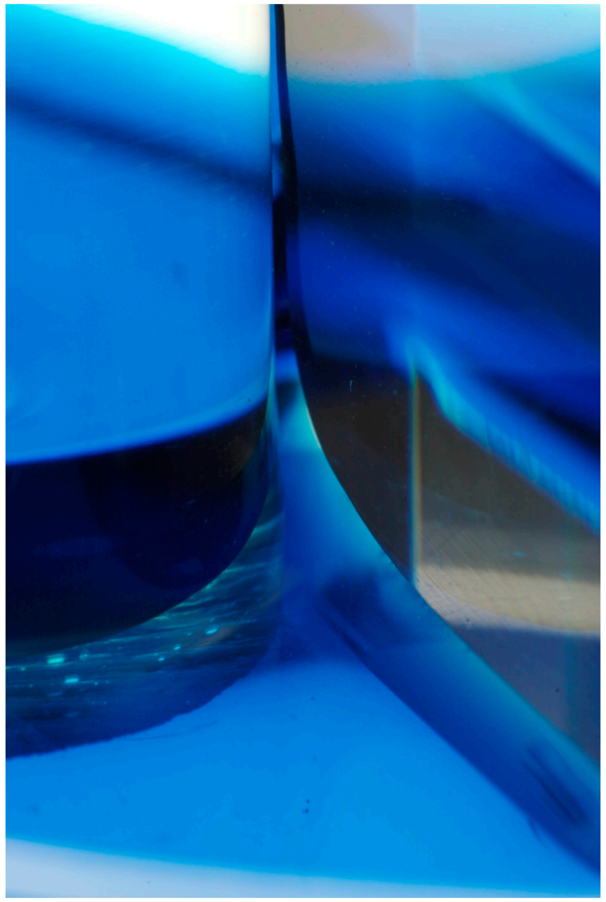
Bounded and Unbounded Capillary Surfaces in a Cusp Domain



Yasunori Aoki
joint work with David Siegel and Hans De Sterck

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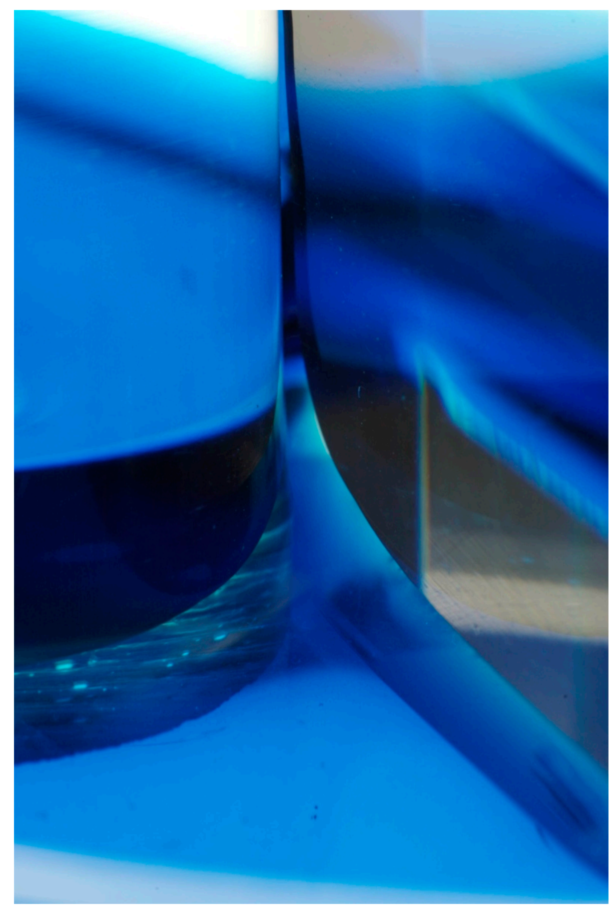

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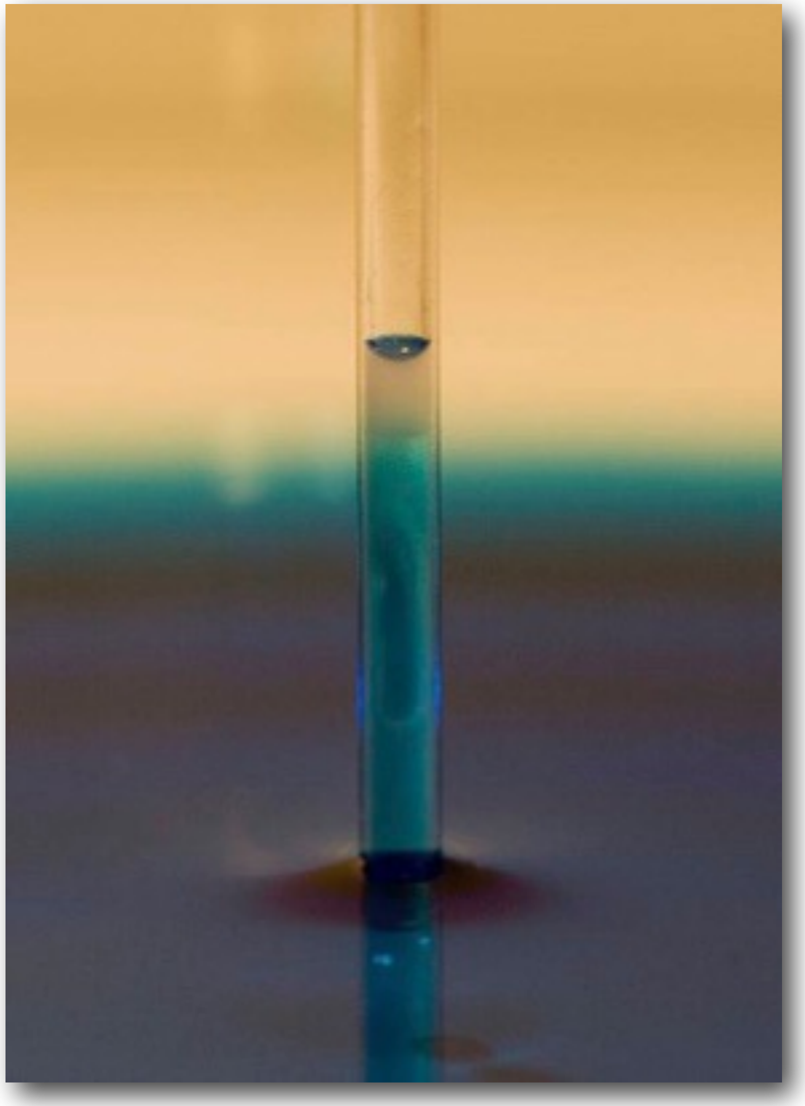
Capillary Surface



Laplace-Young Equations

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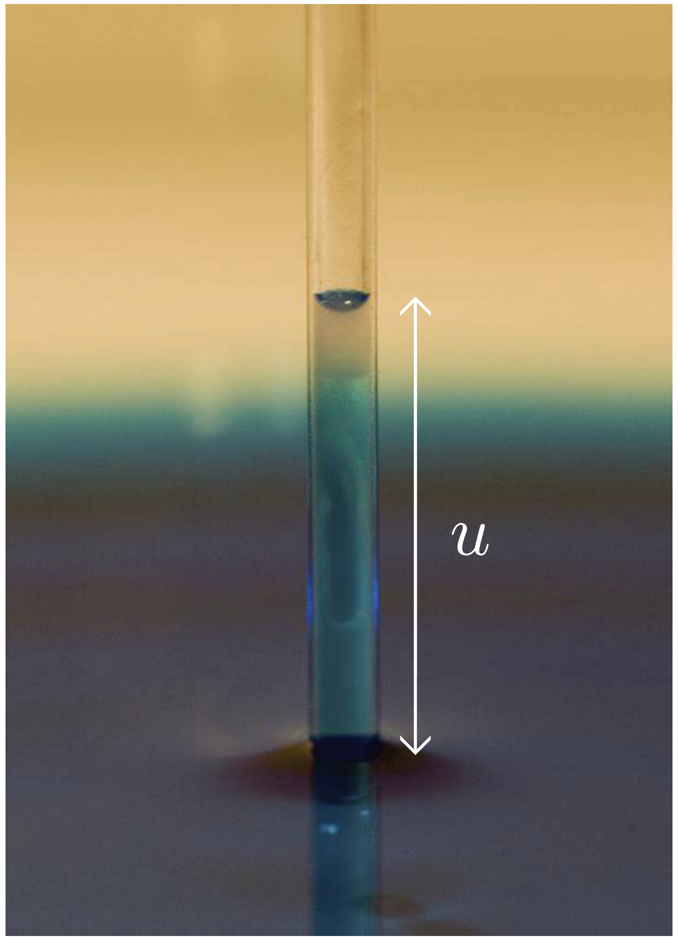
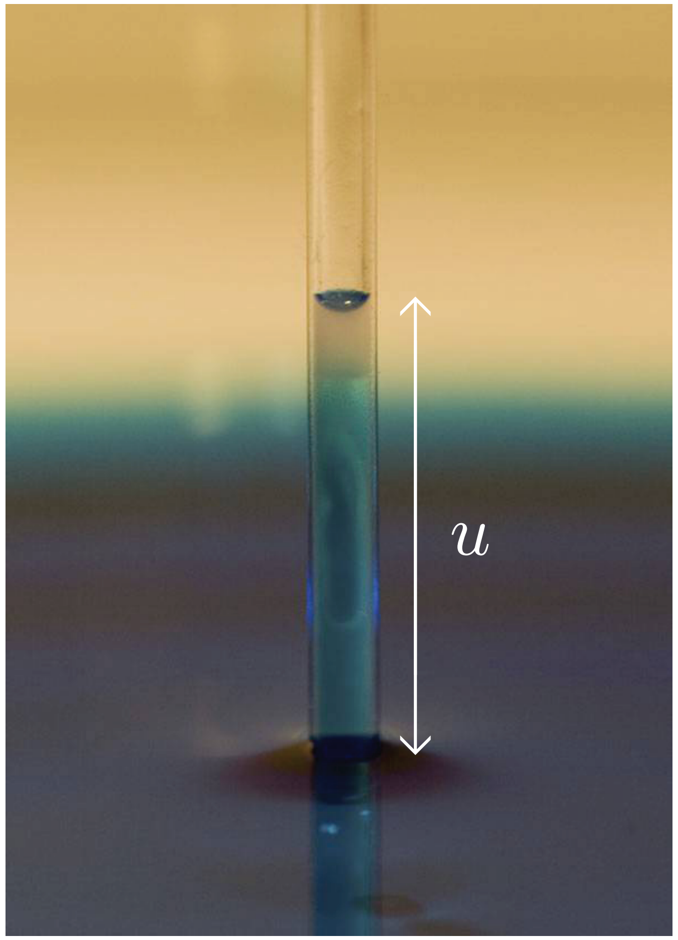


Laplace-Young Equations

$$\nabla \cdot \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = \kappa u \qquad \text{in } \Omega$$



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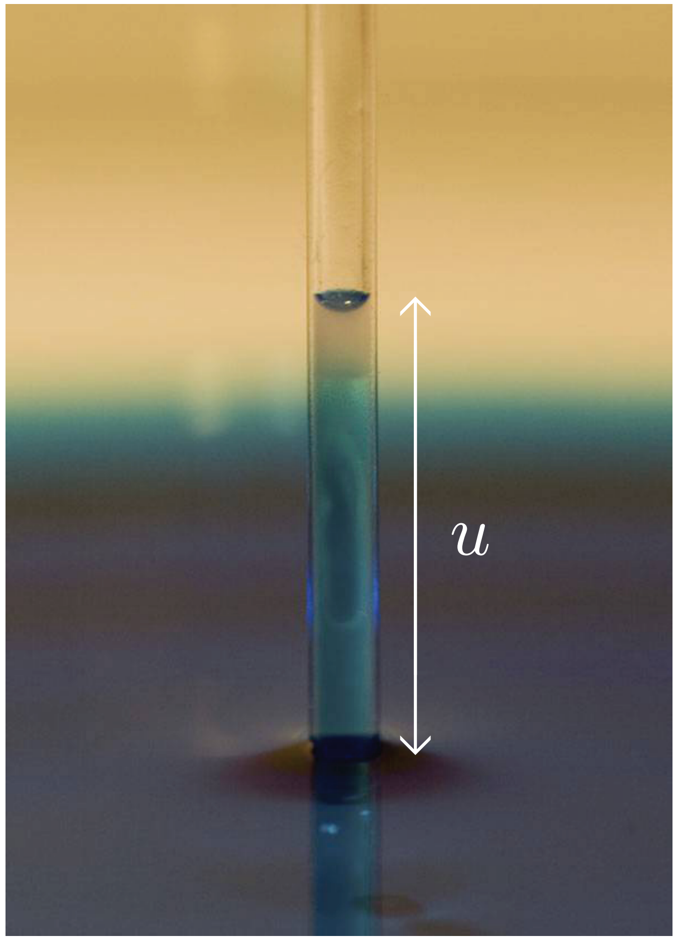
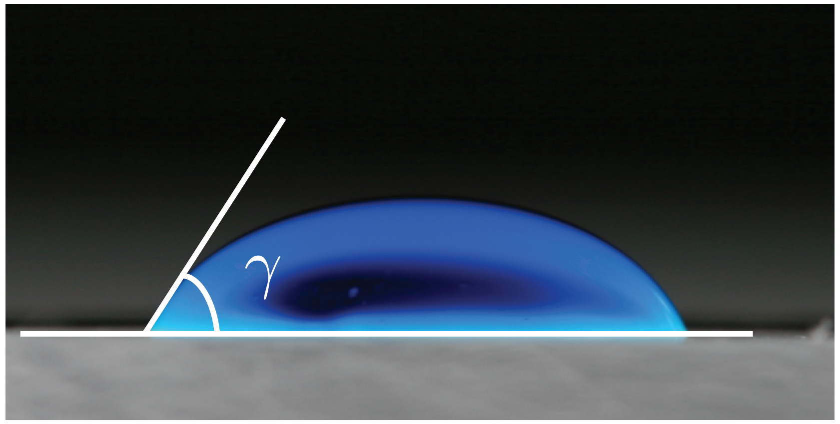
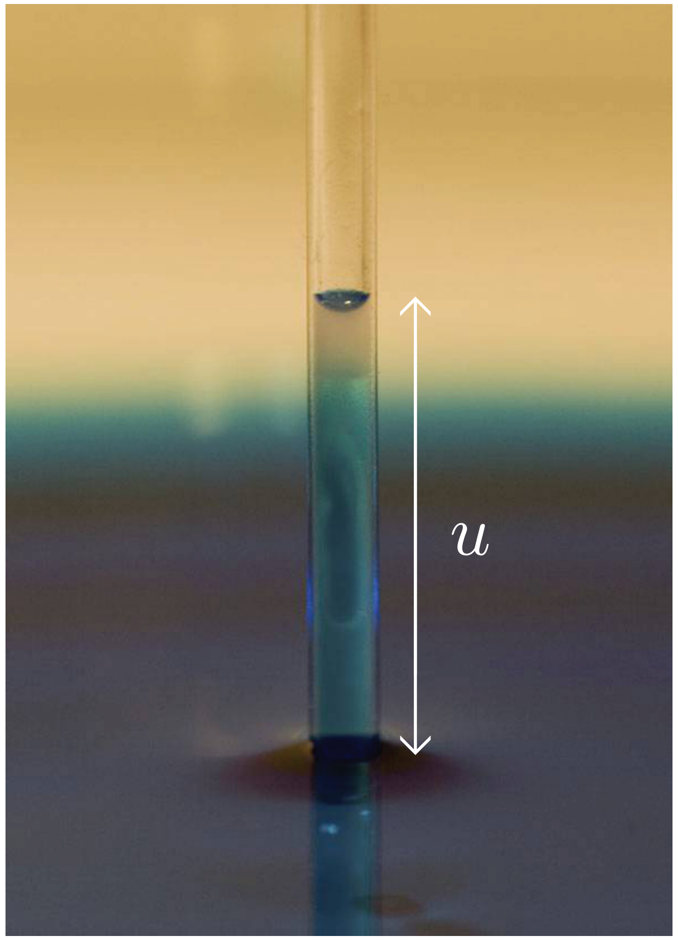


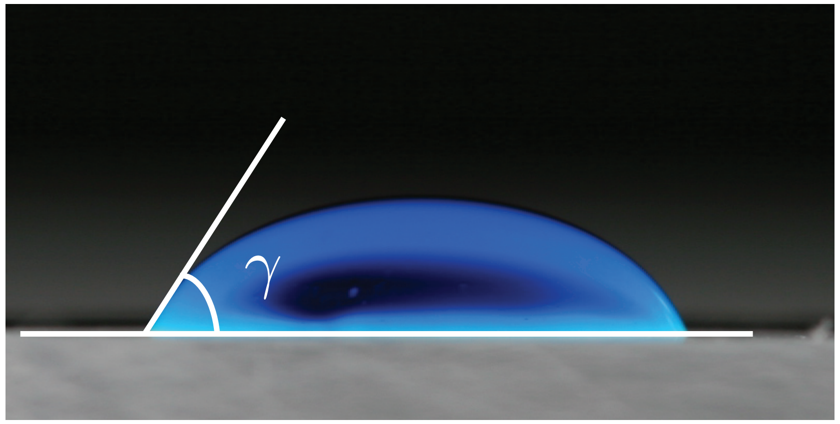


Laplace-Young Equations

$$\nabla \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \kappa u \qquad \text{in } \Omega$$

$$\nu \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \cos\gamma \qquad \text{on } \partial\Omega$$





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$$\kappa = \frac{g(\rho_{\text{liquid}} - \rho_{\text{air}})}{\sigma_{\text{air-liquid}}}$$

Laplace-Young Equations

$$\nabla \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \kappa u \qquad \text{in } \Omega$$

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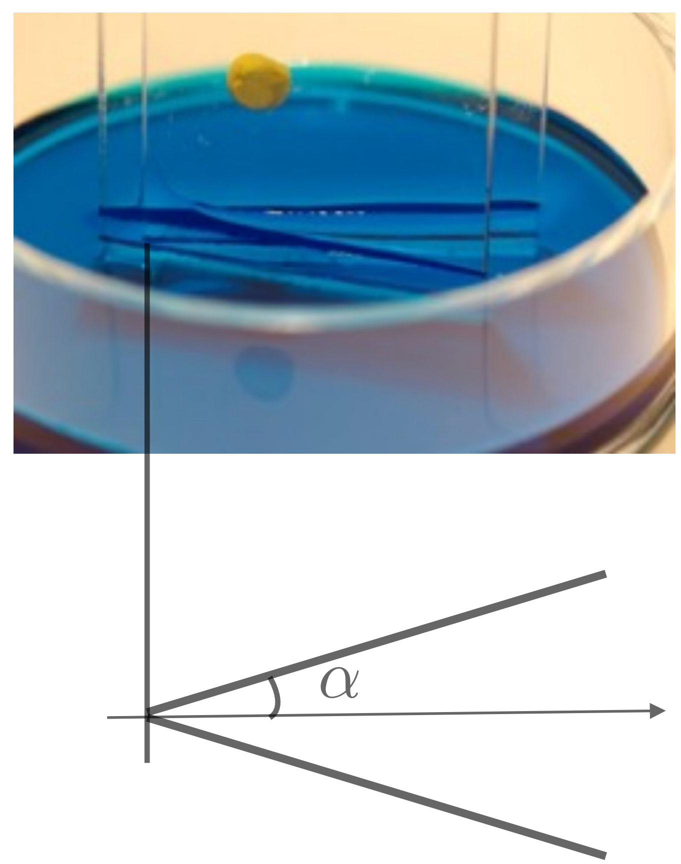
$$\kappa = 1$$

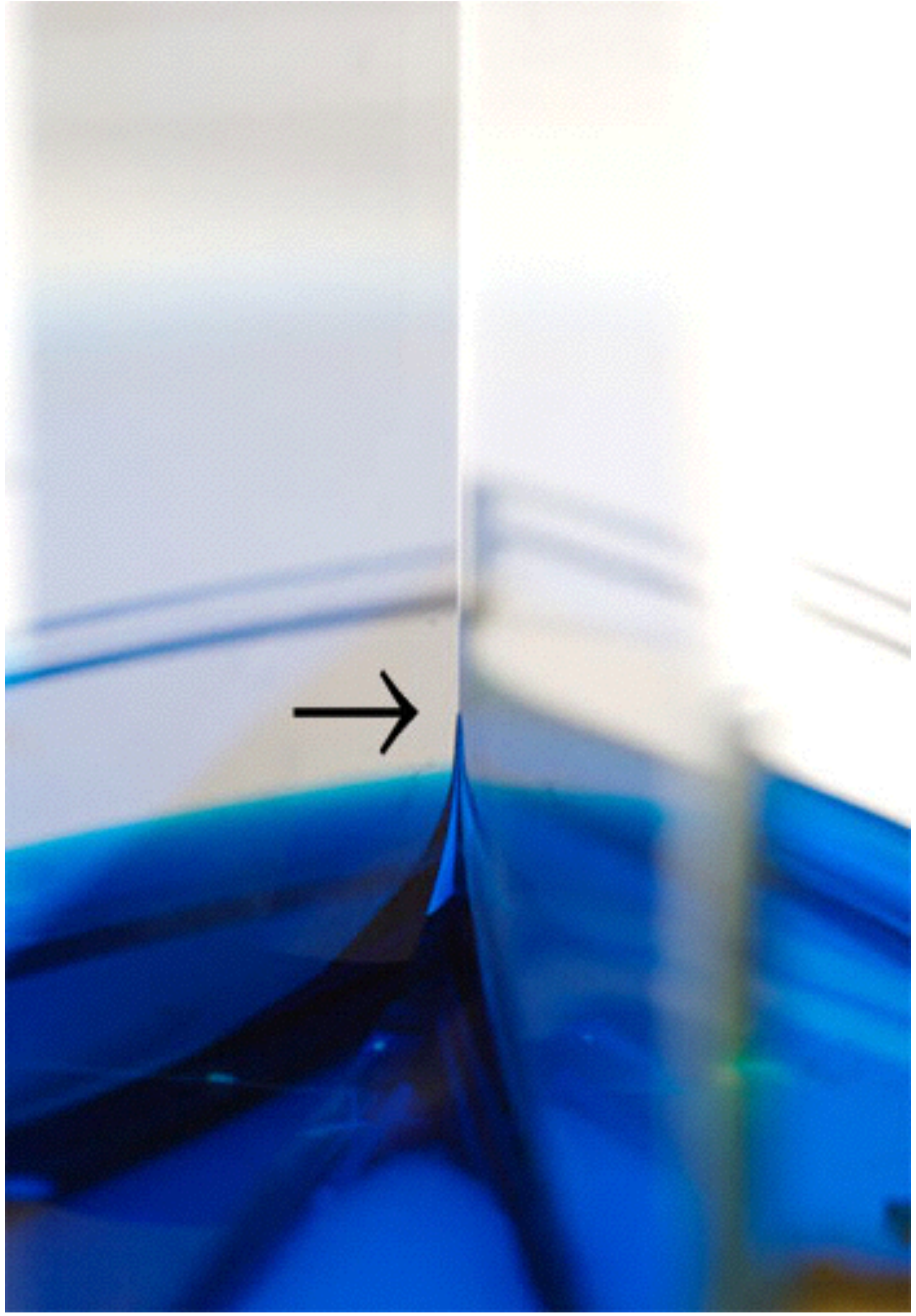
Corner Problem

Height of the capillary surface depends discontinuous to wedge angle.

