{p} \in S^2 \; / \; \overline{C_\Omega} \subset \overline{C_\beta(\hat{p})}, \; p \in \partial C_\beta(\hat{p})$

<u>Proposition</u>

If Ω is a β-convex domain in S^2

then
- Ω is convex (i.e. C_Ω convex)
- $\forall p \in \partial\Omega \quad \exists! \; \hat{p} \; /$ the β-cone condition holds
- The mapping $p \longmapsto \hat{p}$ is continuous

3 ORIENTATION OF THE BOUNDARY OF A β-CONVEX DOMAIN

Ω domain in $\mathbb{S}^2$ with $\partial\Omega \in C^1$

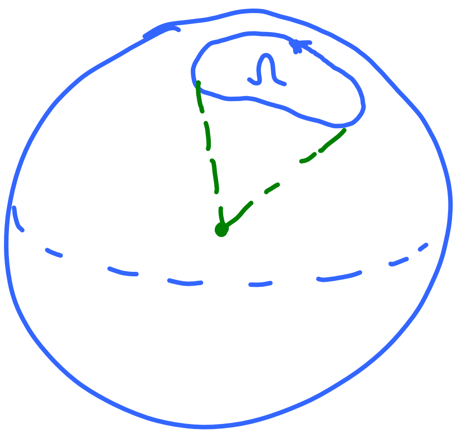
Assume that Ω is β-convex.

Fix a C^1-parametrization $\gamma : \partial B \longrightarrow \partial\Omega$

We say that $\partial\Omega$ is positively oriented by γ if

$$\hat{p}(\gamma(z)) \cdot \gamma(z) \wedge \gamma'(z) > 0 \quad \forall z \in \partial B$$

where $\hat{p}$ is the map of the β-cone condition and $\gamma'(z) = \frac{d}{dt}\left[\gamma(e^{it})\right]$ for $z = e^{it}$.



THEOREM 2 (representation as radial graphs)

Let $\beta \in (0, \frac{\pi}{2})$ and $H : \overline{\mathcal{C}_\beta} \to \mathbb{R}$ of class $C^{1,\alpha}$

(1) $|H(p)|\,|p| \leq \dfrac{\cos\beta}{2(1+\cos\beta)} \quad \forall p \in \overline{\mathcal{C}_\beta}$

(2) $H(p) + \nabla H(p)\cdot p \geq 0 \quad \forall p \in \overline{\mathcal{C}_\beta}$

Let Γ be a $C^{3,\alpha}$ Jordan curve in $\overline{\mathcal{C}_\beta} \setminus \{0\}$ and let X be a stable H-surface spanning Γ

Assume that

(i) Γ is a regular radial graph:
$\Gamma = \{ g(p)\,p \;/\; p \in \partial\Omega \}$ with $g : \partial\Omega \to \mathbb{R}^+$
and $\Omega \subset \mathbb{S}^2$ of class $C^{3,\alpha}$

(ii) Ω is a β-convex domain in $\mathbb{S}^2$

(iii) $\hat{X}|_{\partial B}$ induces a positive orientation on $\partial\Omega$
where $\hat{X} = \dfrac{X}{|X|}$.

Then $\hat{X} : \overline{B} \to \overline{\Omega}$ is a homeomorphism and $X(\overline{B})$ can be represented as a radial graph.

In particular if X is the least energy H-surface given by Theorem 1

Then
- $\hat{X} : \overline{B} \longrightarrow \overline{\Omega}$ is a diffeo
- $X(B) \subset C_\beta$
- X has no branch point (i.e. $X_u \wedge X_v \neq 0$ everywhere in $\overline{B}$)

Remarks

The assumption (2) $H(p) + \nabla H(p) \cdot p \geq 0$ is a monotonicity condition

$$(2) \iff \frac{d}{d\lambda}\left[\lambda H(\lambda p)\right] \geq 0 \quad \forall \lambda > 0$$

The orientation of $\partial\Omega$ must have the same sign appearing in the monotonicity condition.

Outline of the proof

Aim: apply the

GLOBAL INVERSION THEOREM (Hadamard)

$\mathcal{K}_1$ arcwise connected metric space

$\mathcal{K}_2$ simply connected metric space

$\mathcal{F} : \mathcal{K}_1 \longrightarrow \mathcal{K}_2$ continuous, surjective, locally invertible, proper (i.e. $\mathcal{F}^{-1}(\mathcal{K}_0)$ cpt $\forall\, \mathcal{K}_0 \subset \mathcal{K}_2$ cpt)

$\Longrightarrow \mathcal{F}$ homeomorphism of $\mathcal{K}_1$ onto $\mathcal{K}_2$

taking

$$\mathcal{F} = \hat{X} := \frac{X}{|X|} \qquad \mathcal{K}_1 = \overline{B} \qquad \mathcal{K}_2 = \overline{\Omega}$$

We know that $\hat{X}|_{\partial B} : \partial B \longrightarrow \partial\Omega$ with Ω β-convex domain in $\mathbb{S}^2$.