(Robert Finn and Paul Concus)



Corner Problem



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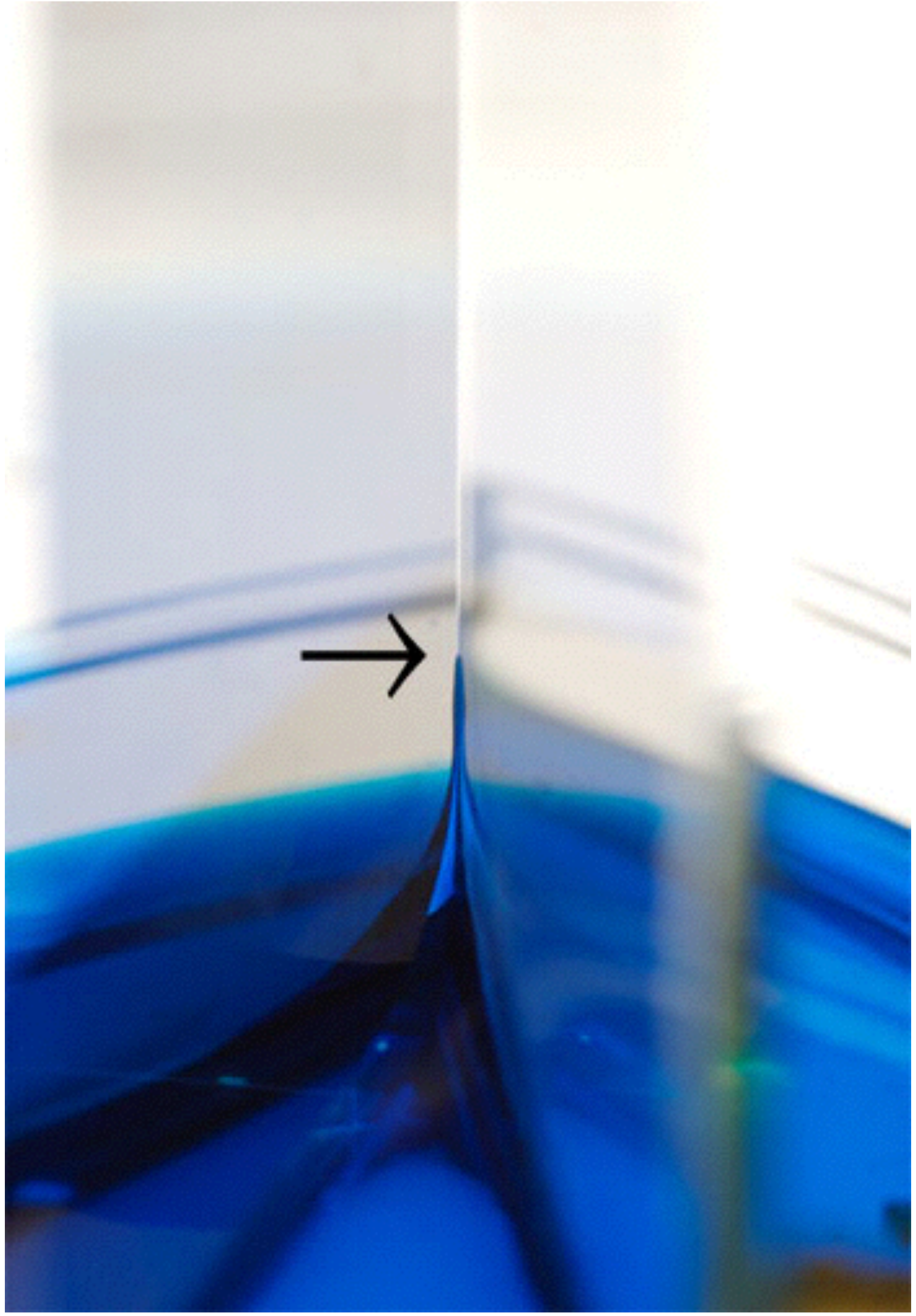

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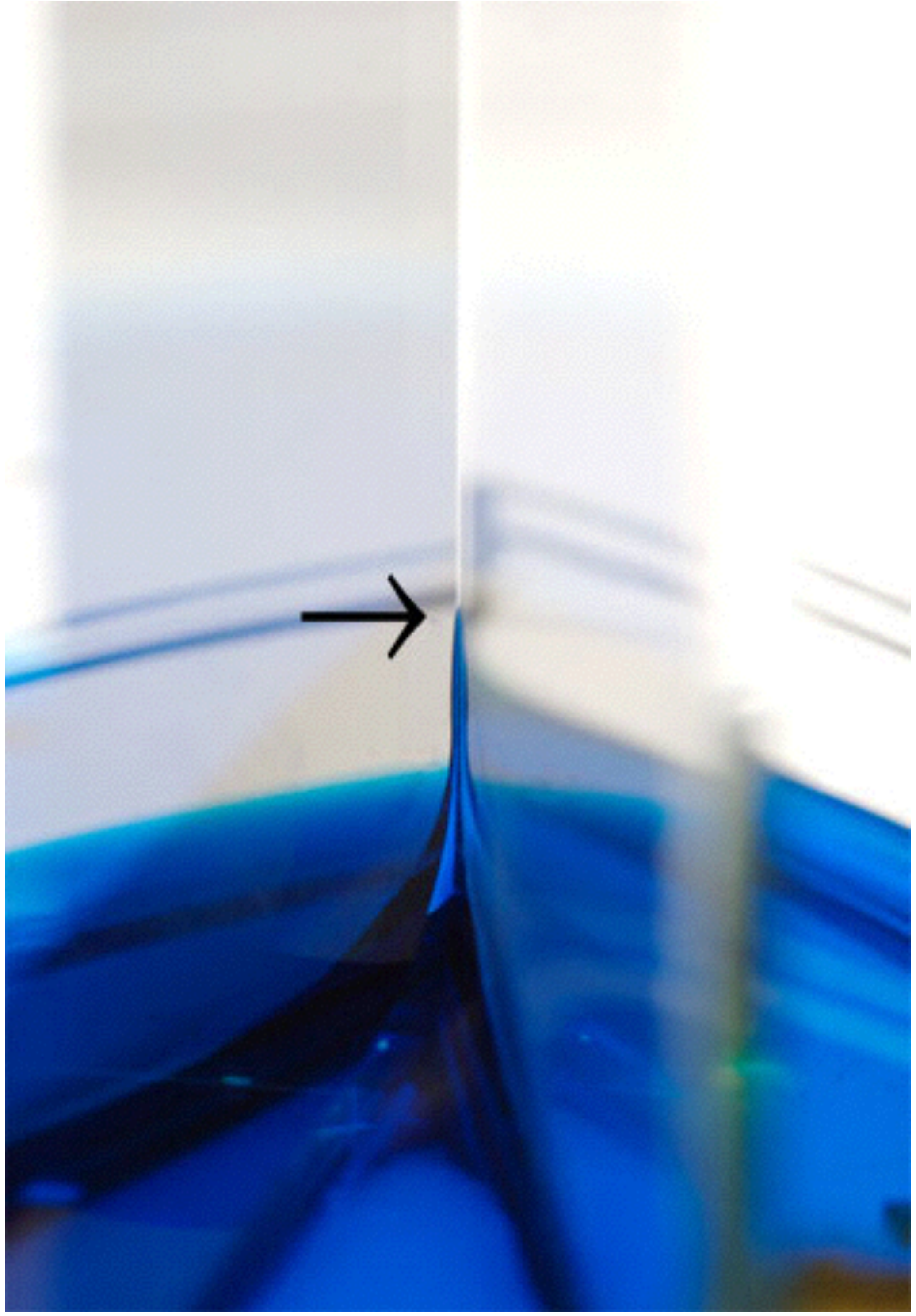

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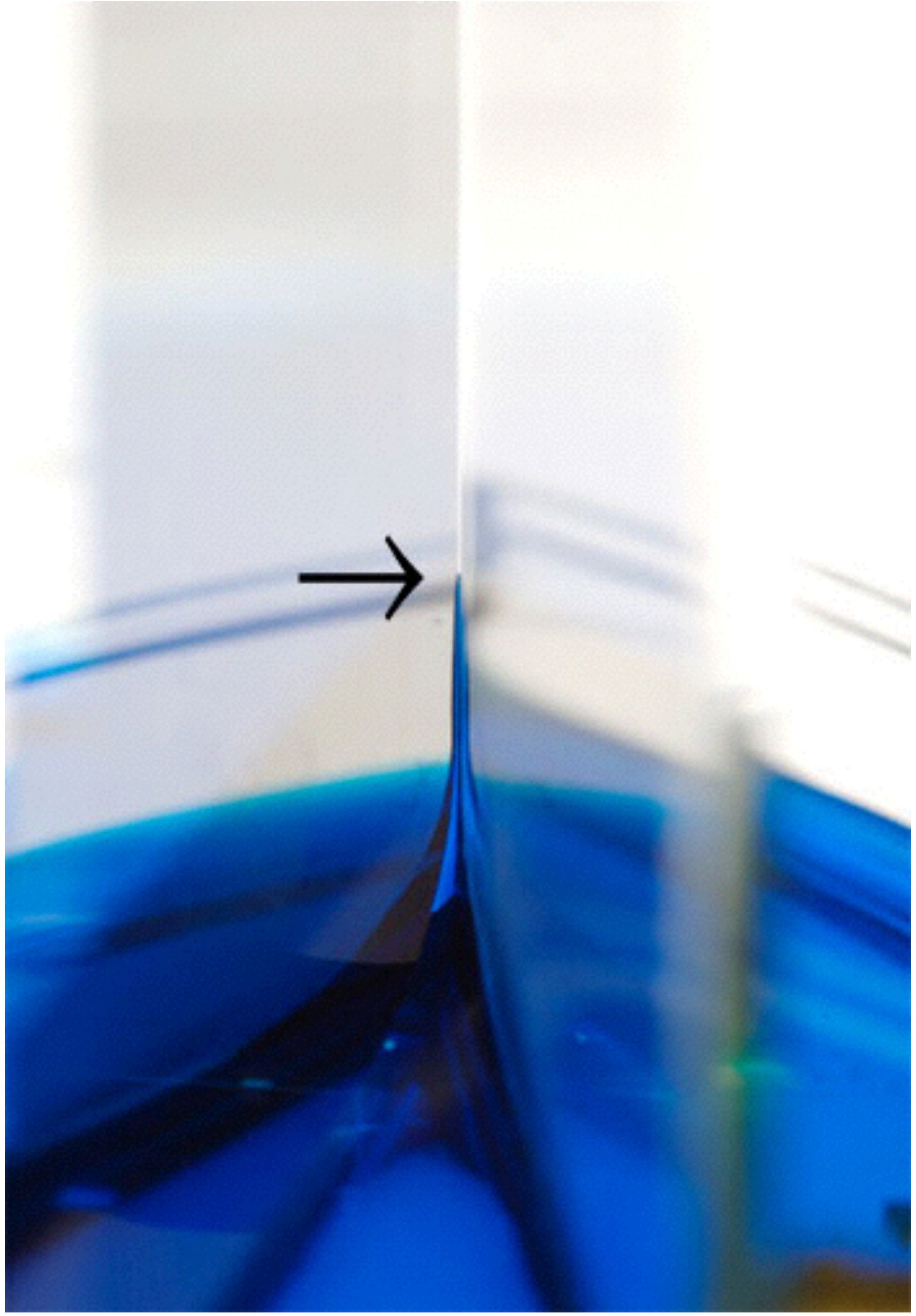

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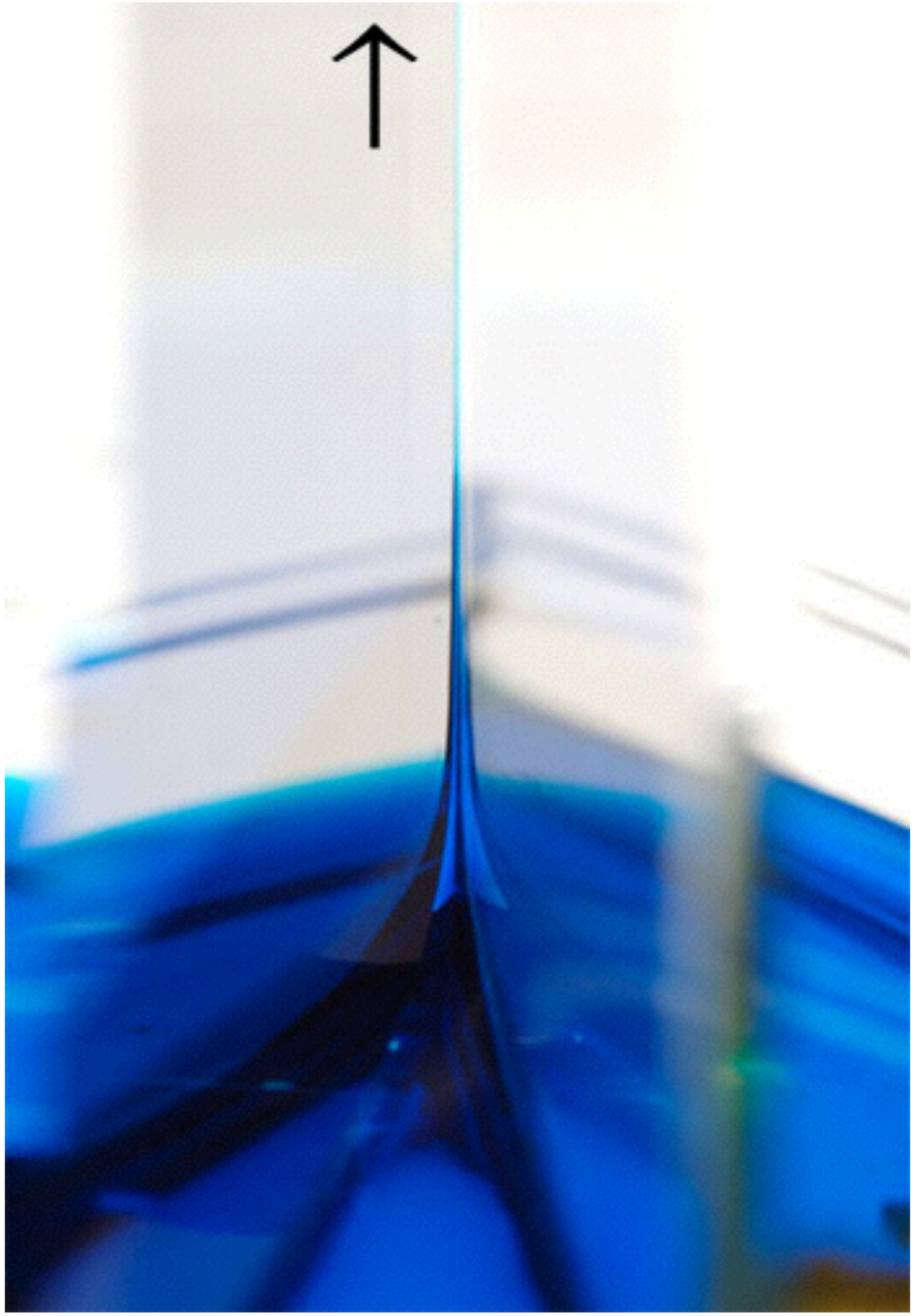


Concus-Finn Condition

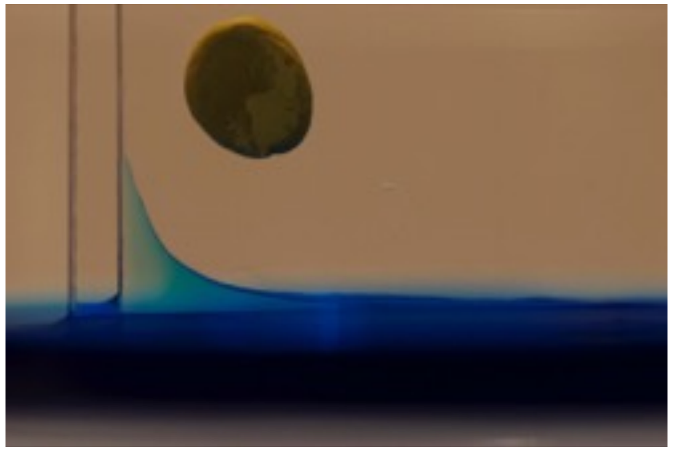
Height of the capillary surface depends <u>discontinuous</u> to wedge angle.

(Robert Finn and Paul Concus)

unbounded surfaces

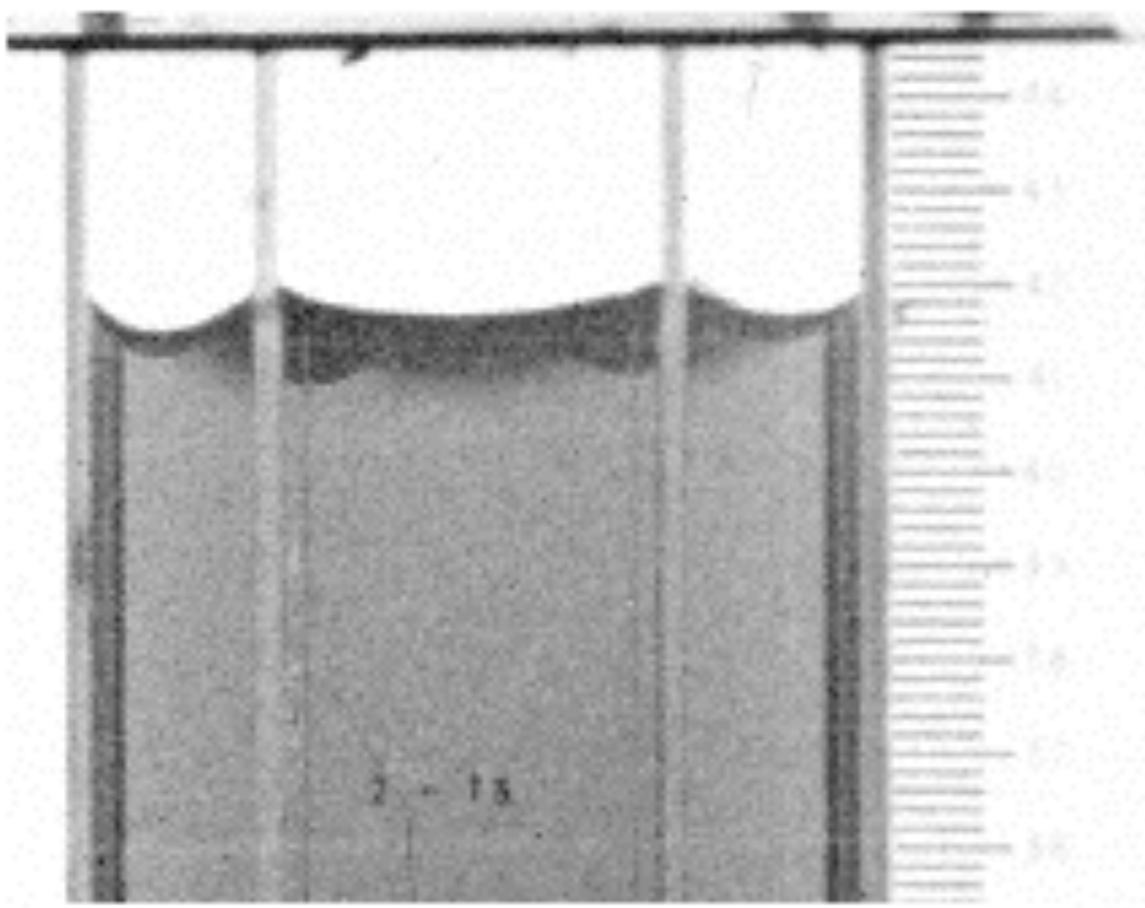
$$\alpha < \frac{\pi}{2} - \gamma$$

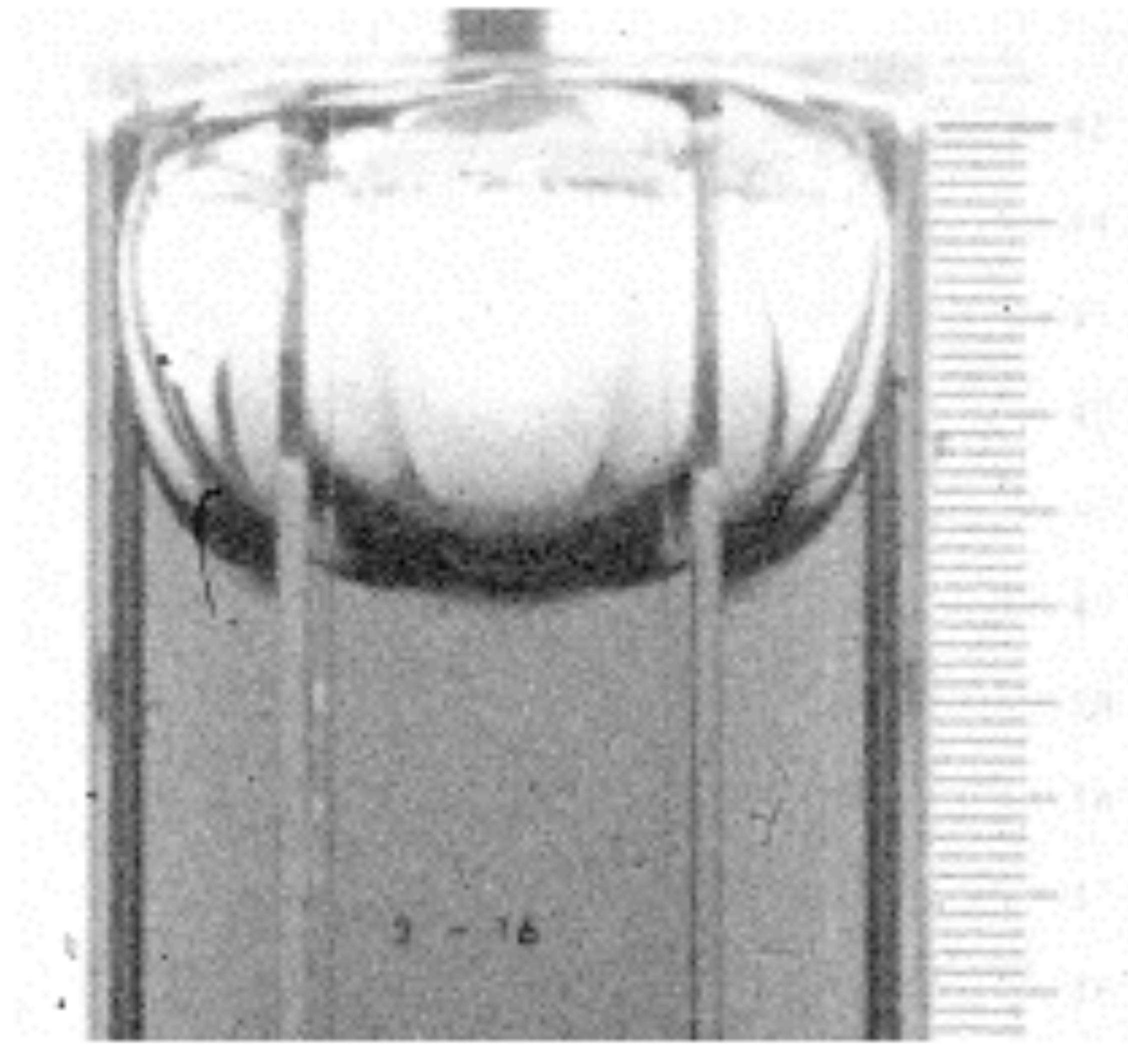
bounded surface



$$\alpha \geq \frac{\pi}{2} - \gamma$$

Design of Fuel Tanks

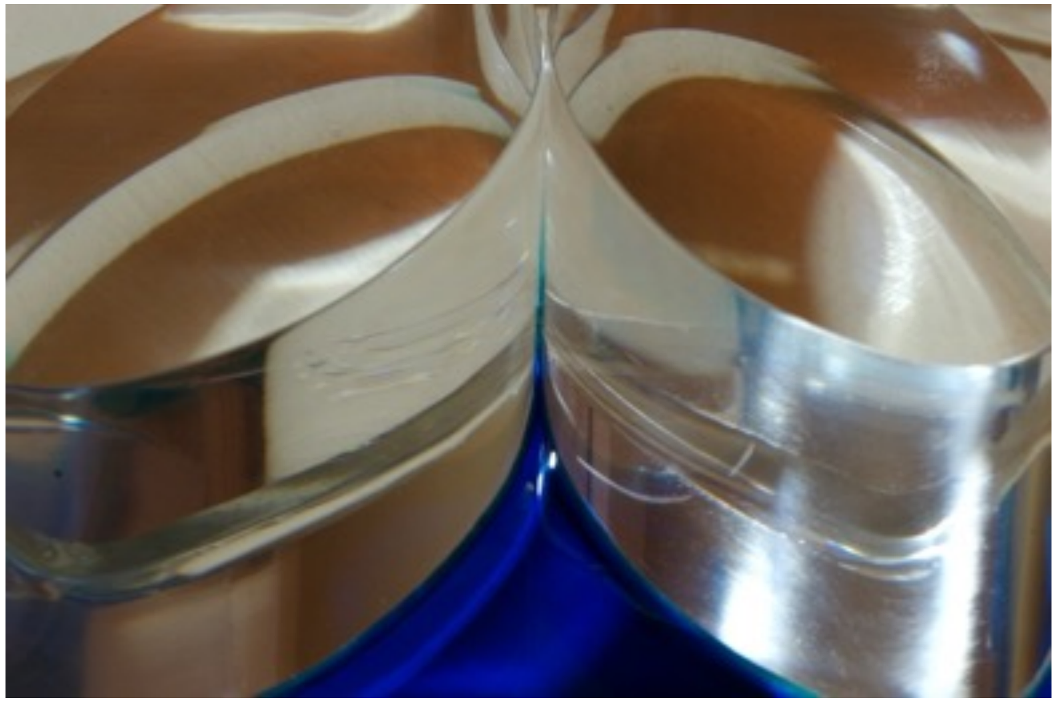




Different fluids in identical hexagonal cylinders during free-fall

Photo and Text from the website of Prof. Kirk Lancaster

Cusp Problem



Cusp Problem

Capillary Surface rises with the same order like the order of contact of two arcs forming the cusp. (Markus Scholz)

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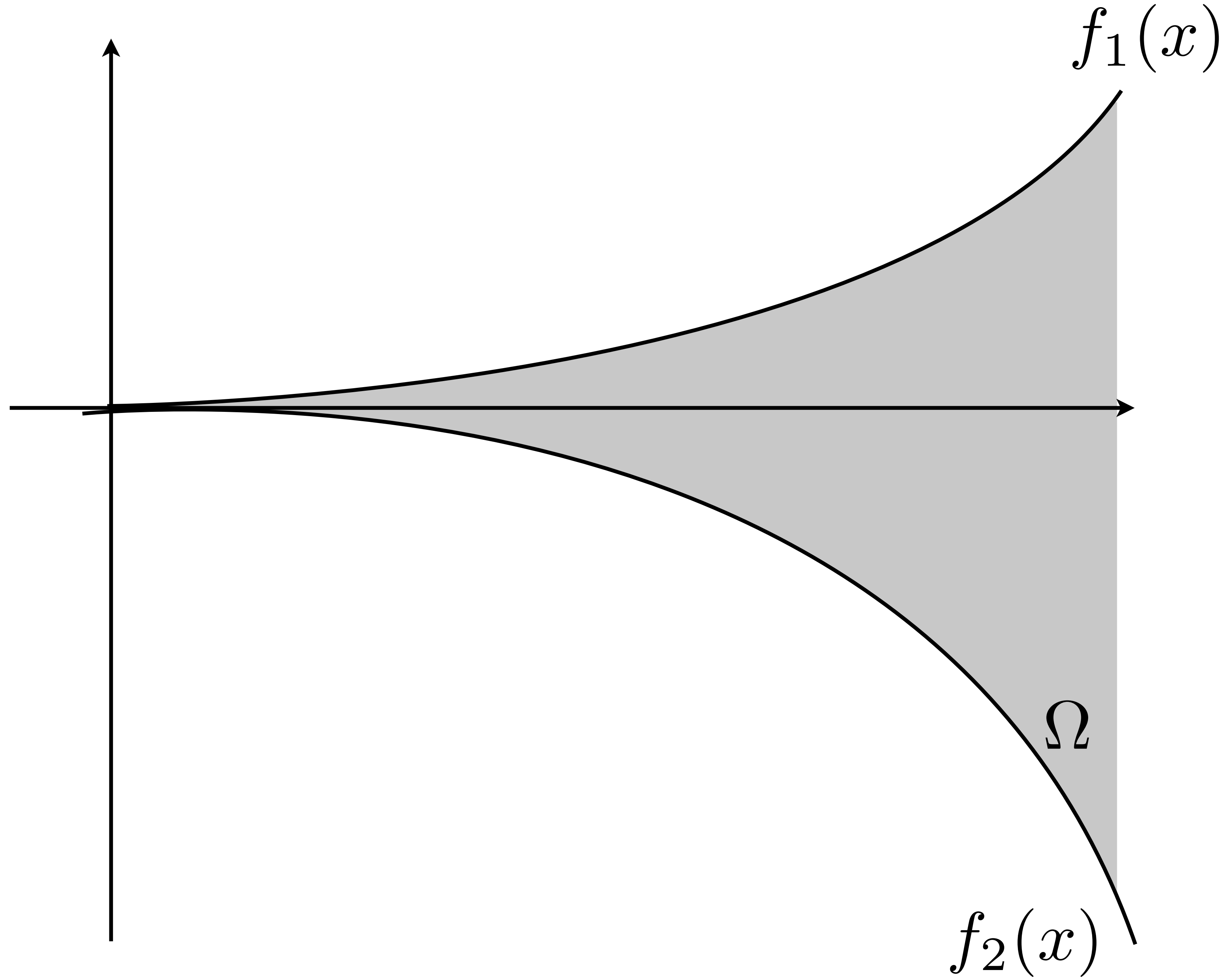

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$$u(x,y) = \Theta\left(\frac{1}{f_1(x) - f_2(x)}\right) \qquad \text{as } x \to 0$$

Cusp Problem

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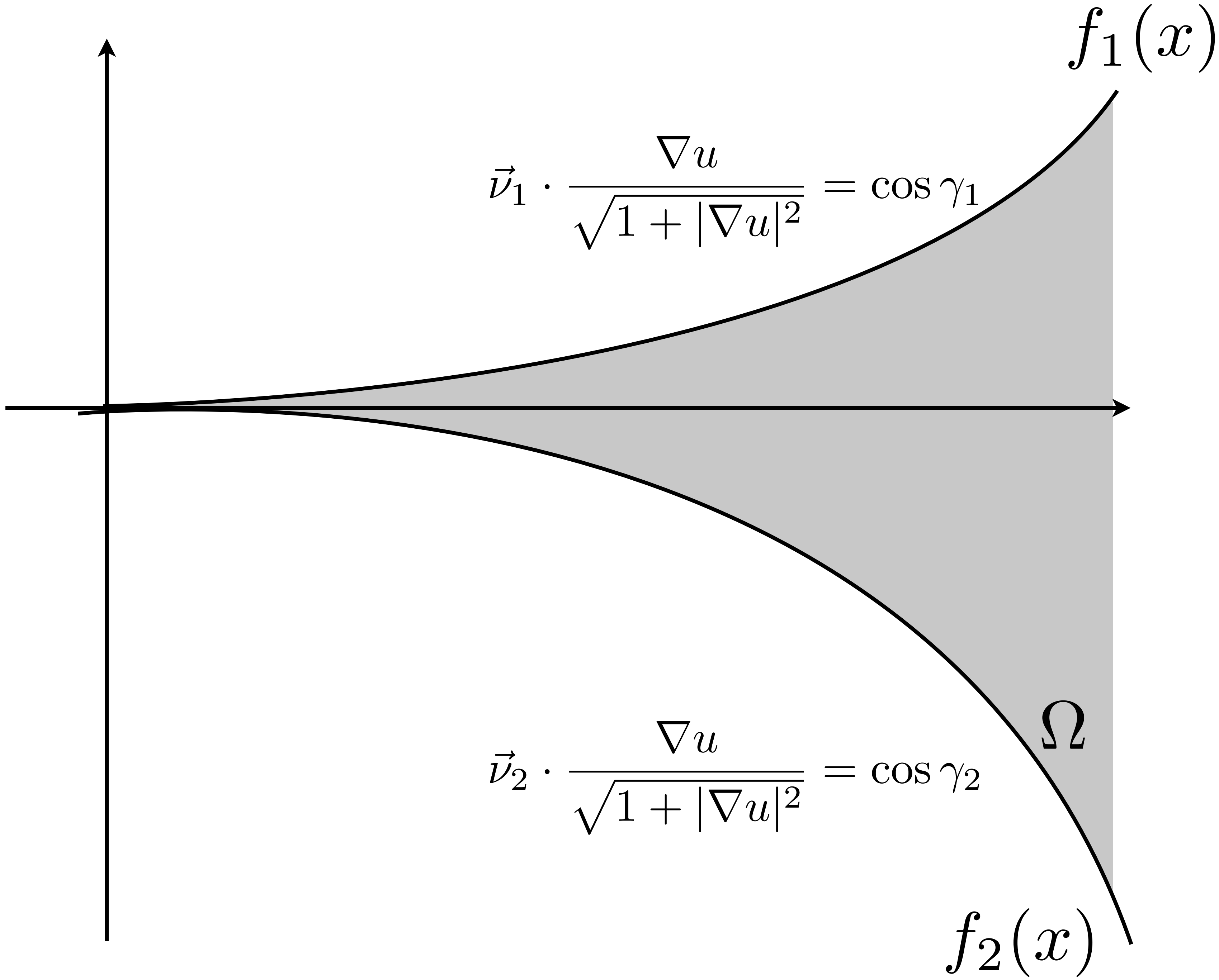
$$u(x,y) = \Theta\left(\frac{1}{f_1(x) - f_2(x)}\right) \qquad \text{as } x \to 0$$

If f_1 and f_2 can be written as **power series** and $\cos\gamma_1 + \cos\gamma_2 \neq 0$.

Cusp Problem

Capillary Surface rises with the same order like the order of contact of two arcs forming the cusp.   (Markus Scholz)



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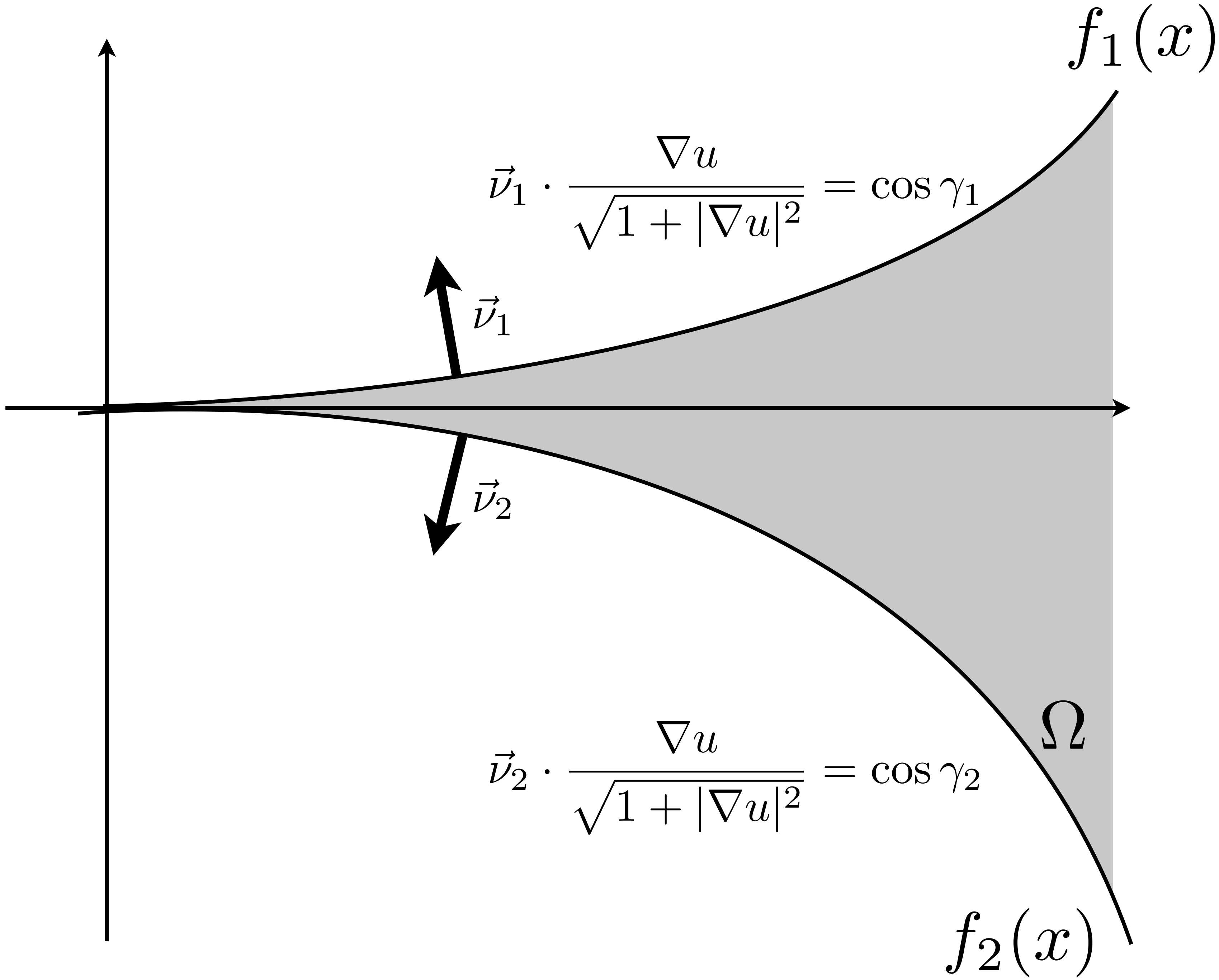

Unbounded Capillary Surface

$$\cos\gamma_1 + \cos\gamma_2 \neq 0$$

Power Series Cusp

(Scholz 2004)

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Open Problems

What if $\cos\gamma_1 + \cos\gamma_2 = 0$?

Unbounded Capillary Surface

$$\cos\gamma_1 + \cos\gamma_2 \neq 0$$

Power Series Cusp
(Scholz 2004)

Open Problems

What if $\cos\gamma_1 + \cos\gamma_2 = 0$?

Non-power Series Cusp?

Theorem 0: u is unbounded if $\cos\gamma_1 + \cos\gamma_2 \neq 0$.

Markus Scholz. Über das Verhalten von Kapillarflächen in Spitzen. PhD thesis, Universität Leipzig, 2001.

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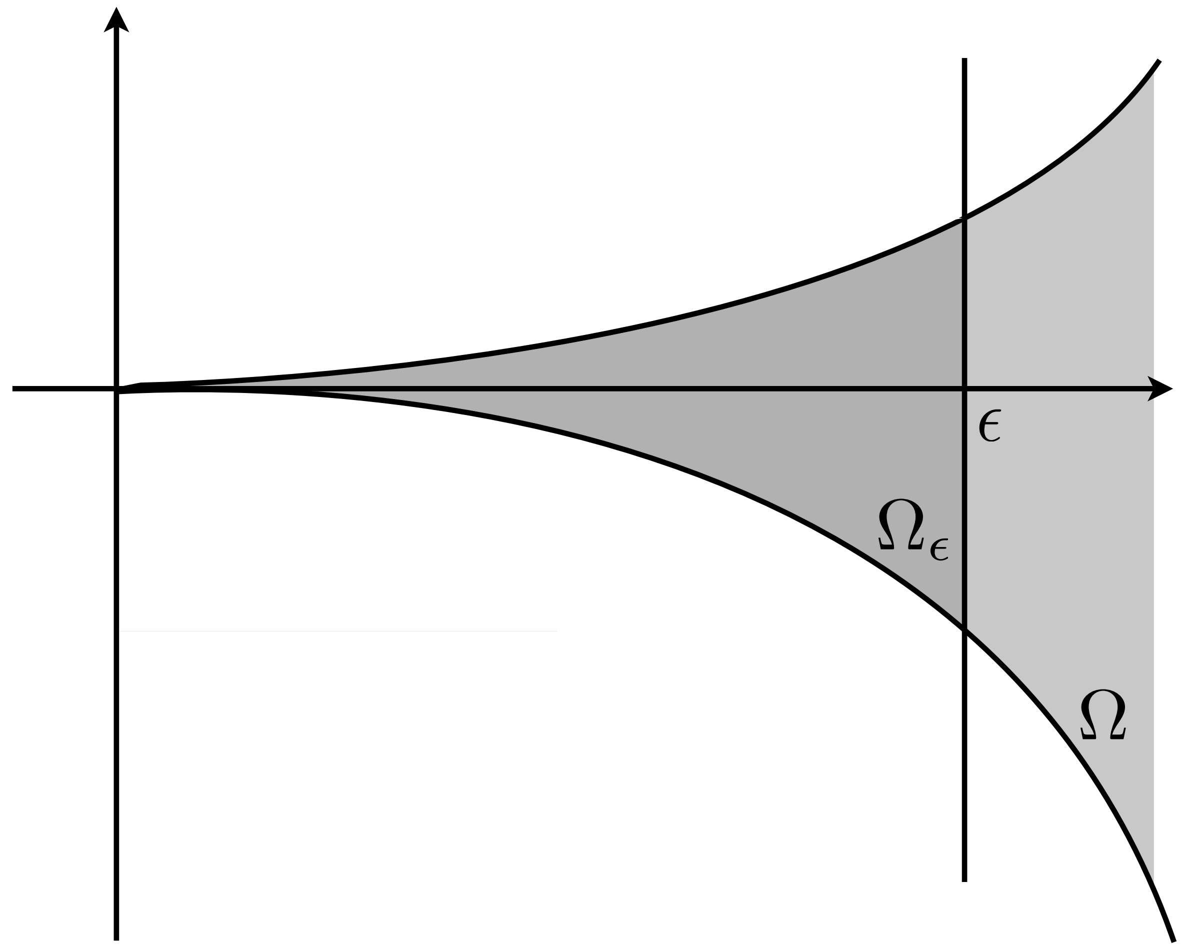

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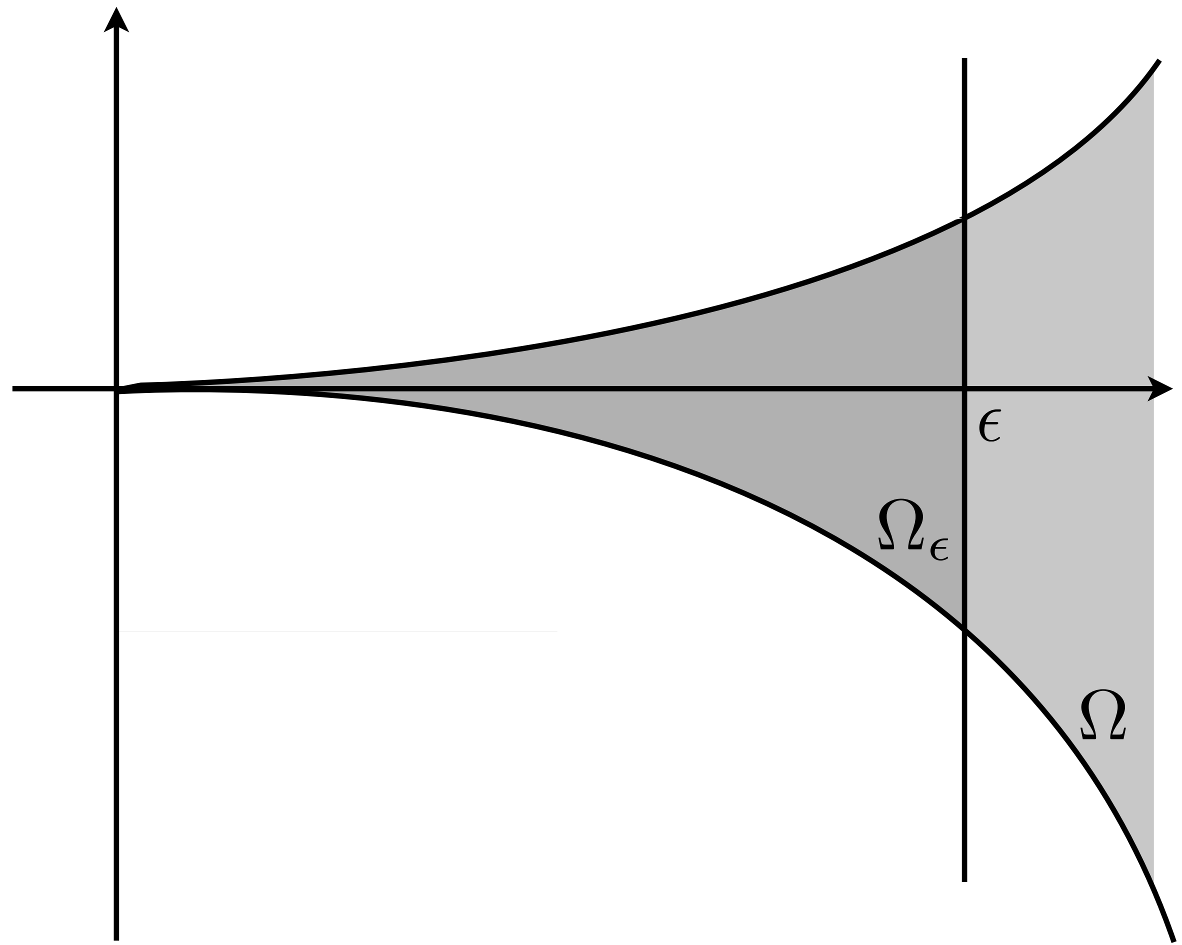
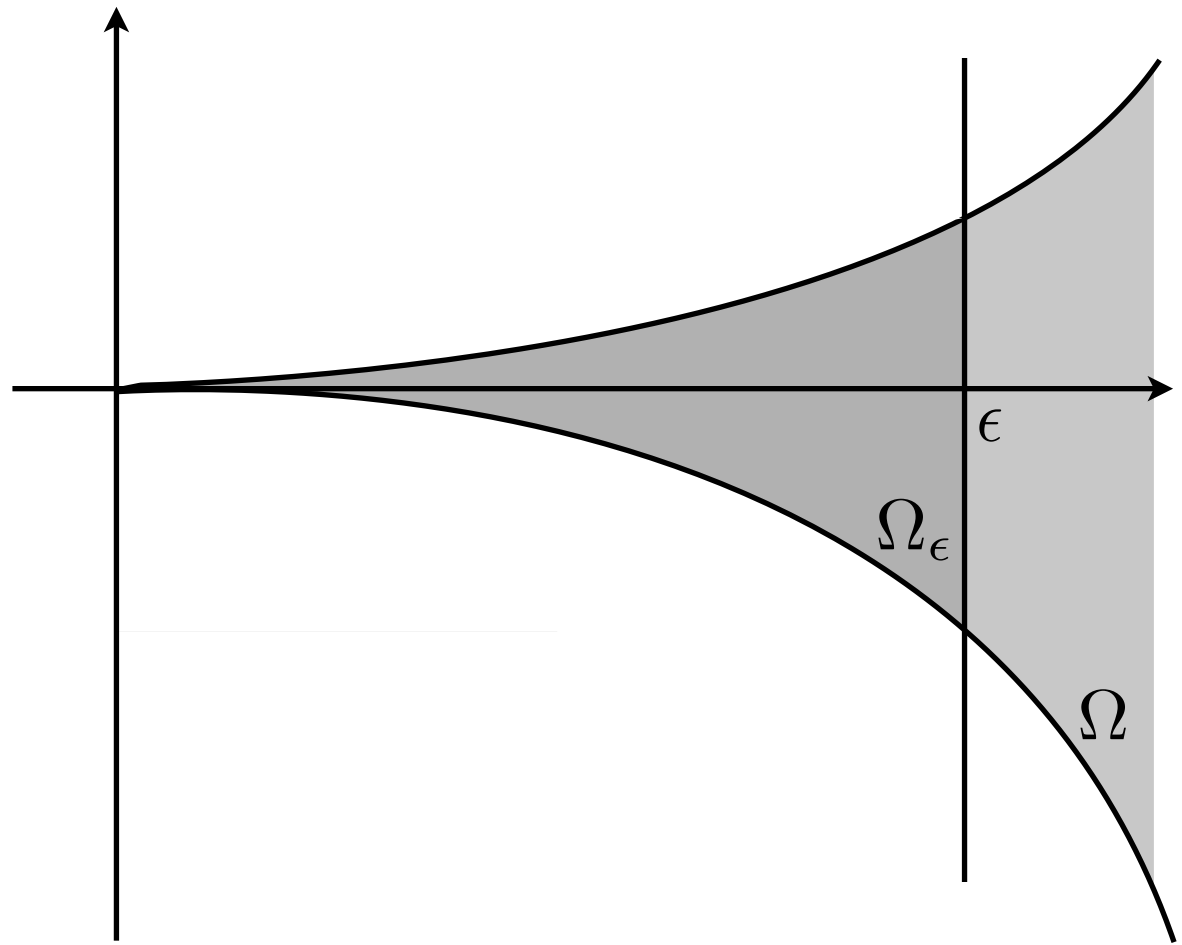


Integrate both sides of the PDE in Ω_ϵ .

$$\int_{\Omega_\epsilon} \nabla \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \mathrm{d}A = \int_{\Omega_\epsilon} u \, \mathrm{d}A$$

Theorem 0: u is unbounded if $\cos\gamma_1 + \cos\gamma_2 \neq 0$.