Step 1 $\hat{X}(B) \subset \Omega$

By contradiction suppose that $\hat{X}(B) \not\subset \Omega$.
Then $\hat{X}(B) \cap \partial\Omega \neq \emptyset$ (connectedness)

Take $p \in \hat{X}(B) \cap \partial\Omega$ and let $\hat{p} \in S^2$ be given by the β-cone property at p

The function $\varphi(u,v) = X(u,v)\cdot\hat{p} - |X(u,v)|\cos\beta$ satisfies

$$\begin{cases} -\Delta\varphi \geq 0 & \text{in } B \quad (\text{use } (1)) \\ \varphi \geq 0 & \text{on } \partial B \quad (\beta\text{-cone property}) \end{cases}$$

Moreover if $p = \hat{X}(u_0, v_0)$, since $p \in \partial C_\beta(\hat{p})$, $\varphi(u_0,v_0) = 0$ and $(u_0,v_0) \in B$.

$\underset{\text{max princ.}}{\Longrightarrow}$ $\varphi \equiv 0$ in B $\Rightarrow$ $X(\overline{B}) \subset \partial C_\beta(\hat{p}) \setminus \{0\}$

Since $|H| <$ mean curvature of $\partial C_\beta(\hat{p}) \setminus \{0\}$ one has $X(B) \cap \partial C_\beta(\hat{p}) = \emptyset$. $\lightning$

In fact, if $X \in C^1(\overline{B}, \mathbb{R}^3)$ then

X has no branch point on ∂B

Step 2 $\quad \hat{X}(\overline{B}) = \overline{\Omega}$ (surjectivity)

$\hat{X}|_{\partial B} : \partial B \longrightarrow \partial\Omega$ homeomorphism

degree argument

Step 3 $\quad \overline{\Omega}$ is simply connected.

Use the Schönflies Theorem (if γ is a planar Jordan curve, then the bounded component of $\mathbb{R}^2 \setminus \gamma$ is homeomorphic to B) and the identification $S^2 \approx \mathbb{R}^2 \cup \{\infty\}$.

Step 4 $\quad$ Local invertibility about regular points

Preliminary study on the sign of the radial component of the Gauss map $N = \dfrac{X_u \wedge X_v}{|X_u \wedge X_v|}$

N admits a continuous extension on $\overline{B}$

$\partial_w X(z_0 + w) = A w^n + o(|w|^m)$ as $w \to 0$

z_0 branch point, $A \in \mathbb{C}^3 \setminus \{0\}$ $\sum A_i^2 = 0$

$$\boxed{N \cdot X > 0 \quad \text{in } \overline{B}}$$

$N \cdot X \neq 0$ on ∂B

$N \cdot X > 0$ on ∂B

(use the positive orientation of $\partial\Omega$)

$N \cdot X > 0$ in B

because $f = N \cdot X$ satisfies

$$-\Delta f - 2q(u,v) f = 2|X_u|^2 \left(\nabla H(X) \cdot X + H(X) \right) \geq 0 \quad \text{in } B$$

and the max. princ. holds for $-\Delta - 2q$

By
$$\boxed{\hat{X}_u \wedge \hat{X}_v \cdot X = \frac{X_u \wedge X_v \cdot X}{|X|^3}}$$

one infers that

at every regular point $\hat{X}_u(z) \wedge \hat{X}_v(z) \neq 0$

Arguing with the help of degree theory the point $\hat{X}(p)$ has just one preimage.

Step 5 Local invertibility about branch points

Let $z_0 = (u_0, v_0) \in \overline{B}$ be a branch point.
Then $z_0 \in B$ and $\exists\, \delta_0 > 0$ s.t. z_0 is the only branch point in $\overline{B_{\delta_0}(z_0)}$.

Assume that $\hat{X}$ is not injective in $B_\delta(z_0)$ for no $\delta \in (0, \delta_0)$.

Then $\exists\; z_i \longrightarrow z_0$, $z_i \neq z_j$, $X(z_i) = X(z_0)\ \forall i$.

Set $p = X(z_0)$ and $\tilde{X}$ = projection of X on the plane orthogonal to p.

$\tilde{X}(z_0 + w) = \ell\, w^{n+1} + o(w^{n+1})$ as $w \to 0$ for some $\ell \in \mathbb{C}$ (use $N(z_0) \cdot X(z_0) > 0$).

$\Rightarrow$ only finitely many preimages of p close to z_0 $\lightning$

COMPARISON WITH KNOWN / SIMILAR RESULTS

On the existence theorem

Main (new) technical difficulty :

the obstacle has a singular point.

On Theorem 2 (radial graph)

Some arguments used in the case of cartesian graphs do not work.

The radial projection is not linear.

Important tools :

- enclosure theorem (geometric version of the maximum principle)
- degree theory

Remark $\quad |H(p)|\,|p| \le \dfrac{\cos\beta}{2(1+\cos\beta)} \quad \forall p \in \overline{C_\beta}$

This bound is not optimal

(Serrin, 1969)