Integrate both sides of the PDE in Ω_ϵ .

$$\int_{\partial\Omega_\epsilon} \nu \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \mathrm{d}s = \int_{\Omega_\epsilon} u \, \mathrm{d}A$$

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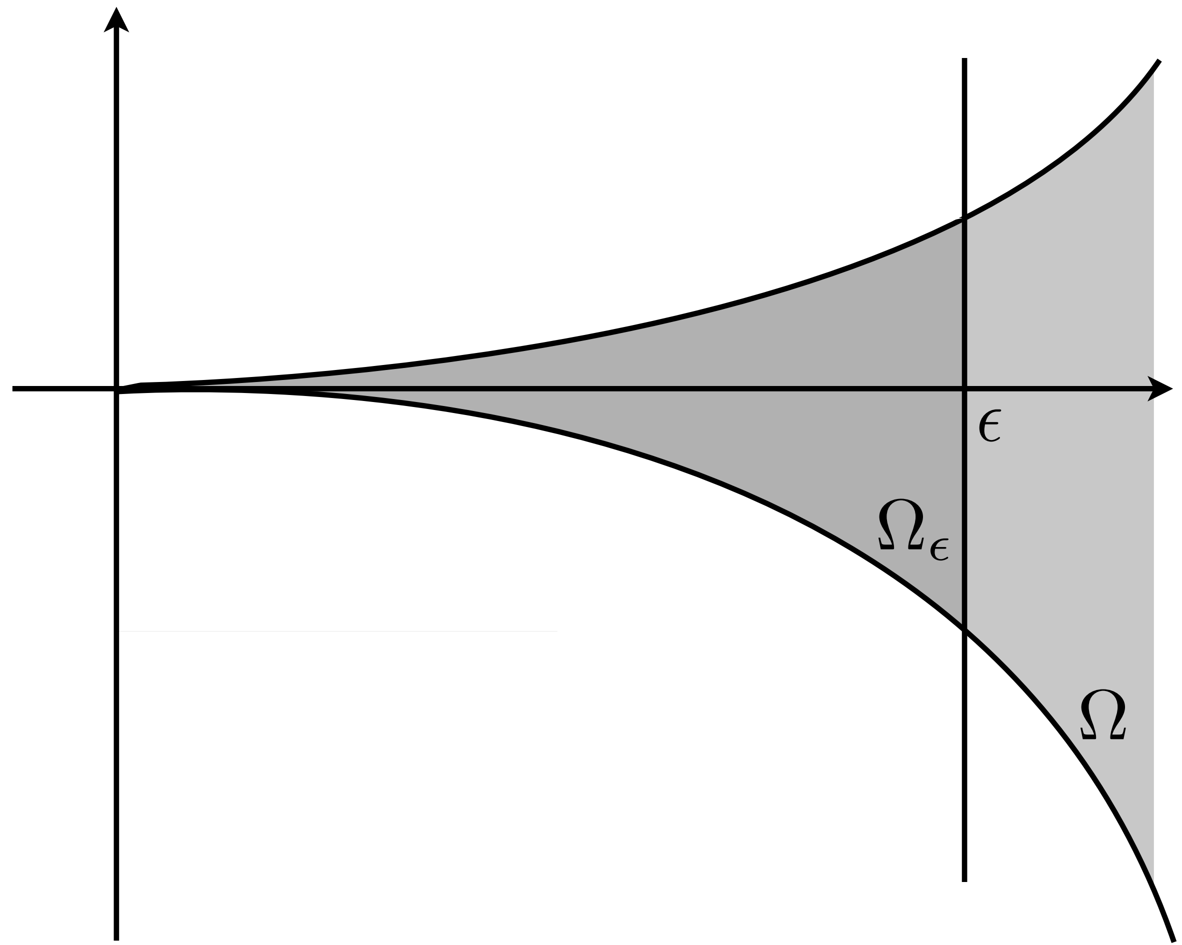

By letting $\epsilon \to 0$

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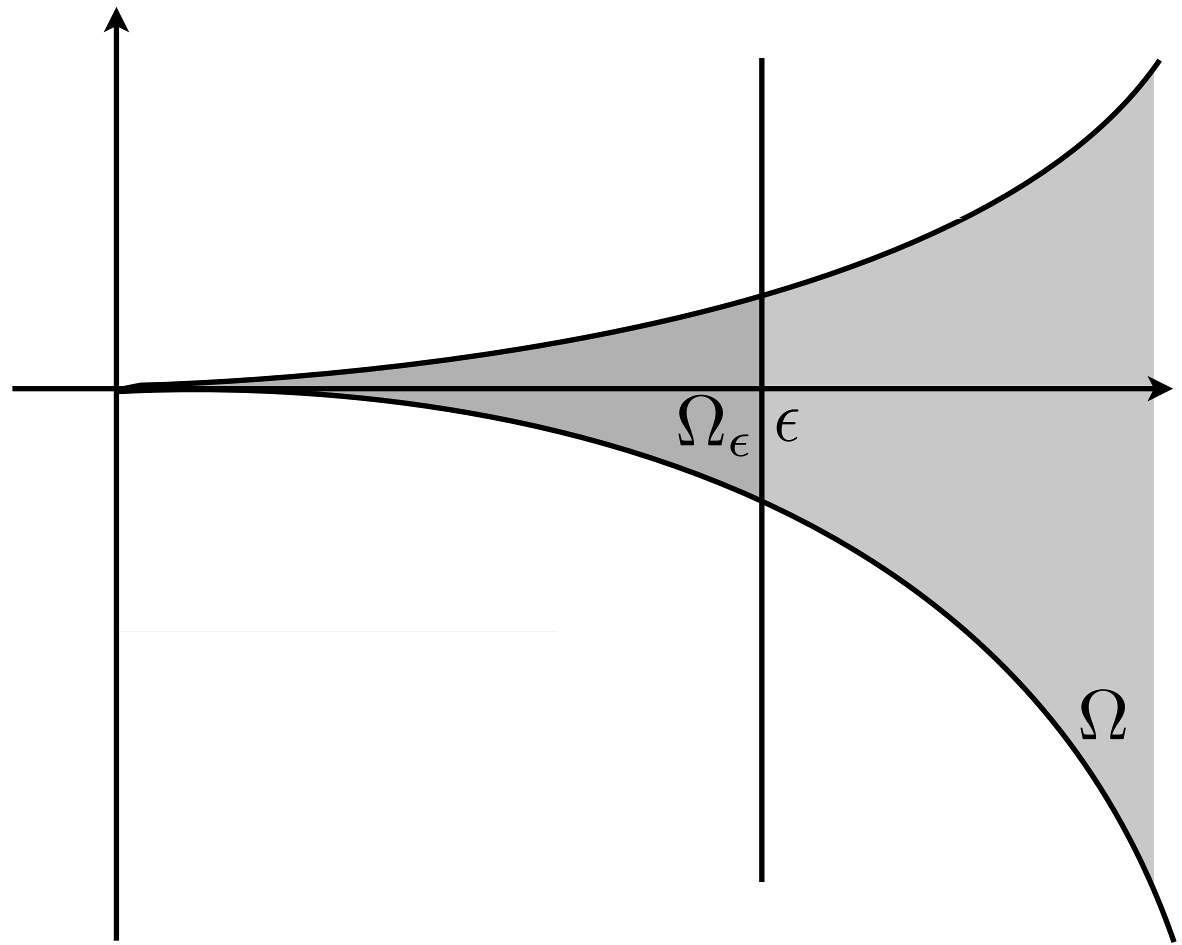

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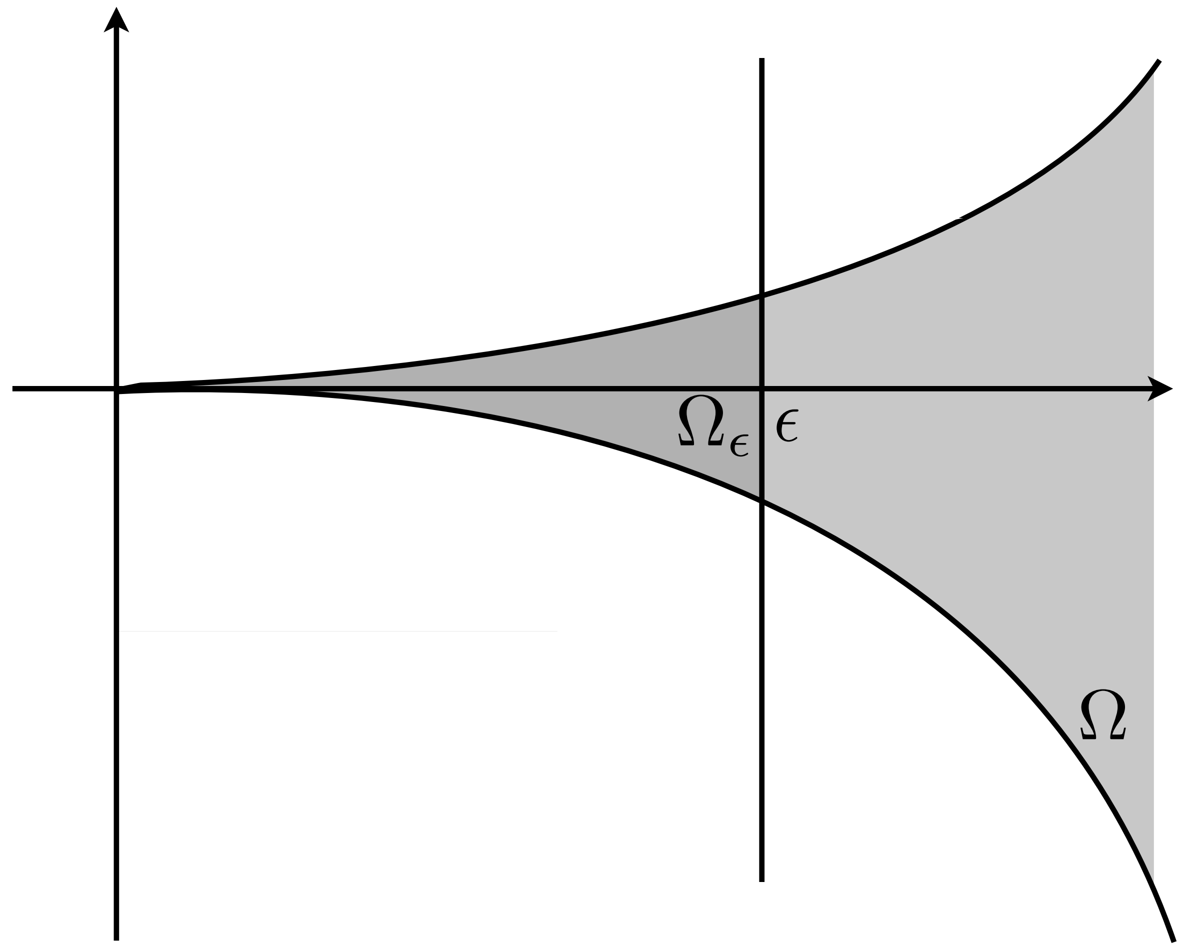

By letting $\epsilon \to 0$

$$(\cos\gamma_1 + \cos\gamma_2) O(\epsilon) = \int_{\Omega_\epsilon} u \, \mathrm{d}A$$

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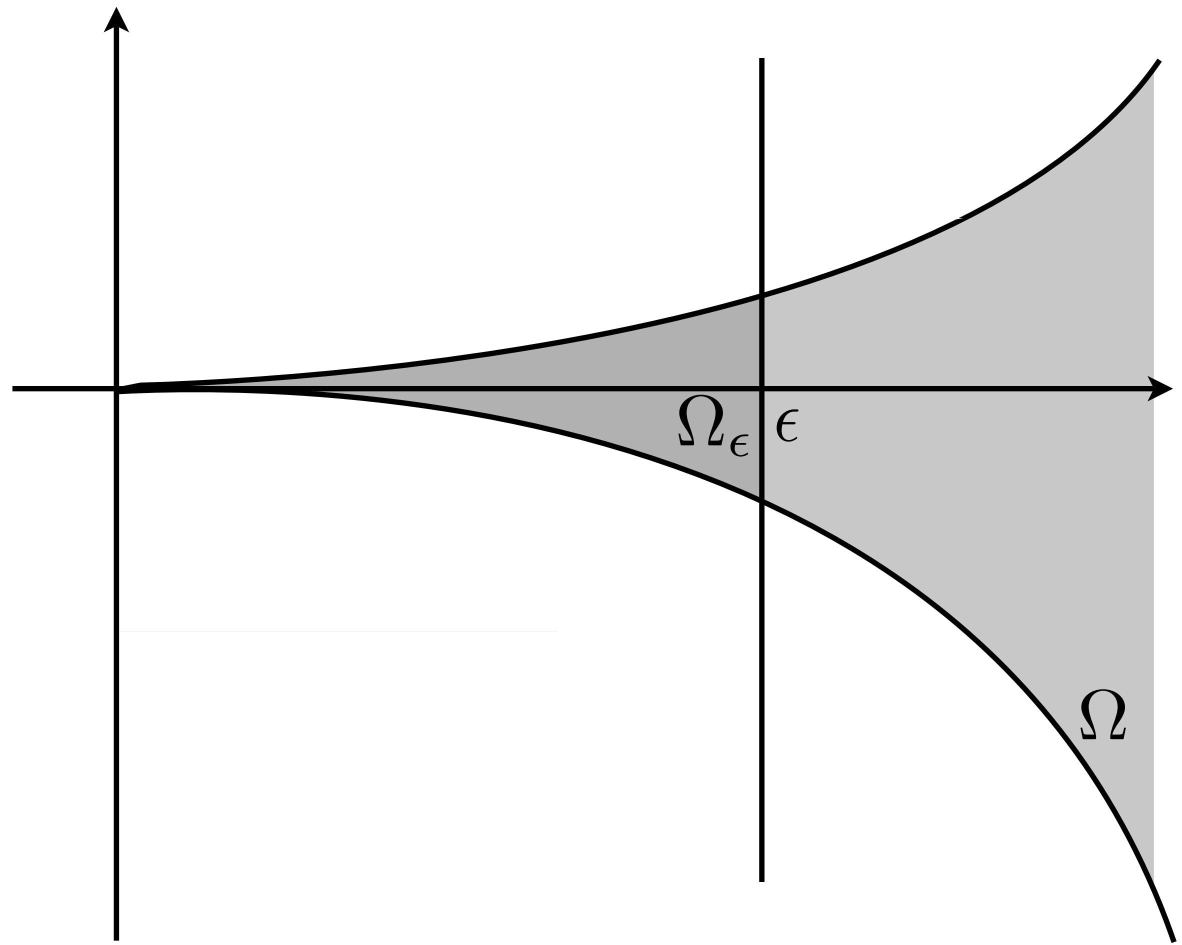

By letting $\epsilon \to 0$

$$(\cos\gamma_1 + \cos\gamma_2)O(\epsilon) = O(\epsilon)o(\epsilon)O(u)$$

Theorem 1: u is bounded if $\cos\gamma_1 + \cos\gamma_2 = 0$,

if the boundaries of the cusp have finite curvature.

Yasunori Aoki and David Siegel:
Journal of Mathematics, Volume 257, Issue 1, 2012, Pages 143-165.

Theorem 1: u is bounded if $\cos\gamma_1 + \cos\gamma_2 = 0$,

if the boundaries of the cusp have finite curvature.

We use the Concus Finn comparison principle

If $$\nabla \cdot \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} - v < 0 \qquad ①$$

$$\vec{\nu}_1 \cdot \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} \geq \cos\gamma_1 \qquad ②$$

$$\vec{\nu}_2 \cdot \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} \geq \cos\gamma_2 \qquad ③$$

Then $v > u \quad \text{in } \Omega$

For the simplicity of the presentation we assume $0 < \gamma_1 < \pi/2$.

Theorem 1: u is bounded if $\cos\gamma_1 + \cos\gamma_2 = 0$,

if the boundaries of the cusp have finite curvature.

We now try to construct v that satisfies ①

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Let

$$v = v_o + C$$

Theorem 1: u is bounded if $\cos\gamma_1 + \cos\gamma_2 = 0$,
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↑
constant

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bounded and finite mean curvature

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Let

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constant

bounded and finite mean curvature

$$\nabla \cdot \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} - v = \nabla \cdot \frac{\nabla v_o}{\sqrt{1+|\nabla v_o|^2}} - v_o - C$$

Theorem 1: u is bounded if $\cos\gamma_1 + \cos\gamma_2 = 0$, if the boundaries of the cusp have finite curvature.

We now try to construct v that satisfies (1)

Let

$$v = v_o + C$$

bounded and finite mean curvature

constant (arrow to C)

choose this big enough (arrow to C)

$$\nabla \cdot \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} - v = \nabla \cdot \frac{\nabla v_o}{\sqrt{1+|\nabla v_o|^2}} - v_o - C$$

Theorem 1: u is bounded if $\cos\gamma_1 + \cos\gamma_2 = 0$,

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We now try to construct v that satisfies (1)

Let

$$v = v_o + C$$

constant

choose this big enough

bounded and finite mean curvature

$$\nabla \cdot \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} - v = \nabla \cdot \frac{\nabla v_o}{\sqrt{1+|\nabla v_o|^2}} - v_o - C < 0$$ (1) ✓

Theorem 1: u is bounded if $\cos\gamma_1 + \cos\gamma_2 = 0$,

if the boundaries of the cusp have finite curvature.

We now try to construct v_o that satisfies ② ③

$$v = v_o + C$$

$$\vec{\nu}_1 \cdot \frac{\nabla v_o}{\sqrt{1 + |\nabla v_o|^2}} \geq \cos\gamma_1$$

$$\vec{\nu}_2 \cdot \frac{\nabla v_o}{\sqrt{1 + |\nabla v_o|^2}} \geq \cos\gamma_2$$

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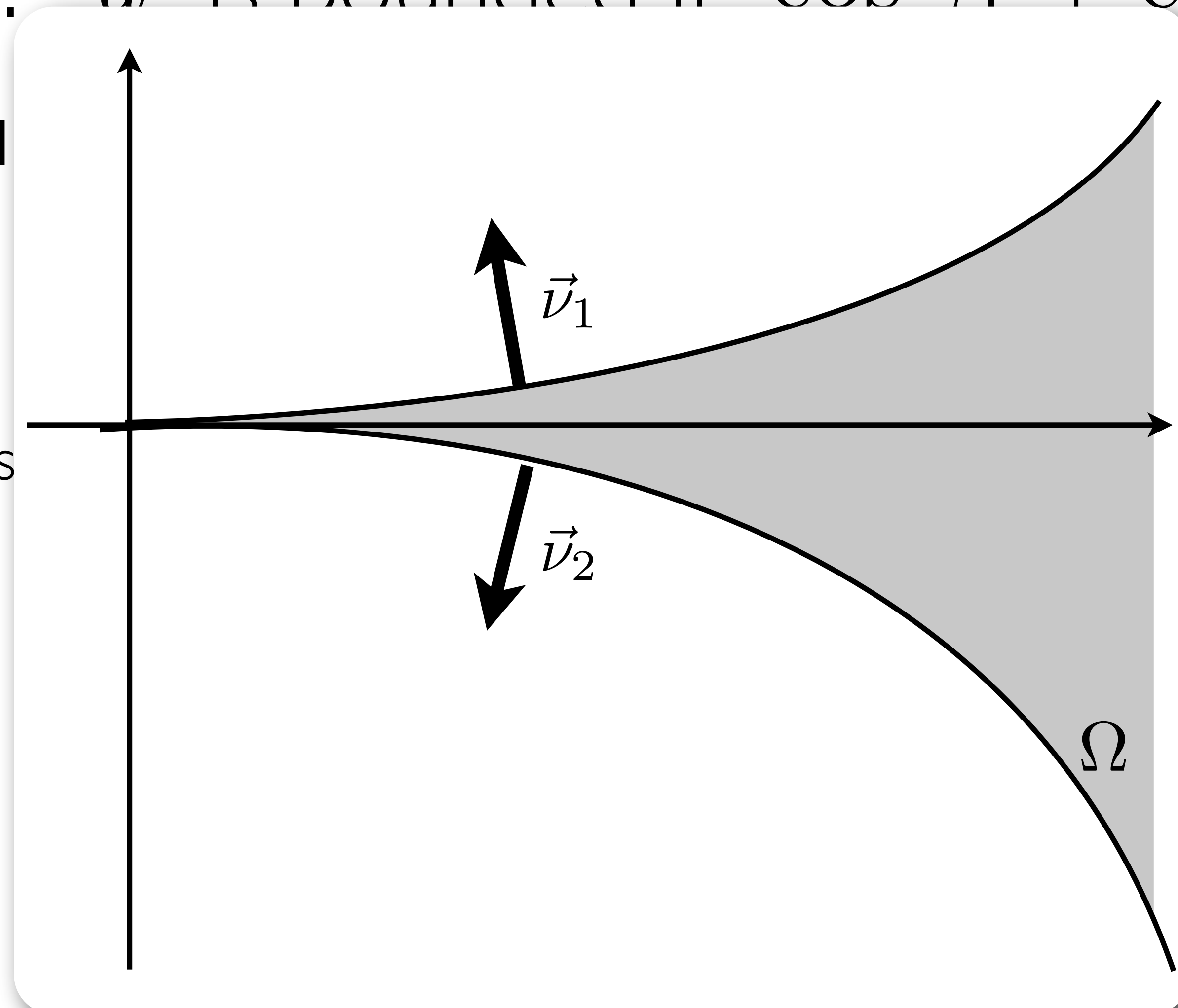
Theorem 1: u is bounded if $\cos\gamma_1 + \cos\gamma_2 = 0$,

if the bou curvature.



We now try to cons

$$\vec{\nu}_1 \cdot \frac{\nabla v_o}{\sqrt{1+|\nabla v_o|^2}} \geq \cos\gamma_1$$

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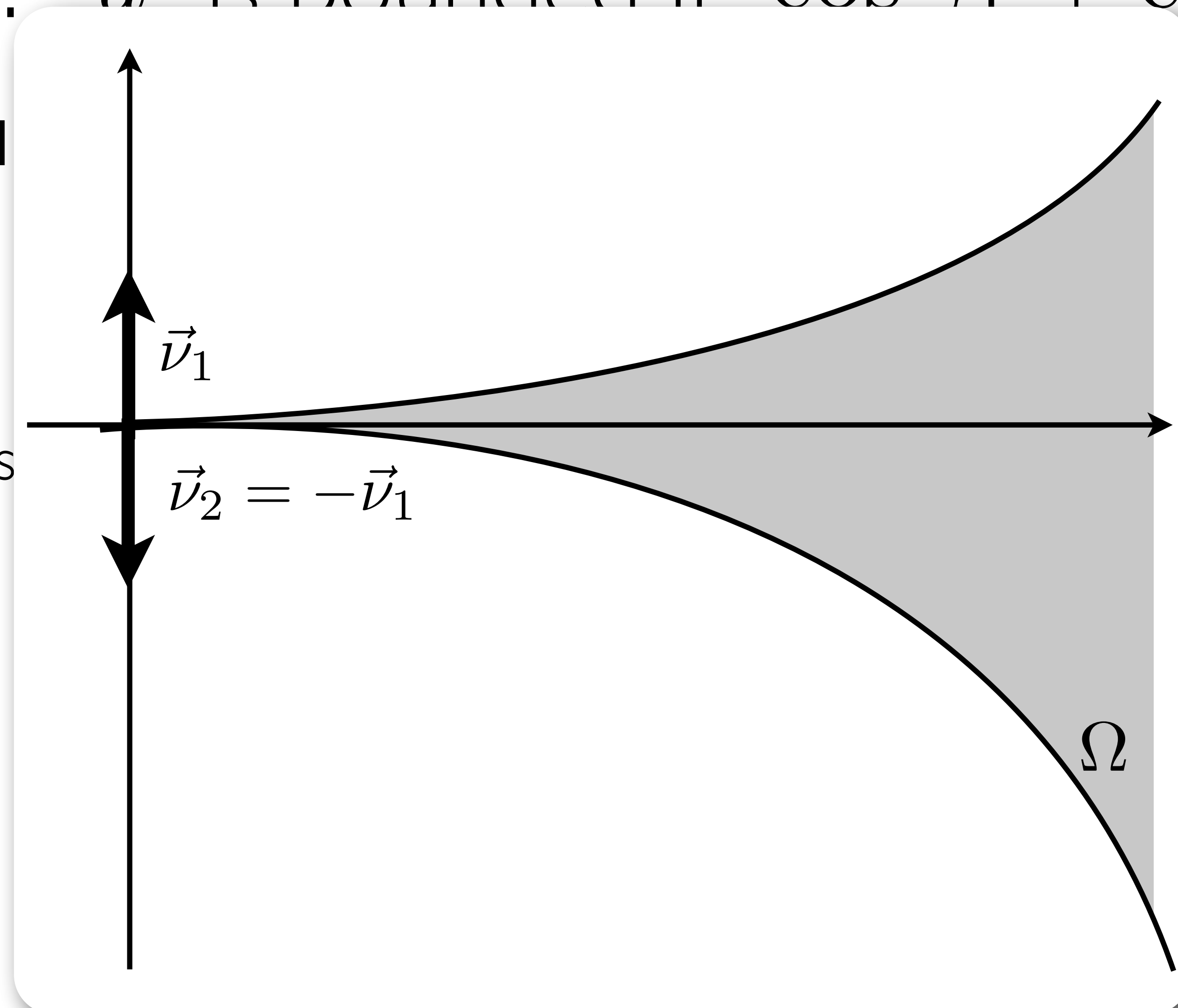

$$\vec{\nu}_1 \cdot \frac{\nabla v_o}{\sqrt{1+|\nabla v_o|^2}} \geq \cos\gamma_1$$

$$-\vec{\nu}_1 \cdot \frac{\nabla v_o}{\sqrt{1+|\nabla v_o|^2}} \geq \cos\gamma_2 \quad = -\cos\gamma_1$$

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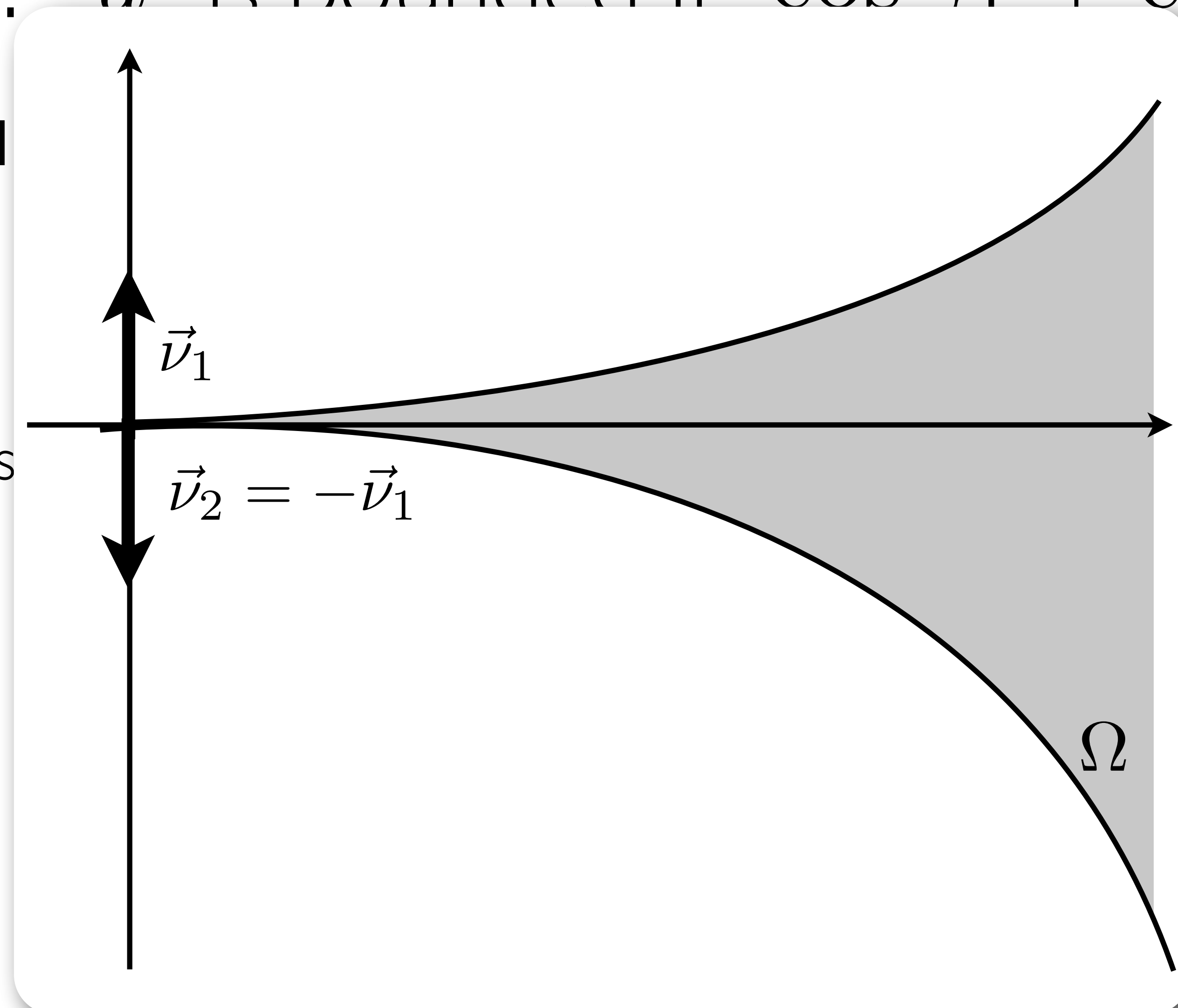


We now try to cons

$$\lim_{x \to 0} \vec{\nu}_1 \cdot \frac{\nabla v_o}{\sqrt{1 + |\nabla v_o|^2}} = \cos\gamma_1$$

Theorem 1: u is bounded if $\cos\gamma_1 + \cos\gamma_2 = 0$,

if the boundaries of the cusp have finite curvature.

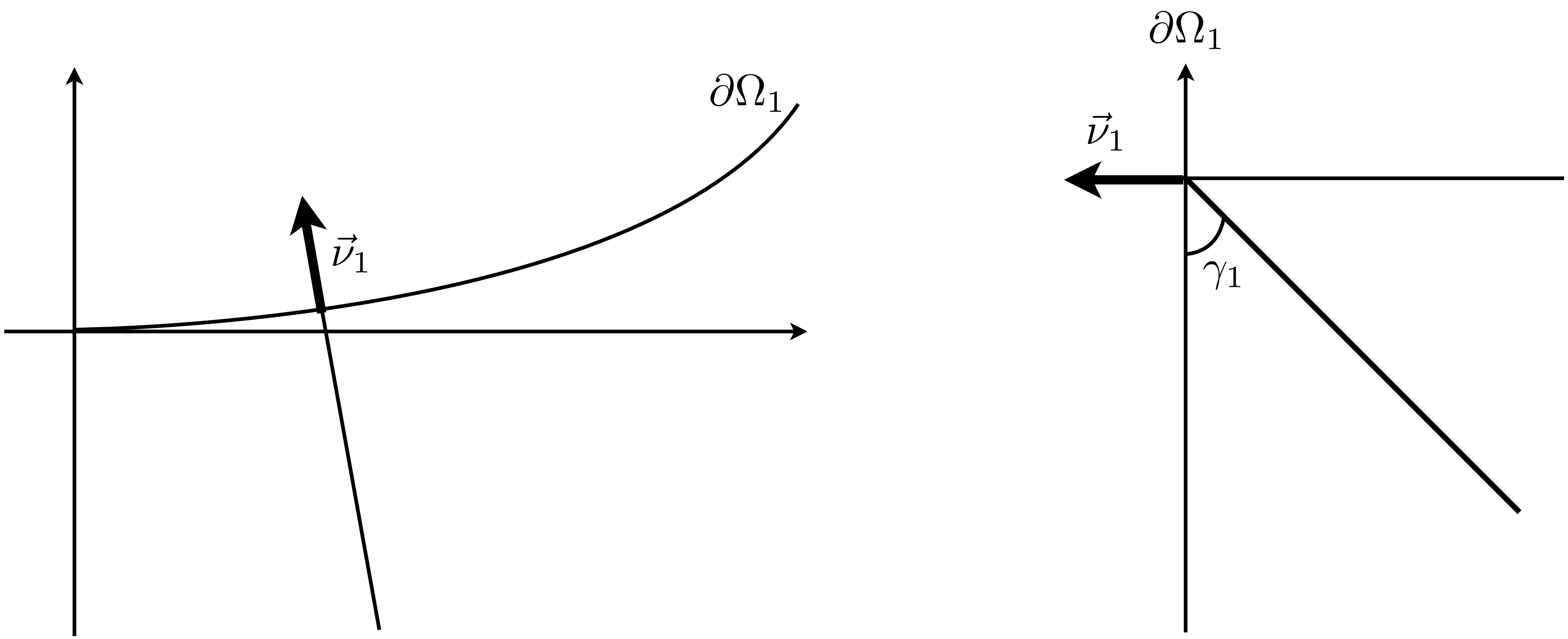
The easiest way to construct v_o that satisfies $\lim_{x \to 0} \vec{\nu}_1 \cdot \frac{\nabla v_o}{\sqrt{1 + |\nabla v_o|^2}} = \cos\gamma_1$ is to impose

$$\vec{\nu}_1 \cdot \frac{\nabla v_o}{\sqrt{1 + |\nabla v_o|^2}} = \cos\gamma_1 \qquad \text{on } \partial\Omega_1$$

Theorem 1: u is bounded if $\cos\gamma_1 + \cos\gamma_2 = 0$, if the boundaries of the cusp have finite curvature.

We choose $v_o = -\cot\gamma_1\, t$ where t is the distance from the boundary $\partial\Omega_1$.

$$\vec{\nu}_1 \cdot \frac{\nabla v_o}{\sqrt{1+|\nabla v_o|^2}} = \cos\gamma_1 \qquad \text{on } \partial\Omega_1$$



Theorem I: u is bounded if $\cos\gamma_1 + \cos\gamma_2 = 0$,

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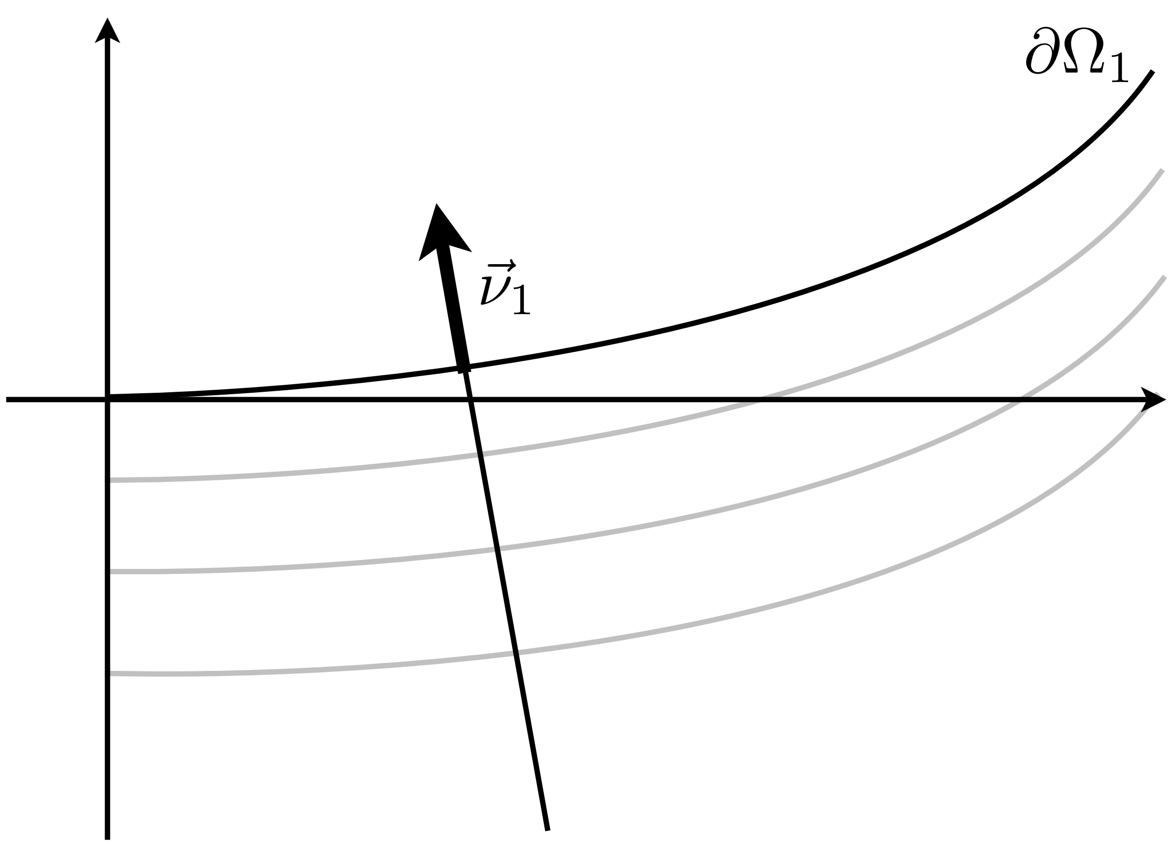

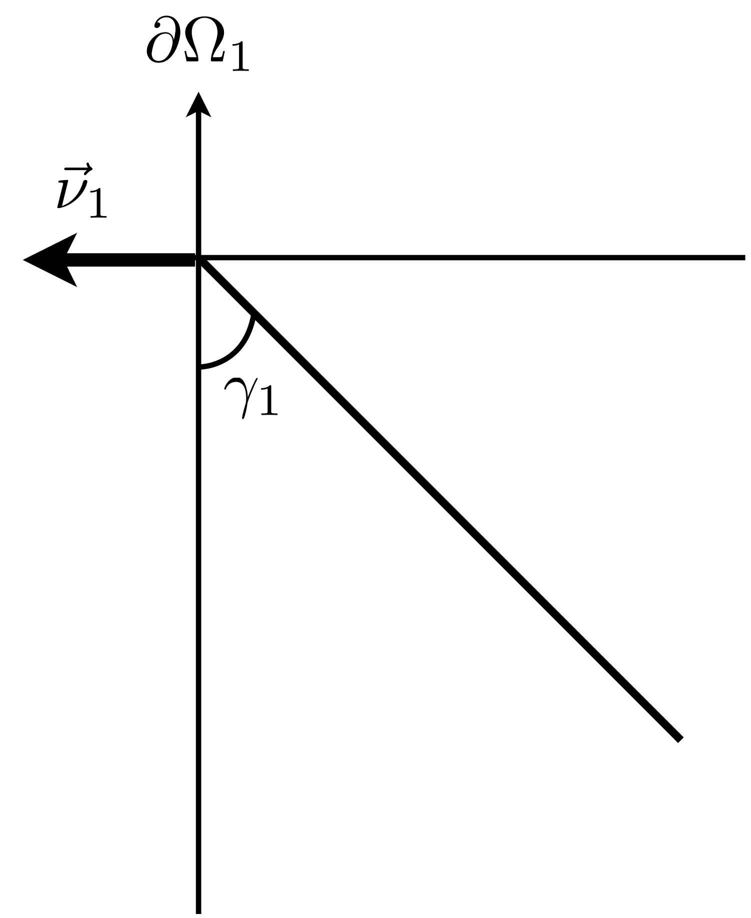

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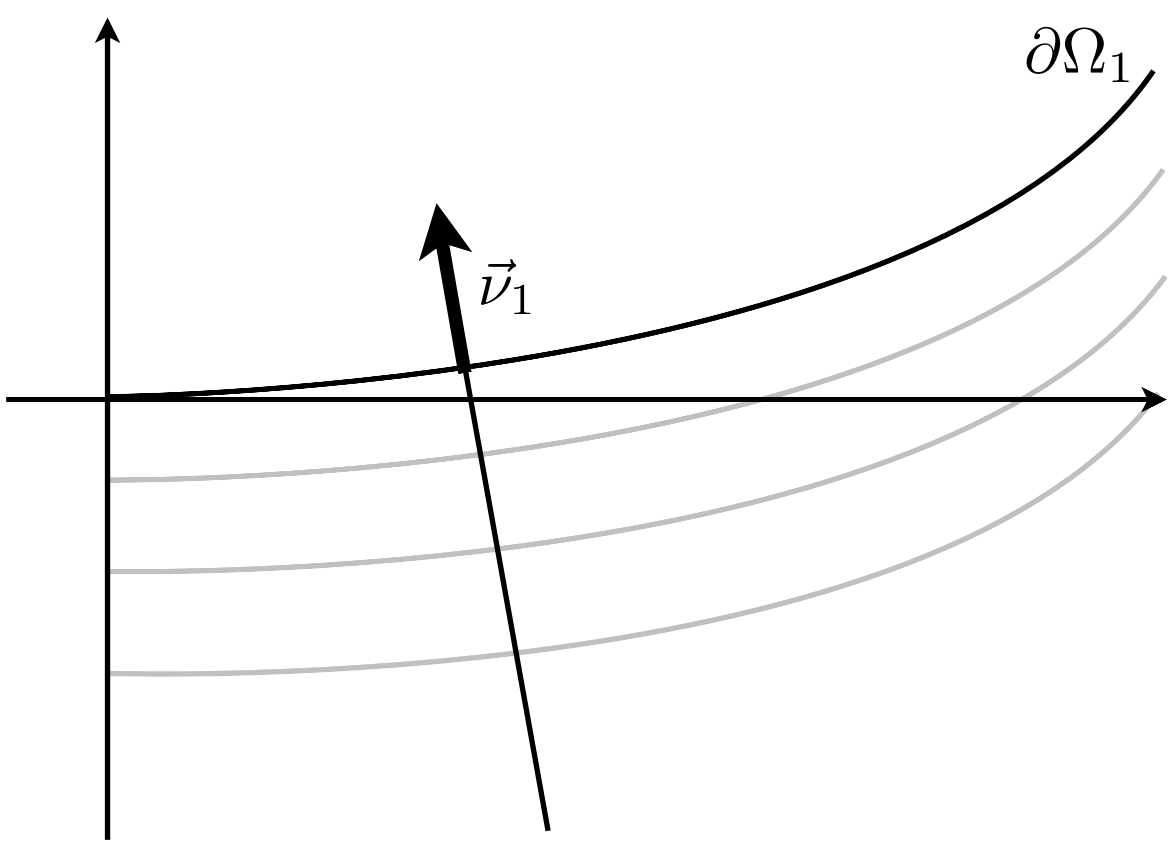
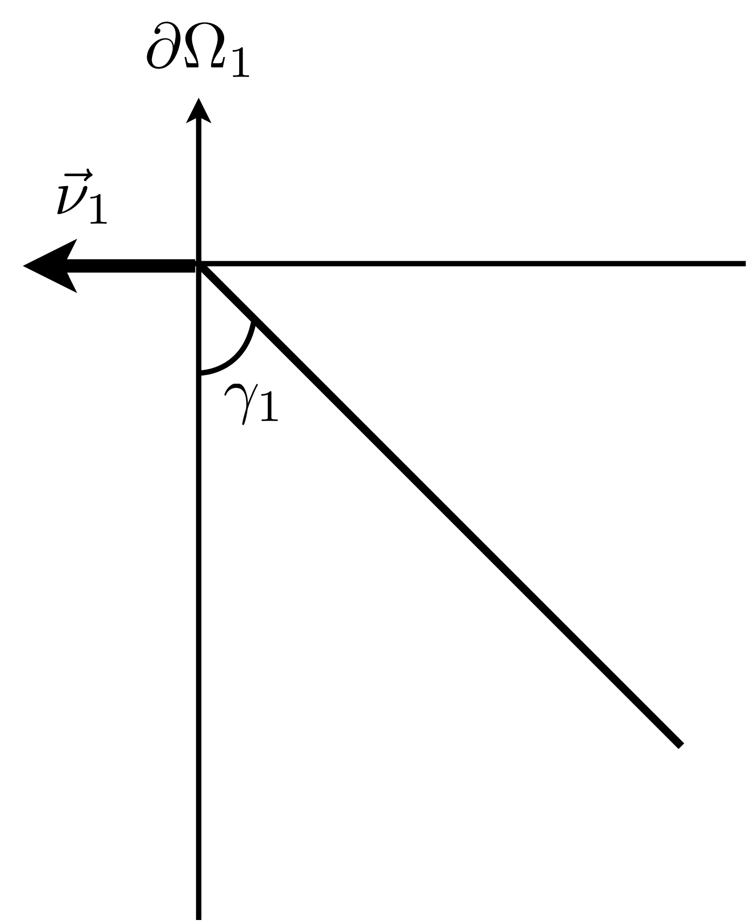
$$\vec{\nu}_1 \cdot \frac{\nabla v_o}{\sqrt{1+|\nabla v_o|^2}} = \cos\gamma_1 \qquad \text{on } \partial\Omega_1 \qquad (2)$$

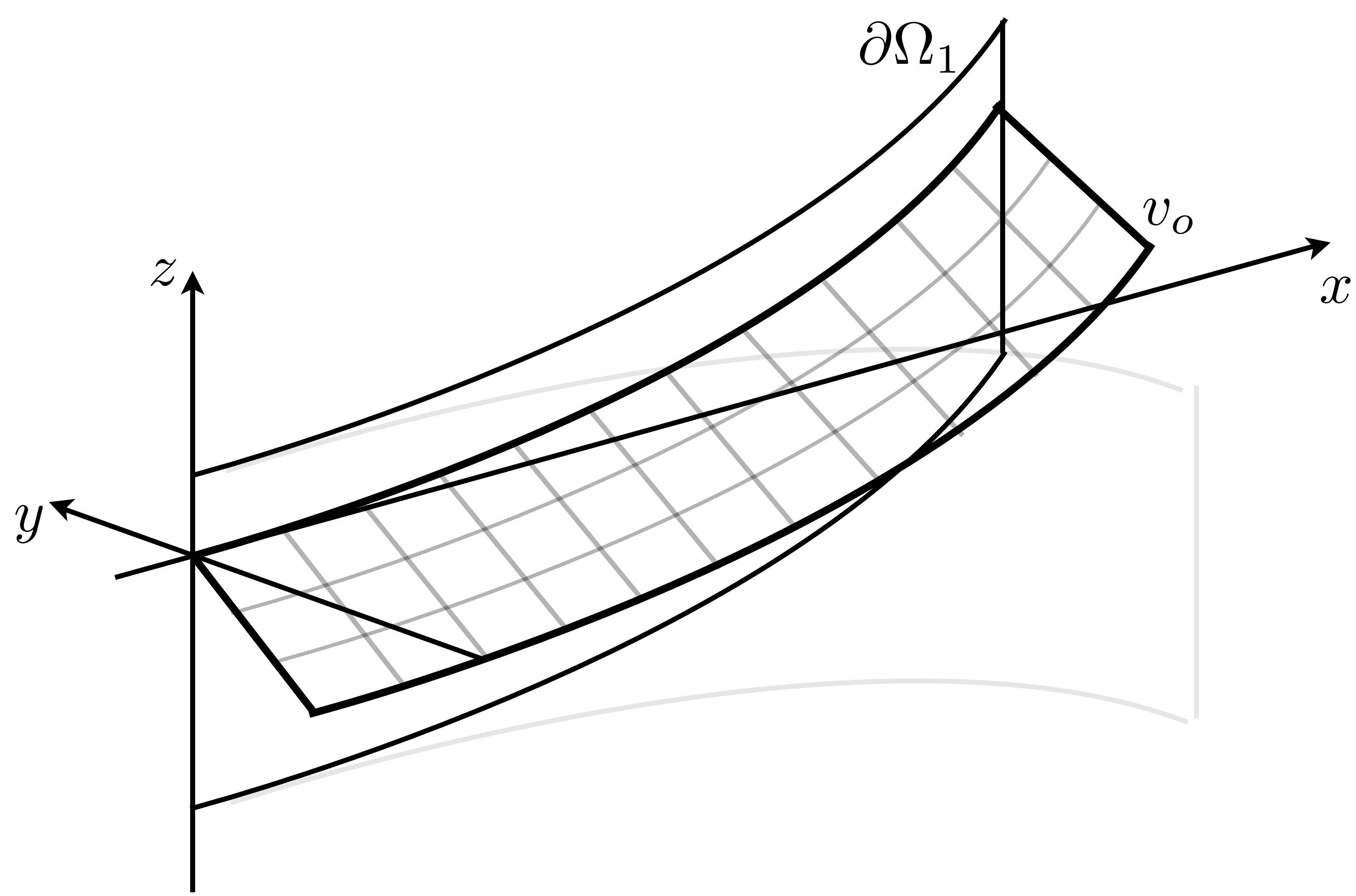




Theorem 1: u is bounded if $\cos\gamma_1 + \cos\gamma_2 = 0$,
if the boundaries of the cusp have finite curvature.

We choose $v_o = -\cot\gamma_1\, t$ where t is the distance from the boundary $\partial\Omega_1$.

If the curvature of $\partial\Omega_1$ is finite then v_o exists and has a finite mean curvature.



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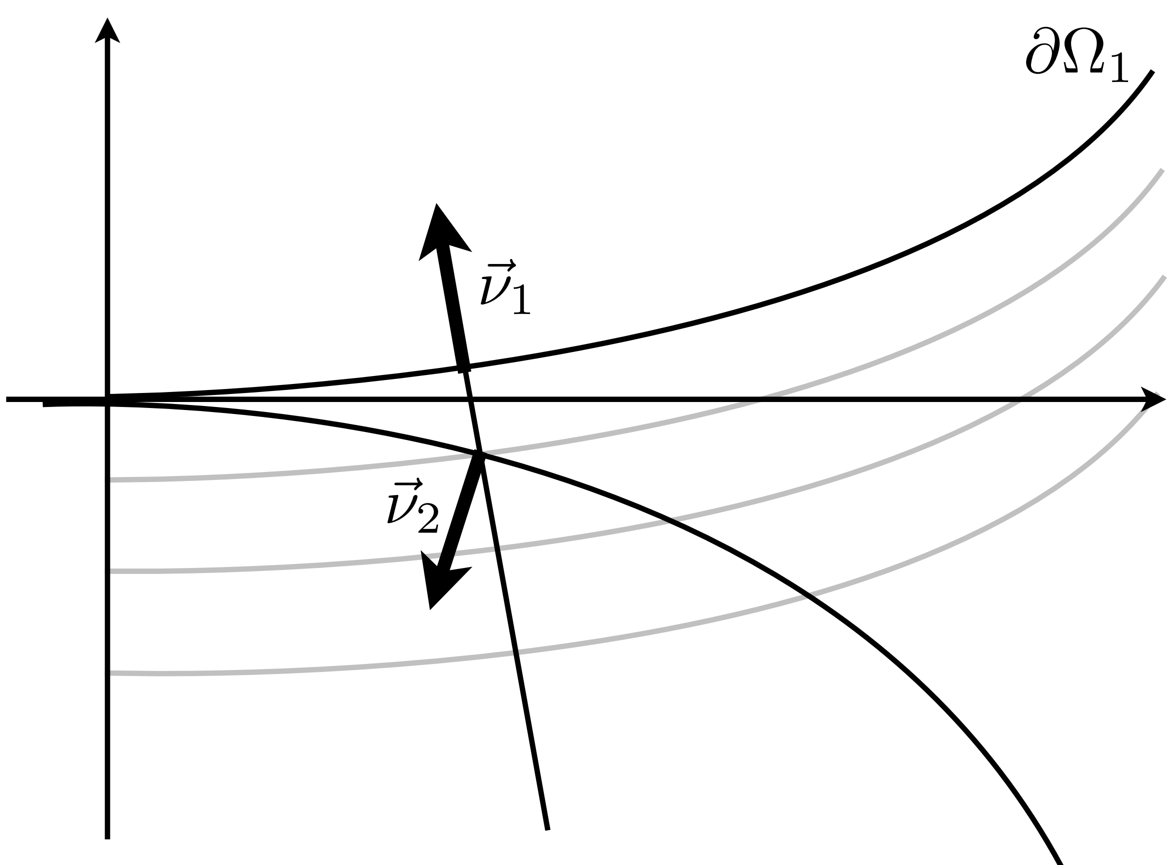
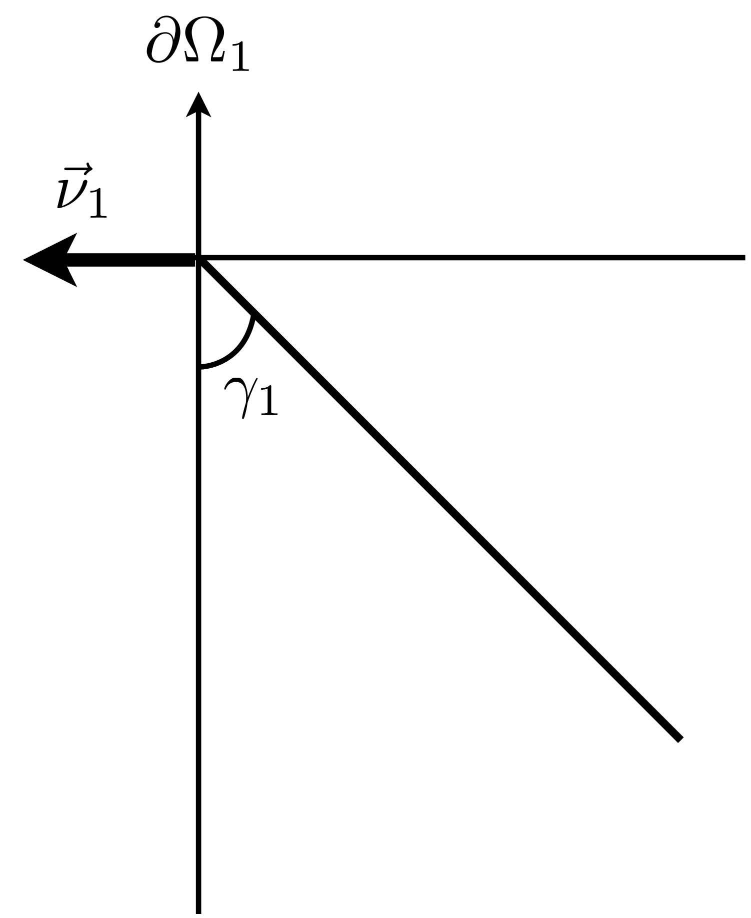
We choose $v_o = -\cot\gamma_1\, t$ where t is the distance from the boundary $\partial\Omega_1$.

$$\vec{\nu}_1 \cdot \frac{\nabla v_o}{\sqrt{1+|\nabla v_o|^2}} = \cos\gamma_1 \qquad \text{in } \bar{\Omega} \qquad \Rightarrow \qquad -\cos\gamma_1 \leq \vec{\nu}_2 \cdot \frac{\nabla v_o}{\sqrt{1+|\nabla v_o|^2}} \leq \cos\gamma_1$$





Theorem 1: u is bounded if $\cos\gamma_1 + \cos\gamma_2 = 0$,

if the boundaries of the cusp have finite curvature.

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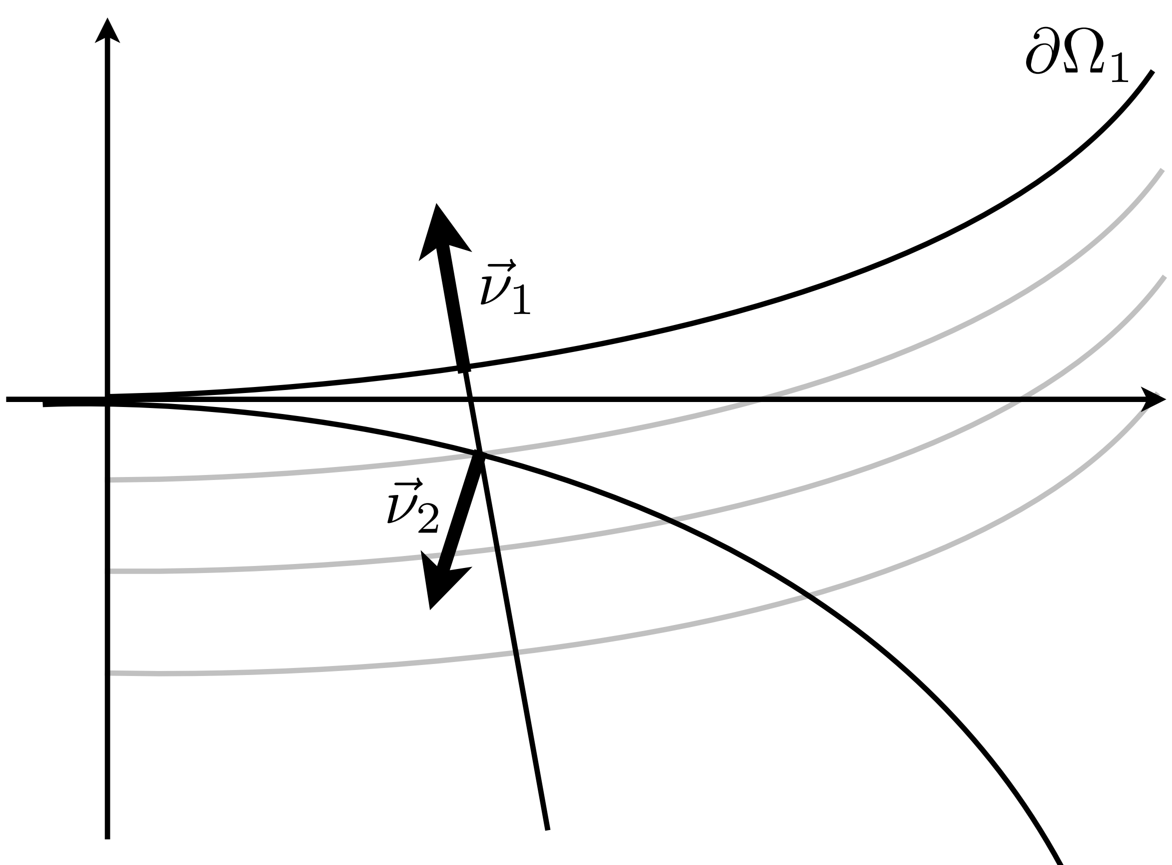
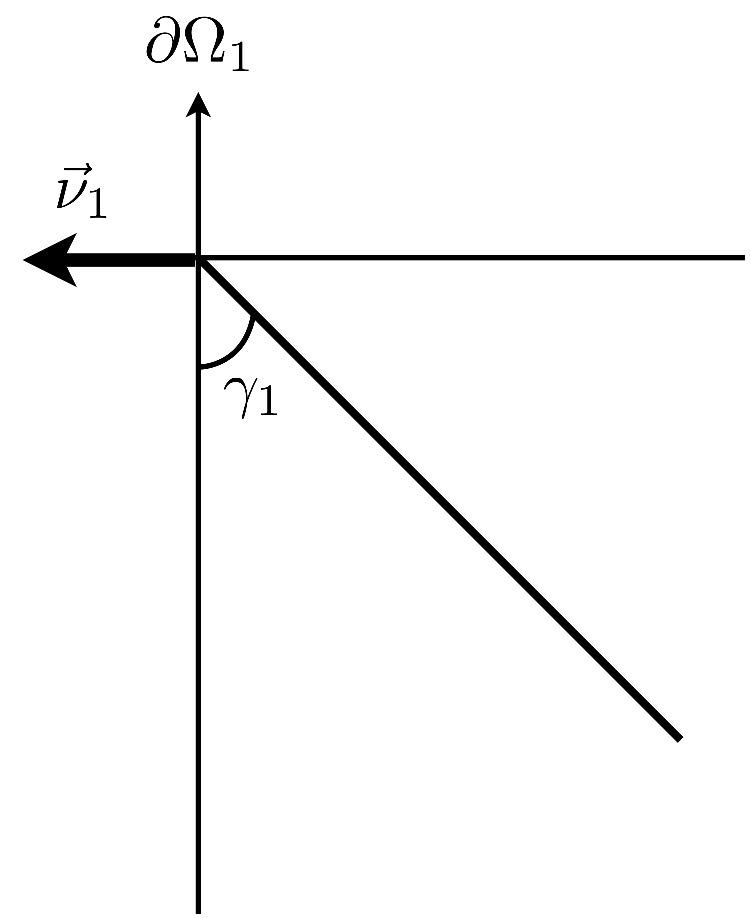
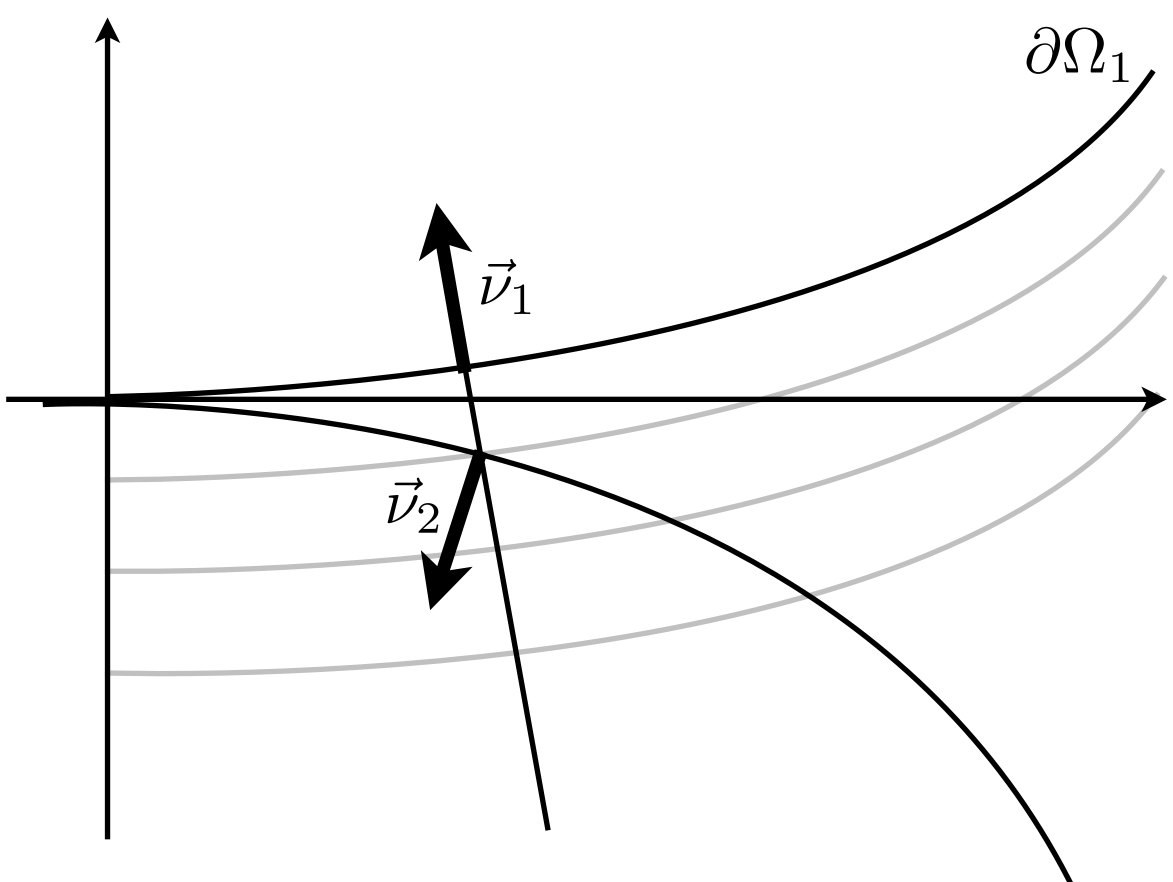


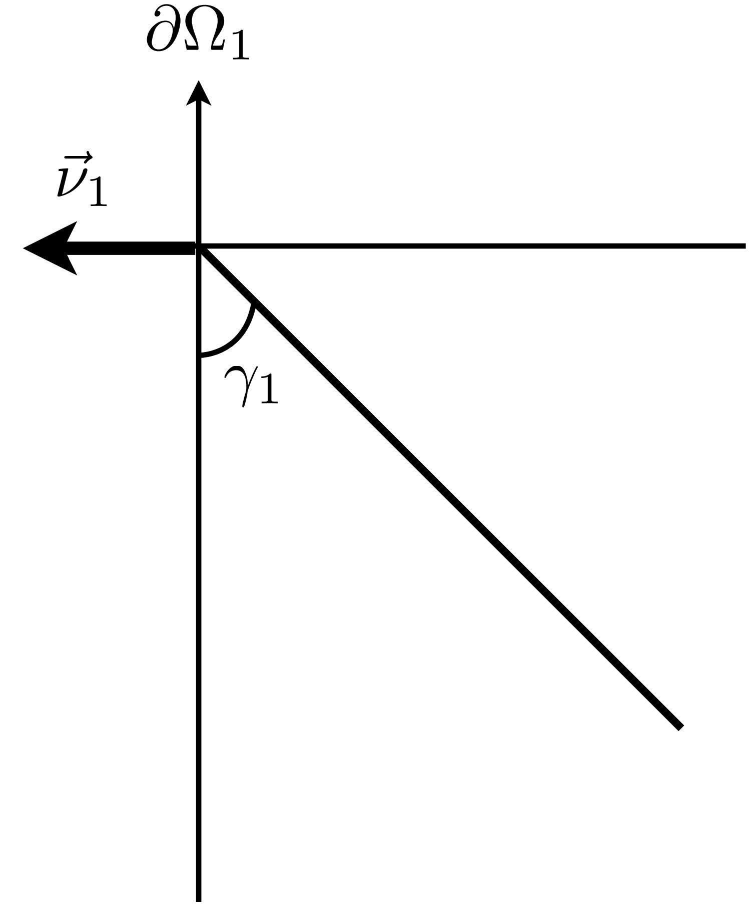


Theorem I: u is bounded if $\cos\gamma_1 + \cos\gamma_2 = 0$, if the boundaries of the cusp have finite curvature.

We choose $v_o = -\cot\gamma_1\, t$ where t is the distance from the boundary $\partial\Omega_1$.

$$\vec{\nu}_2 \cdot \frac{\nabla v_o}{\sqrt{1+|\nabla v_o|^2}} \geq \cos\gamma_2 \qquad \text{on } \partial\Omega_2 \quad (3)$$





Theorem 1: u is bounded if $\cos\gamma_1 + \cos\gamma_2 = 0$,

if the boundaries of the cusp have finite curvature.

There exists a constant C s.t.

$$\nabla \cdot \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} - v < 0 \qquad (1)$$

$$\vec{\nu}_1 \cdot \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} \geq \cos\gamma_1 \qquad (2)$$

$$\vec{\nu}_2 \cdot \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} \geq \cos\gamma_2 \qquad (3)$$

where $$v = -\cot\gamma_1\, t + C$$

Thus by the Concus Finn comparison principle, the solution u can be bounded above

$$-\cot\gamma_1\, t + C > u \qquad \text{in } \Omega$$

Unbounded Capillary Surface

$\cos\gamma_1 + \cos\gamma_2 \neq 0$

Power Series Cusp
(Scholz 2004)

Open Problems

What if $\cos\gamma_1 + \cos\gamma_2 = 0$?

Non-power Series Cusp?

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Finite Curvature Cusp

(Aoki and Siegel 2012)

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Theorem 2: $$u = \frac{\cos\gamma_1 + \cos\gamma_2}{f_1(x) - f_2(x)} + O\left(\frac{f_1'(x) - f_2'(x)}{f_1(x) - f_2(x)}\right)$$

in a non-osculatory cusp domain if $\cos\gamma_1 + \cos\gamma_2 \neq 0$.

Yasunori Aoki and David Siegel:
Journal of Mathematics, Volume 257, Issue 1, 2012, Pages 143-165.

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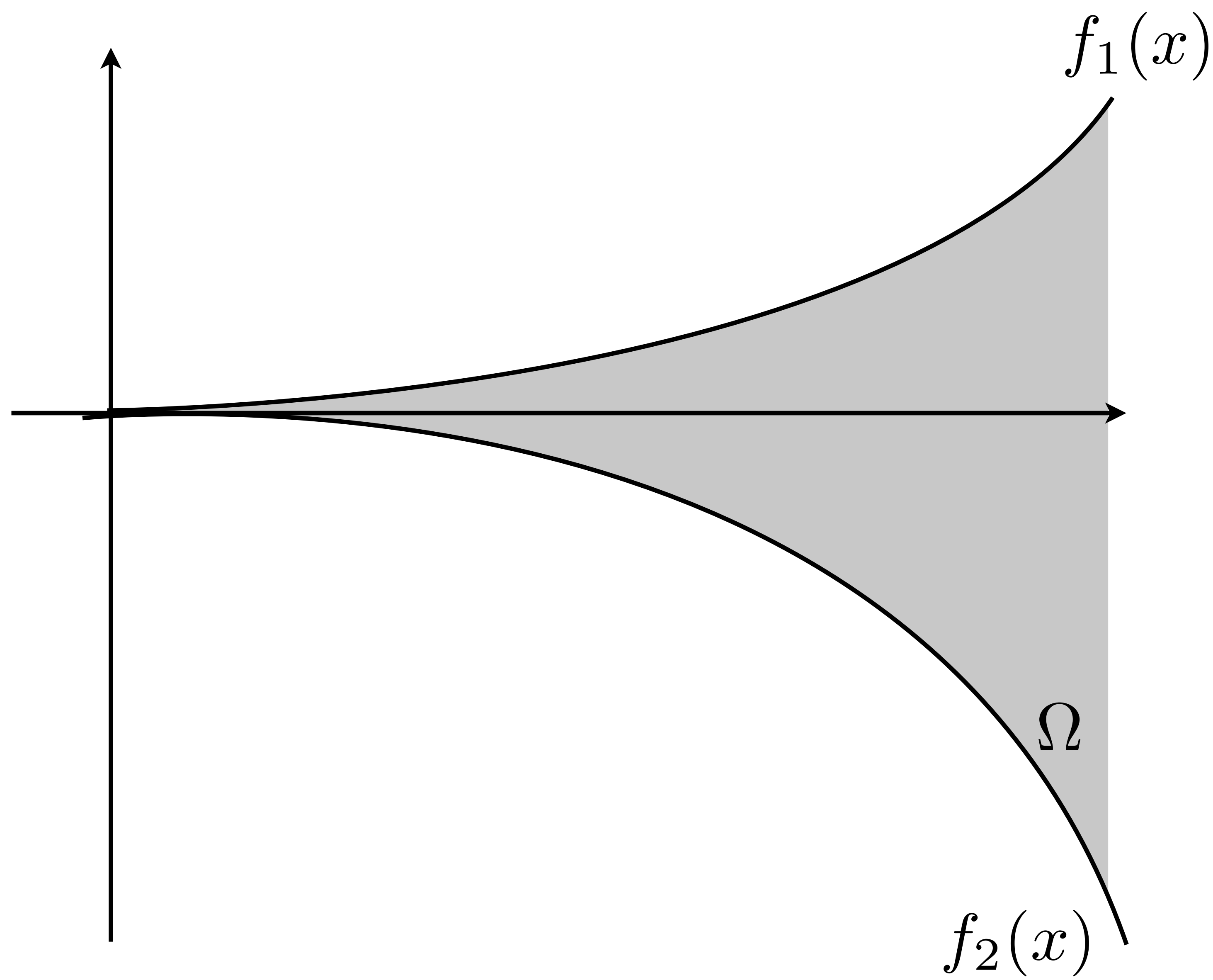

Asymptotic Notation

(big O)

$$u = \frac{\cos\gamma_1 + \cos\gamma_2}{f_1(x) - f_2(x)} + O\left(\frac{f_1'(x) - f_2'(x)}{f_1(x) - f_2(x)}\right) \qquad \text{as } x \to 0$$

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$$\Updownarrow$$

There exist positive constants M and x_o , s.t.

$$\left| u - \frac{\cos\gamma_1 + \cos\gamma_2}{f_1(x) - f_2(x)} \right| \leq M \left| \frac{f_1'(x) - f_2'(x)}{f_1(x) - f_2(x)} \right| \qquad \text{for all } 0 < x < x_o$$

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$$\Updownarrow$$

There exist positive constants M and x_o , s.t.

$$\frac{\cos\gamma_1 + \cos\gamma_2}{f_1(x) - f_2(x)} - M\left|\frac{f_1'(x) - f_2'(x)}{f_1(x) - f_2(x)}\right| \leq u \leq \frac{\cos\gamma_1 + \cos\gamma_2}{f_1(x) - f_2(x)} + M\left|\frac{f_1'(x) - f_2'(x)}{f_1(x) - f_2(x)}\right|$$

$$\text{for all } 0 < x < x_o$$

Asymptotic Notation

(big O)

$$u = \frac{\cos\gamma_1 + \cos\gamma_2}{f_1(x) - f_2(x)} + O\left(\frac{f_1'(x) - f_2'(x)}{f_1(x) - f_2(x)}\right) \qquad \text{as } x \to 0$$

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$$\frac{\cos\gamma_1 + \cos\gamma_2}{f_1(x) - f_2(x)} - M\left|\frac{f_1'(x) - f_2'(x)}{f_1(x) - f_2(x)}\right| \le u \le \boxed{\frac{\cos\gamma_1 + \cos\gamma_2}{f_1(x) - f_2(x)} + M\left|\frac{f_1'(x) - f_2'(x)}{f_1(x) - f_2(x)}\right|}$$

Super Solution v^+

$\text{for all } 0 < x < x_o$

First find the formal asymptotic series

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$$v = \frac{\cos\gamma_1 + \cos\gamma_2}{f_1(x) - f_2(x)} + g(x,y)\frac{f_1'(x) - f_2'(x)}{f_1(x) - f_2(x)} + h(x,y)\frac{(f_1'(x) - f_2'(x))^2}{f_1(x) - f_2(x)}$$

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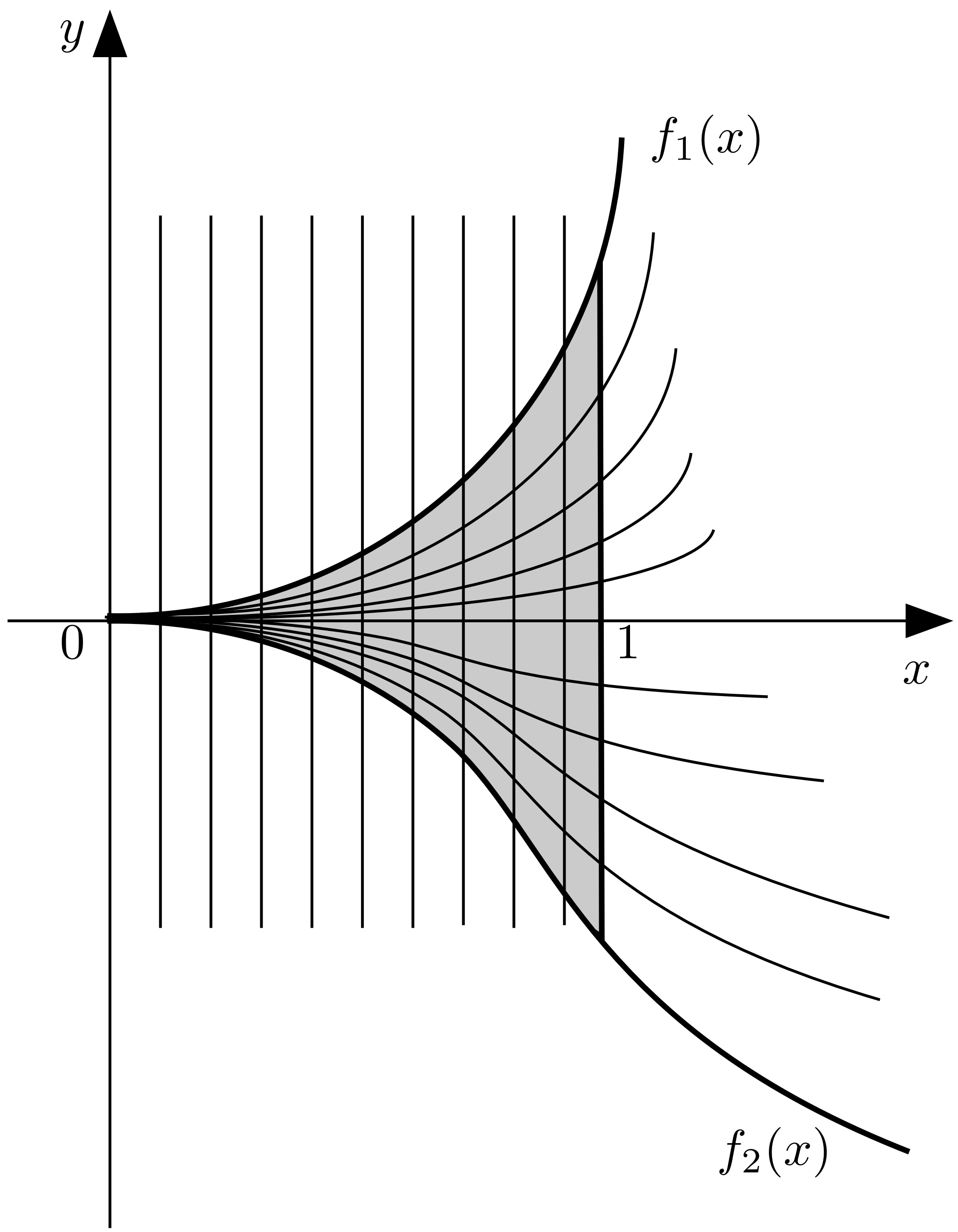

$$t = \frac{2y - (f_1(x) + f_2(x))}{f_1(x) - f_2(x)}$$

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$$g(x,y) = -\sqrt{1 - \left(\frac{\cos\gamma_1(t+1) + \cos\gamma_2(t-1)}{2}\right)^2}$$

$$t = \frac{2y - (f_1(x) + f_2(x))}{f_1(x) - f_2(x)}$$

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$$v = \frac{\cos\gamma_1 + \cos\gamma_2}{f_1(x) - f_2(x)} + g(x,y)\frac{f_1'(x) - f_2'(x)}{f_1(x) - f_2(x)} + h(x,y)\frac{(f_1'(x) - f_2'(x))^2}{f_1(x) - f_2(x)}$$

$$g(x,y) = -\sqrt{1 - \left(\frac{\cos\gamma_1(t+1) + \cos\gamma_2(t-1)}{2}\right)^2}$$

$$h(x,y) = -\frac{\cos\gamma_1 + \cos\gamma_2}{4}\left(\delta t + \frac{t^2}{2}\right) + \frac{1-\alpha}{2(\cos\gamma_1 + \cos\gamma_2)}g(x,y)^2$$

$$t = \frac{2y - (f_1(x) + f_2(x))}{f_1(x) - f_2(x)}$$

Third order formal asymptotic series for the general cusp

July 9, 2008

The comparison function $v(s,t)$ is chosen as the following:

$$v(s,t) = \frac{A}{f_1(s)-f_2(s)} + g(t)\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)} + h(t)\frac{(f_1'(s)-f_2'(s))^2}{f_1(s)-f_2(s)} + C\,, \tag{1}$$

where

$$\begin{aligned} A &= \cos\gamma_1 + \cos\gamma_2 \\ g(t) &= -\left(\sqrt{1-\left(\frac{\cos\gamma_1(t+1)+\cos\gamma_2(t-1)}{2}\right)^2} + k\right) \\ k &: \text{a constant to be determined} \\ h(t) &: \text{a function to be determined} \\ s &= x\,, \quad (2) \\ t &= \frac{2y-(f_1(x)+f_2(x))}{f_1(x)-f_2(x)}\,. \quad (3) \end{aligned}$$

With the following assumptions

$$\frac{f_1''(s)-f_2''(s)}{f_1(s)-f_2(s)} \sim a\,\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^2} \tag{4}$$

$$\frac{f_1'''(s)-f_2'''(s)}{f_1'(s)-f_2'(s)} \sim b\,\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^2} \tag{5}$$

$$\frac{(f_1'(s)-f_2'(s))^3}{(f_1(s)-f_2(s))^4} >> 1 \tag{6}$$

$$f_1'(s)+f_2'(s) \sim c\,(f_1'(s)-f_2'(s)) \tag{7}$$

$$\Rightarrow \quad f_1'(s) \sim \frac{c+1}{2}\,(f_1'(s)-f_2'(s)) \tag{8}$$

$$\Rightarrow \quad f_2'(s) \sim \frac{c-1}{2}\,(f_1'(s)-f_2'(s)) \tag{9}$$

$$f_1''(s)+f_2''(s) \sim c\,(f_1''(s)-f_2''(s)) \tag{10}$$

(11)

$$\begin{aligned}
\frac{ds}{dx} &= 1\,, && (12)\\
\frac{ds}{dy} &= 0\,, && (13)\\
\frac{dt}{dx} &= \frac{-(f_1'(x)+f_2'(x))}{f_1(x)-f_2(x)} - \frac{2y-(f_1(x)+f_2(x))}{(f_1(x)-f_2(x))^2}(f_1'(x)-f_2'(x)) && (14)\\
&= \frac{-(f_1'(x)+f_2'(x))}{f_1(x)-f_2(x)} - t\frac{(f_1'(x)-f_2'(x))}{(f_1(x)-f_2(x))} && (15)\\
&= -\frac{(f_1'(x)+f_2'(x))+t(f_1'(x)-f_2'(x))}{f_1(x)-f_2(x)} && (16)\\
&\sim -(c+t)\frac{f_1'(x)-f_2'(x)}{f_1(x)-f_2(x)} && (17)\\
\frac{dt}{dy} &= \frac{2}{f_1(x)-f_2(x)} && (18)
\end{aligned}$$

Each derivatives can be calculated as the following:

$$\begin{aligned}
v_s &= -A\frac{f_1'(x)-f_2'(x)}{(f_1(x)-f_2(x))^2} + g(t)\frac{f_1''(s)-f_2''(s)}{f_1(s)-f_2(s)} - g(t)\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^2}\\
&\quad +h(t)\frac{(f_1'(s)-f_2'(s))(f_1''(s)-f_2''(s))}{f_1(s)-f_2(s)} - h(t)\frac{(f_1'(s)-f_2'(s))^3}{(f_1(s)-f_2(s))^2} && (19)\\
&\sim -A\frac{f_1'(x)-f_2'(x)}{(f_1(x)-f_2(x))^2} - (1-a)g(t)\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^2} - (1-a)h(t)\frac{(f_1'(s)-f_2'(s))^3}{(f_1(s)-f_2(s))^2} && (20)\\
&\sim -A\frac{f_1'(x)-f_2'(x)}{(f_1(x)-f_2(x))^2} - (1-a)g(t)\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^2} && (21)
\end{aligned}$$

$$v_t = g'(t)\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)} + h'(t)\frac{(f_1'(s)-f_2'(s))^2}{f_1(s)-f_2(s)} \qquad (22)$$

$$\begin{aligned}
v_x &= \frac{\partial v}{\partial s}\frac{ds}{dx} + \frac{\partial v}{\partial t}\frac{dt}{dx} && (23)\\
&= -A\frac{f_1'(x)-f_2'(x)}{(f_1(x)-f_2(x))^2} + g(t)\frac{f_1''(s)-f_2''(s)}{f_1(s)-f_2(s)} - g(t)\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^2}\\
&\quad +h(t)\frac{(f_1'(s)-f_2'(s))(f_1''(s)-f_2''(s))}{f_1(s)-f_2(s)} - h(t)\frac{(f_1'(s)-f_2'(s))^3}{(f_1(s)-f_2(s))^2}\\
&\quad -\left(g'(t)\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)} + h'(t)\frac{(f_1'(s)-f_2'(s))^2}{f_1(s)-f_2(s)}\right)\frac{(f_1'(x)+f_2'(x))+t(f_1'(x)-f_2'(x))}{f_1(x)-f_2(x)} && (24)\\
&\sim -A\frac{f_1'(x)-f_2'(x)}{(f_1(x)-f_2(x))^2} - (1-a)g(t)\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^2}\\
&\quad -(c+t)\frac{f_1'(x)-f_2'(x)}{f_1(x)-f_2(x)}\left(g'(t)\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)} + h'(t)\frac{(f_1'(s)-f_2'(s))^2}{f_1(s)-f_2(s)}\right) && (25)\\
v_x &\sim -A\frac{f_1'(x)-f_2'(x)}{(f_1(x)-f_2(x))^2} - ((1-a)g(t)+(c+t)g'(t))\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^2} && (26)
\end{aligned}$$

$$v_y = \frac{\partial v}{\partial s}\frac{ds}{dy} + \frac{\partial v}{\partial t}\frac{dt}{dy} \tag{27}$$

$$= \left(g'(t)\frac{f_1'(s) - f_2'(s)}{f_1(s) - f_2(s)} + h'(t)\frac{(f_1'(s) - f_2'(s))^2}{f_1(s) - f_2(s)} \right) \frac{2}{f_1(x) - f_2(x)} \tag{28}$$

$$v_y = 2g'(t)\frac{f_1'(s) - f_2'(s)}{(f_1(s) - f_2(s))^2} + 2h'(t)\frac{(f_1'(s) - f_2'(s))^2}{(f_1(s) - f_2(s))^2} \tag{29}$$

$$\begin{aligned} v_x^2 + v_y^2 \sim & \left(-A\frac{f_1'(x) - f_2'(x)}{(f_1(x) - f_2(x))^2} - ((1-a)g(t) + (c+t)g'(t))\, \frac{(f_1'(s) - f_2'(s))^2}{(f_1(s) - f_2(s))^2} \right)^2 \\ & + \left(2g'(t)\frac{f_1'(s) - f_2'(s)}{(f_1(s) - f_2(s))^2} + 2h'(t)\frac{(f_1'(s) - f_2'(s))^2}{(f_1(s) - f_2(s))^2} \right)^2 \end{aligned} \tag{30}$$

$$\sim \left(A^2 + 4(g'(t))^2\right) \frac{(f_1'(s) - f_2'(s))^2}{(f_1(s) - f_2(s))^4} + \left[2A\left\{(1-a)g(t) + (c+t)g'(t)\right\} + 8g'(t)h'(t)\right] \frac{(f_1'(s) - f_2'(s))^3}{(f_1(s) - f_2(s))^4} \tag{31}$$

$$v_x^2 + v_y^2 \sim \left(A^2 + 4(g'(t))^2\right) \left(1 + \frac{\left[2A\left\{(1-a)g(t) + (c+t)g'(t)\right\} + 8g'(t)h'(t)\right]}{A^2 + 4(g'(t))^2}(f_1'(s) - f_2'(s)) \right) \frac{(f_1'(s) - f_2'(s))^2}{(f_1(s) - f_2(s))^4} \tag{32}$$

We now aim to calculate the asymptotic solution to the boundary equations $\nu \cdot Tv$

$$\nu \cdot Tv|_{t=1} = \frac{(-f_1'(s), 1)}{\sqrt{1 + f_1'^2(s)}} \cdot \frac{(v_x, v_y)}{\sqrt{1 + v_x^2 + v_y^2}} \sim \cos\gamma_1\,, \tag{33}$$

$$\nu \cdot Tv|_{t=-1} = \frac{(f_2'(s), 1)}{\sqrt{1 + f_2'^2(s)}} \cdot \frac{(v_x, v_y)}{\sqrt{1 + v_x^2 + v_y^2}} \sim \cos\gamma_2\,. \tag{34}$$

Here we use the binomial series

$$\frac{1}{\sqrt{1+\epsilon^2}} = 1 - \frac{1}{2}\epsilon^2 + \frac{3}{8}\epsilon^4 + O(\epsilon^6) \qquad \text{as } \epsilon \to 0\,, \tag{35}$$

$$\frac{1}{\sqrt{1+\xi}} = \frac{1}{\sqrt{\xi}} - \frac{1}{2}\frac{1}{\xi^{3/2}} + O\left(\frac{1}{\xi^{5/2}}\right) \qquad \text{as } \xi \to \infty\,, \tag{36}$$

and obtain the following:

$$\nu \cdot Tv|_{t=1} = (-f_1'(s), 1)\cdot(v_x, v_y)\left(1 - \frac{1}{2}f_1'^2(s)\right)\left(\frac{1}{\sqrt{v_x^2 + v_y^2}} - \frac{1}{2}\frac{1}{(v_x^2 + v_y^2)^{3/2}} \right), \tag{37}$$

$$\nu \cdot Tv|_{t=-1} = (f_2'(s), -1)\cdot(v_x, v_y)\left(1 - \frac{1}{2}f_2'^2(s)\right)\left(\frac{1}{\sqrt{v_x^2 + v_y^2}} - \frac{1}{2}\frac{1}{(v_x^2 + v_y^2)^{3/2}} \right). \tag{38}$$

First compute the following

$$
\begin{aligned}
(-f_1'(s),1)\cdot(v_x,v_y) &\sim A\frac{f_1'(s)-f_2'(s)}{(f_1(s)-f_2(s))^2}f_1' + 2g'(t)\frac{f_1'(s)-f_2'(s)}{(f_1(s)-f_2(s))^2} + 2h'(t)\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^2}, &(39)\\
&\sim 2g'(t)\frac{f_1'(s)-f_2'(s)}{(f_1(s)-f_2(s))^2} + \left(2h'(t)+A\frac{c+1}{2}\right)\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^2}, &(40)\\
(f_2'(s),-1)\cdot(v_x,v_y) &\sim -A\frac{f_1'(s)-f_2'(s)}{(f_1(s)-f_2(s))^2}f_2' - 2g'(t)\frac{f_1'(s)-f_2'(s)}{(f_1(s)-f_2(s))^2} - 2h'(t)\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^2} &(41)\\
&\sim -2g'(t)\frac{f_1'(s)-f_2'(s)}{(f_1(s)-f_2(s))^2} - \left(2h'(t)+A\frac{c-1}{2}\right)\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^2} &(42)
\end{aligned}
$$

$$
\begin{aligned}
\nu\cdot Tv|_{t=1} &\sim \frac{2g'(t)\frac{f_1'(s)-f_2'(s)}{(f_1(s)-f_2(s))^2} + \left(2h'(t)+A\frac{c+1}{2}\right)\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^2}}{\sqrt{(A^2+4(g'(t))^2)\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^4}}}\\
&\quad\left(1-\frac{1}{2}\left(\frac{[2A\{(1-a)g(t)+(c+t)g'(t)\}+8g'(t)h'(t)]}{A^2+4(g'(t))^2}(f_1'(s)-f_2'(s))\right)\right) &(43)\\
&\sim \frac{2g'(t)+\left(2h'(t)+A\frac{c+1}{2}\right)(f_1'(s)-f_2'(s))}{\sqrt{(A^2+4(g'(t))^2)}}\\
&\quad\left(1-\frac{1}{2}\left(\frac{[2A\{(1-a)g(t)+(c+t)g'(t)\}+8g'(t)h'(t)]}{A^2+4(g'(t))^2}(f_1'(s)-f_2'(s))\right)\right) &(44)\\
&\sim \frac{2g'(t)+\left(2h'(t)+A\frac{c+1}{2}\right)(f_1'(s)-f_2'(s))}{\sqrt{(A^2+4(g'(t))^2)}}\\
&\quad -g'(t)\left(\frac{[2A\{(1-a)g(t)+(c+t)g'(t)\}+8g'(t)h'(t)]}{(A^2+4(g'(t))^2)^{3/2}}(f_1'(s)-f_2'(s))\right) &(45)\\
&\sim \cos\gamma_1\frac{\left(2h'(t)+A\frac{c+1}{2}\right)(f_1'(s)-f_2'(s))}{\sqrt{(A^2+4(g'(t))^2)}}\\
&\quad -g'(t)\left(\frac{[2A\{(1-a)g(t)+(c+t)g'(t)\}+8g'(t)h'(t)]}{(A^2+4(g'(t))^2)^{3/2}}(f_1'(s)-f_2'(s))\right) &(46)\\
&\sim \cos\gamma_1\\
&\quad +\left(\frac{\left(2h'(t)+A\frac{c+1}{2}\right)}{\sqrt{(A^2+4(g'(t))^2)}} - g'(t)\left(\frac{[2A\{(1-a)g(t)+(c+1)g'(t)\}+8g'(t)h'(t)]}{(A^2+4(g'(t))^2)^{3/2}}\right)\right)(f_1'(s)-f_2'(s))\\
&&(47)
\end{aligned}
$$

$$
\begin{aligned}
\nu\cdot Tv|_{t=-1} &\sim \cos\gamma_2\\
&\quad -\left(\frac{\left(2h'(t)+A\frac{c-1}{2}\right)}{\sqrt{(A^2+4(g'(t))^2)}} - g'(t)\left(\frac{[2A\{(1-a)g(t)+(c-1)g'(t)\}+8g'(t)h'(t)]}{(A^2+4(g'(t))^2)^{3/2}}\right)\right)(f_1'(s)-f_2'(s))\\
&&(48)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial s} v_x \quad = \quad & -A\left(\frac{f_1''(x)-f_2''(x)}{(f_1(x)-f_2(x))^2} - 2\frac{(f_1'(x)-f_2'(x))^2}{(f_1(x)-f_2(x))^3}\right) \\
& +g(t)\left(\frac{f_1'''(s)-f_2'''(s)}{f_1(s)-f_2(s)} - \frac{(f_1''(s)-f_2''(s))(f_1'(s)-f_2'(s))}{(f_1(s)-f_2(s))^2}\right) \\
& -g(t)\left(2\frac{(f_1''(s)-f_2''(s))(f_1'(s)-f_2'(s))}{(f_1(s)-f_2(s))^2} - 2\frac{(f_1'(s)-f_2'(s))^3}{(f_1(s)-f_2(s))^3}\right) \\
& +h(t)\left(\frac{(f_1''(s)-f_2''(s))^2}{f_1(s)-f_2(s)} + \frac{(f_1'(s)-f_2'(s))(f_1'''(s)-f_2'''(s))}{f_1(s)-f_2(s)} - \frac{(f_1'(s)-f_2'(s))^2(f_1''(s)-f_2''(s))}{(f_1(s)-f_2(s))^2}\right) \\
& -h(t)\left(\frac{3(f_1'(s)-f_2'(s))^2(f_1''(s)-f_2''(s))}{(f_1(s)-f_2(s))^2} - 2\frac{(f_1'(s)-f_2'(s))^4}{(f_1(s)-f_2(s))^3}\right) \\
& -\left(g'(t)\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)} + h'(t)\frac{(f_1'(s)-f_2'(s))^2}{f_1(s)-f_2(s)}\right) \\
& \left(\frac{(f_1''(x)+f_2''(x))+t(f_1''(x)-f_2''(x))}{f_1(x)-f_2(x)} - (f_1'(x)-f_2'(x))\frac{(f_1'(x)+f_2'(x))+t(f_1'(x)-f_2'(x))}{(f_1(x)-f_2(x))^2}\right) \qquad (49) \\
& -\left(g'(t)\left(\frac{f_1''(s)-f_2''(s)}{f_1(s)-f_2(s)} - \frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^2}\right) + h'(t)\left(2\frac{(f_1'(s)-f_2'(s))(f_1''(s)-f_2''(s))}{f_1(s)-f_2(s)} - \frac{(f_1'(s)-f_2'(s))^3}{(f_1(s)-f_2(s))^2}\right)\right) \\
& \frac{(f_1'(x)+f_2'(x))+t(f_1'(x)-f_2'(x))}{f_1(x)-f_2(x)} \qquad (50) \\
\sim \quad & -A\left((a-2)\frac{(f_1'(x)-f_2'(x))^2}{(f_1(x)-f_2(x))^3}\right) \\
& +g(t)\left((b-a)\frac{(f_1'(s)-f_2'(s))^3}{(f_1(s)-f_2(s))^3}\right) \\
& -g(t)\left(2(a-1)\frac{(f_1'(s)-f_2'(s))^3}{(f_1(s)-f_2(s))^3}\right) \\
& +h(t)\left((a^2+b-a)\frac{(f_1'(s)-f_2'(s))^4}{(f_1(s)-f_2(s))^3}\right) \\
& -h(t)\left((3a-2)\frac{(f_1'(s)-f_2'(s))^4}{(f_1(s)-f_2(s))^3}\right) \\
& -\left(g'(t)\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)} + h'(t)\frac{(f_1'(s)-f_2'(s))^2}{f_1(s)-f_2(s)}\right)\left((a-1)(c+t)\frac{(f_1'(x)-f_2'(x))^2}{(f_1(x)-f_2(x))^2}\right) \\
& -\left(g'(t)\left((a-1)\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^2}\right) + h'(t)\left((2a-1)\frac{(f_1'(s)-f_2'(s))^3}{(f_1(s)-f_2(s))^2}\right)\right)\frac{(c+t)(f_1'(x)-f_2'(x))}{f_1(x)-f_2(x)} \qquad (51) \\
\sim \quad & -A(a-2)\frac{(f_1'(x)-f_2'(x))^2}{(f_1(x)-f_2(x))^3} + \left[g(t)\left\{(b-a)-2(a-1)\right\} - 2\left(g'(t)\right)(a-1)(c+t)\right]\frac{(f_1'(s)-f_2'(s))^3}{(f_1(s)-f_2(s))^3} \qquad (52)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial t}v_x &= g'(t)\frac{f_1''(s)-f_2''(s)}{f_1(s)-f_2(s)} - g'(t)\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^2} \\
&\quad + h'(t)\frac{(f_1'(s)-f_2'(s))(f_1''(s)-f_2''(s))}{f_1(s)-f_2(s)} - h'(t)\frac{(f_1'(s)-f_2'(s))^3}{(f_1(s)-f_2(s))^2} \\
&\quad - \left(g''(t)\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)} + h''(t)\frac{(f_1'(s)-f_2'(s))^2}{f_1(s)-f_2(s)}\right)\frac{(f_1'(x)+f_2'(x))+t(f_1'(x)-f_2'(x))}{f_1(x)-f_2(x)} \\
&\quad - \left(g'(t)\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)} + h'(t)\frac{(f_1'(s)-f_2'(s))^2}{f_1(s)-f_2(s)}\right)\frac{(f_1'(x)-f_2'(x))}{f_1(x)-f_2(x)} \qquad (53) \\
&\sim (a-1)g'(t)\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^2} \\
&\quad + (a-1)h'(t)\frac{(f_1'(s)-f_2'(s))^3}{(f_1(s)-f_2(s))^2} \\
&\quad - \left(g''(t)\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)} + h''(t)\frac{(f_1'(s)-f_2'(s))^2}{f_1(s)-f_2(s)}\right)\frac{(c+t)(f_1'(x)-f_2'(x))}{f_1(x)-f_2(x)} \\
&\quad - \left(g'(t)\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)} + h'(t)\frac{(f_1'(s)-f_2'(s))^2}{f_1(s)-f_2(s)}\right)\frac{(f_1'(x)-f_2'(x))}{f_1(x)-f_2(x)} \qquad (54) \\
&= \{(a-1)g'(t)-(c+t)g''(t)-g'(t)\}\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^2} \\
&\quad + \{(a-1)h'(t)-(c+t)h''(t)-h'(t)\}\frac{(f_1'(s)-f_2'(s))^3}{(f_1(s)-f_2(s))^2} \qquad (55)
\end{aligned}
$$

$$
\begin{aligned}
v_{xx} &= \frac{\partial}{\partial s}v_x\frac{ds}{dx} + \frac{\partial}{\partial t}v_x\frac{dt}{dx} \qquad (56) \\
&\sim -A(a-2)\frac{(f_1'(x)-f_2'(x))^2}{(f_1(x)-f_2(x))^3} + \left[g(t)\{(b-a)-2(a-1)\} - 2(g'(t))(a-1)(c+t)\right]\frac{(f_1'(s)-f_2'(s))^3}{(f_1(s)-f_2(s))^3} \\
&\quad - \left[\{(a-1)g'(t)-(c+t)g''(t)-g'(t)\}\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^2}\right. \\
&\quad \left. + \{(a-1)h'(t)-(c+t)h''(t)-h'(t)\}\frac{(f_1'(s)-f_2'(s))^3}{(f_1(s)-f_2(s))^2}\right](c+t)\frac{f_1'(x)-f_2'(x)}{f_1(x)-f_2(x)} \qquad (57) \\
&\sim -A(a-2)\frac{(f_1'(x)-f_2'(x))^2}{(f_1(x)-f_2(x))^3} + \left[g(t)\{(b-a)-2(a-1)\} - 2(g'(t))(a-1)(c+t)\right]\frac{(f_1'(s)-f_2'(s))^3}{(f_1(s)-f_2(s))^3} \\
&\quad - \{(a-1)g'(t)-(c+t)g''(t)-g'(t)\}(c+t)\frac{(f_1'(s)-f_2'(s))^3}{(f_1(s)-f_2(s))^3} \qquad (58) \\
&\sim -A(a-2)\frac{(f_1'(x)-f_2'(x))^2}{(f_1(x)-f_2(x))^3} \\
&\quad + \left[g(t)\{(b-a)-2(a-1)\} - 3(g'(t))(a-1)(c+t) + (c+t)^2g''(t) + (c+t)g'(t)\right]\frac{(f_1'(s)-f_2'(s))^3}{(f_1(s)-f_2(s))^3} \\
&\qquad (59)
\end{aligned}
$$

$$
\begin{aligned}
v_{xy} &= \frac{\partial}{\partial t} v_x \frac{dt}{dy} && (60)\\
&\sim \left[\{(a-1)g'(t)-(c+t)g''(t)-g'(t)\}\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^2}\right.\\
&\quad \left. +\{(a-1)h'(t)-(c+t)h''(t)-h'(t)\}\frac{(f_1'(s)-f_2'(s))^3}{(f_1(s)-f_2(s))^2}\right]\frac{2}{f_1(x)-f_2(x)} && (61)\\
&= 2\{(a-2)g'(t)-(c+t)g''(t)\}\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^3}+2\{(a-2)h'(t)-(c+t)h''(t)\}\frac{(f_1'(s)-f_2'(s))^3}{(f_1(s)-f_2(s))^3} && (62)
\end{aligned}
$$

$$
\begin{aligned}
v_{yy} &= \frac{\partial}{\partial t} v_y \frac{dt}{dy}, && (63)\\
&= \left(2g''(t)\frac{f_1'(s)-f_2'(s)}{(f_1(s)-f_2(s))^2}+2h''(t)\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^2}\right)\frac{2}{f_1(x)-f_2(x)} && (64)\\
&= 4g''(t)\frac{f_1'(s)-f_2'(s)}{(f_1(s)-f_2(s))^3}+4h''(t)\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^3} && (65)
\end{aligned}
$$

$$
\begin{aligned}
(1+v_x^2) &\sim 1+\left(-A\frac{f_1'(x)-f_2'(x)}{(f_1(x)-f_2(x))^2}-((1-a)g(t)+(c+t)g'(t))\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^2}\right)^2 && (66)\\
&\sim A^2\frac{(f_1'(x)-f_2'(x))^2}{(f_1(x)-f_2(x))^4}+2A((1-a)g(t)+(c+t)g'(t))\frac{(f_1'(s)-f_2'(s))^3}{(f_1(s)-f_2(s))^4} && (67)
\end{aligned}
$$

$$
\begin{aligned}
(1+v_y^2) &= 1+\left(2g'(t)\frac{f_1'(s)-f_2'(s)}{(f_1(s)-f_2(s))^2}+2h'(t)\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^2}\right)^2 && (68)\\
&\sim 4(g'(t))^2\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^4}+8g'(t)h'(t)\frac{(f_1'(s)-f_2'(s))^3}{(f_1(s)-f_2(s))^4} && (69)
\end{aligned}
$$

$$
\begin{aligned}
v_{yy}(1+v_x^2) &\sim \left(4g''(t)\frac{f_1'(s)-f_2'(s)}{(f_1(s)-f_2(s))^3}+4h''(t)\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^3}\right)\\
&\quad \left(A^2\frac{(f_1'(x)-f_2'(x))^2}{(f_1(x)-f_2(x))^4}+2A((1-a)g(t)+(c+t)g'(t))\frac{(f_1'(s)-f_2'(s))^3}{(f_1(s)-f_2(s))^4}\right) && (70)\\
&\sim \left(4A^2g''(t)\frac{(f_1'(s)-f_2'(s))^3}{(f_1(s)-f_2(s))^7}+\{8g''(t)A((1-a)g(t)+(c+t)g'(t))+4A^2h''(t)\}\frac{(f_1'(s)-f_2'(s))^4}{(f_1(s)-f_2(s))^7}\right) && (71)
\end{aligned}
$$

$$
v_{xx}(1+v_y^2) \sim -4(g'(t))^2A(a-2)\frac{(f_1'(x)-f_2'(x))^4}{(f_1(x)-f_2(x))^7} \qquad (72)
$$

$$
\begin{aligned}
2v_{xy}v_xv_y &\sim 2\left(2\{(a-2)g'(t)-(c+t)g''(t)\}\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^3}+2\{(a-2)h'(t)-(c+t)h''(t)\}\frac{(f_1'(s)-f_2'(s))^3}{(f_1(s)-f_2(s))^3}\right)\\
&\quad \left(2g'(t)\frac{f_1'(s)-f_2'(s)}{(f_1(s)-f_2(s))^2}+2h'(t)\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^2}\right)\\
&\quad \left(-A\frac{f_1'(x)-f_2'(x)}{(f_1(x)-f_2(x))^2}-((1-a)g(t)+(c+t)g'(t))\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^2}\right) && (73)\\
&\sim -8g'(t)A\{(a-2)g'(t)-(c+t)g''(t)\}\frac{(f_1'(s)-f_2'(s))^4}{(f_1(s)-f_2(s))^7} && (74)
\end{aligned}
$$

$$
\begin{aligned}
& v_{xx}(1+v_y^2)+v_{yy}(1+v_x^2)-2v_{xy}v_xv_y \\
\sim\;& 4A^2g''(t)\frac{(f_1'(s)-f_2'(s))^3}{(f_1(s)-f_2(s))^7} \\
& +\left[\left\{8g''(t)A\left((1-a)g(t)+(c+t)g'(t)\right)+4A^2h''(t)\right\}+8g'(t)A\left\{(a-2)g'(t)-(c+t)g''(t)\right\}-4(g'(t))^2A(a-2)\right] \\
& \frac{(f_1'(s)-f_2'(s))^4}{(f_1(s)-f_2(s))^7} \qquad (75)\\
\sim\;& 4A^2g''(t)\frac{(f_1'(s)-f_2'(s))^3}{(f_1(s)-f_2(s))^7}+\left[8A(1-a)g(t)g''(t)+4A^2h''(t)+4(g'(t))^2A(a-2)\right]\frac{(f_1'(s)-f_2'(s))^4}{(f_1(s)-f_2(s))^7} \qquad (76)
\end{aligned}
$$

$$
\begin{aligned}
\nabla\cdot Tv \;=\;& \frac{v_{xx}(1+v_y^2)+v_{yy}(1+v_x^2)-2v_xv_yv_{xy}}{\left(1+v_x^2+v_y^2\right)^{3/2}} \\
\sim\;& \frac{4A^2g''(t)\frac{(f_1'(s)-f_2'(s))^3}{(f_1(s)-f_2(s))^7}+\left[8A(1-a)g(t)g''(t)+4A^2h''(t)+4(g'(t))^2A(a-2)\right]\frac{(f_1'(s)-f_2'(s))^4}{(f_1(s)-f_2(s))^7}}{\left(1+(A^2+4(g'(t))^2)\left(1+\frac{[2A\{(1-a)g(t)+(c+t)g'(t)\}+8g'(t)h'(t)]}{A^2+4(g'(t))^2}(f_1'(s)-f_2'(s))\right)\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^4}\right)^{3/2}} \qquad (77)\\
\sim\;& \frac{4A^2g''(t)\frac{1}{f_1(s)-f_2(s)}+\left[8A(1-a)g(t)g''(t)+4A^2h''(t)+4(g'(t))^2A(a-2)\right]\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)}}{\left((A^2+4(g'(t))^2)\left(1+\frac{[2A\{(1-a)g(t)+(c+t)g'(t)\}+8g'(t)h'(t)]}{A^2+4(g'(t))^2}(f_1'(s)-f_2'(s))\right)\right)^{3/2}} \qquad (78)\\
\sim\;& \frac{4A^2g''(t)\frac{1}{f_1(s)-f_2(s)}}{\left((A^2+4(g'(t))^2)\left(1+\frac{[2A\{(1-a)g(t)+(c+t)g'(t)\}+8g'(t)h'(t)]}{A^2+4(g'(t))^2}(f_1'(s)-f_2'(s))\right)\right)^{3/2}} \\
& +\frac{\left[8A(1-a)g(t)g''(t)+4A^2h''(t)+4(g'(t))^2A(a-2)\right]}{(A^2+4(g'(t))^2)^{3/2}}\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)} \qquad (79)\\
\sim\;& \frac{4A^2g''(t)}{(A^2+4(g'(t))^2)^{3/2}}\left(1-\frac{3}{2}\frac{[2A\{(1-a)g(t)+(c+t)g'(t)\}+8g'(t)h'(t)]}{A^2+4(g'(t))^2}(f_1'(s)-f_2'(s))\right)\frac{1}{f_1(s)-f_2(s)} \\
& +\frac{\left[8A(1-a)g(t)g''(t)+4A^2h''(t)+4(g'(t))^2A(a-2)\right]}{(A^2+4(g'(t))^2)^{3/2}}\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)} \qquad (80)\\
\sim\;& \frac{A}{f_1(s)-f_2(s)}-\frac{3}{2}A\frac{[2A\{(1-a)g(t)+(c+t)g'(t)\}+8g'(t)h'(t)]}{A^2+4(g'(t))^2}\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)} \\
& +\frac{\left[8A(1-a)g(t)g''(t)+4A^2h''(t)+4(g'(t))^2A(a-2)\right]}{(A^2+4(g'(t))^2)^{3/2}}\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)} \qquad (81)
\end{aligned}
$$

We choose $h'(t)$ to be

$$
\begin{aligned}
h'(t) &= -\frac{A}{4}(c+t)+H(t) \qquad (82)\\
h''(t) &= -\frac{A}{4}+H'(t) \qquad (83)
\end{aligned}
$$

$$\nu \cdot Tv|_{t=1} \sim \cos\gamma_1 + \left(\frac{\left(2h'(t) + A\frac{c+1}{2}\right)}{\sqrt{(A^2 + 4(g'(t))^2)}} - g'(t) \left(\frac{[2A\left\{(1-a)g(t) + (c+1)g'(t)\right\} + 8g'(t)h'(t)]}{(A^2 + 4(g'(t))^2)^{3/2}} \right) \right) (f_1'(s) - f_2'(s)) \tag{84}$$

$$= \cos\gamma_1 + \left(\frac{2H(1)}{\sqrt{(A^2 + 4(g'(1))^2)}} - g'(1) \left(\frac{[2A(1-a)g(1) + 8g'(1)H(1)]}{(A^2 + 4(g'(1))^2)^{3/2}} \right) \right) (f_1'(s) - f_2'(s)) \tag{85}$$

$$= \cos\gamma_1 + \left(\frac{2H(1)}{\sqrt{(A^2 + 4(g'(1))^2)}} - \left(\frac{2A(1-a)g'(1)g(1) + 8(g'(1))^2 H(1)}{(A^2 + 4(g'(1))^2)^{3/2}} \right) \right) (f_1'(s) - f_2'(s)) \tag{86}$$

$$= \cos\gamma_1 + \left(\frac{2A^2 H(1) - 2A(1-a)g'(1)g(1)}{\left((A^2 + 4(g'(1))^2)\right)^{3/2}} \right) (f_1'(s) - f_2'(s)) \tag{87}$$

$$\nu \cdot Tv|_{t=-1} \sim \cos\gamma_2 - \left(\frac{\left(2h'(t) + A\frac{c-1}{2}\right)}{\sqrt{(A^2 + 4(g'(t))^2)}} - g'(t) \left(\frac{[2A\left\{(1-a)g(t) + (c-1)g'(t)\right\} + 8g'(t)h'(t)]}{(A^2 + 4(g'(t))^2)^{3/2}} \right) \right) (f_1'(s) - f_2'(s)) \tag{88}$$

$$= \cos\gamma_2 - \left(\frac{2H(-1)}{\sqrt{(A^2 + 4(g'(-1))^2)}} - g'(-1) \left(\frac{[2A(1-a)g(-1) + 8g'(-1)H(-1)]}{(A^2 + 4(g'(-1))^2)^{3/2}} \right) \right) (f_1'(s) - f_2'(s)) \tag{89}$$

$$= \cos\gamma_2 - \left(\frac{2A^2 H(-1) - 2A(1-a)g'(-1)g(-1)}{\left((A^2 + 4(g'(-1))^2)\right)^{3/2}} \right) (f_1'(s) - f_2'(s)) \tag{90}$$

$$\nabla \cdot Tv \sim \frac{A}{f_1(s) - f_2(s)} - \frac{3}{2}A\frac{[2A\left\{(1-a)g(t) + (c+t)g'(t)\right\} + 8g'(t)h'(t)]}{A^2 + 4(g'(t))^2}\frac{f_1'(s) - f_2'(s)}{f_1(s) - f_2(s)} + \frac{\left[8A(1-a)g(t)g''(t) + 4A^2 h''(t) + 4(g'(t))^2 A(a-2)\right]}{(A^2 + 4(g'(t))^2)^{3/2}}\frac{f_1'(s) - f_2'(s)}{f_1(s) - f_2(s)} \tag{91}$$

$$= \frac{A}{f_1(s) - f_2(s)} - \frac{3}{2}A\frac{[2A\left\{(1-a)g(t)\right\} + 8g'(t)H(t)]}{A^2 + 4(g'(t))^2}\frac{f_1'(s) - f_2'(s)}{f_1(s) - f_2(s)} + \frac{\left[8A(1-a)g(t)g''(t) - A^3 + 4A^2 H'(t) + 4(g'(t))^2 A(a-2)\right]}{(A^2 + 4(g'(t))^2)^{3/2}}\frac{f_1'(s) - f_2'(s)}{f_1(s) - f_2(s)} \tag{92}$$

$$= \frac{A}{f_1(s) - f_2(s)} - \frac{3}{2}A\frac{[2A\left\{(1-a)g(t)\right\} + 8g'(t)H(t)]}{A^2 + 4(g'(t))^2}\frac{f_1'(s) - f_2'(s)}{f_1(s) - f_2(s)} + \left(\frac{\left[8A(1-a)g(t)g''(t) + 4(g'(t))^2 A(a-1) + 4A^2 H'(t)\right]}{(A^2 + 4(g'(t))^2)^{3/2}} - \frac{A}{\sqrt{A^2 + 4(g'(t))^2}} \right) \frac{f_1'(s) - f_2'(s)}{f_1(s) - f_2(s)} \tag{93}$$

We choose $H(t)$ to be

$$H(t) = \frac{1-a}{A}g'(t)g(t) + I(t) \tag{94}$$

$$H'(t) = \frac{1-a}{A}\left(g''(t)g(t) + (g'(t))^2\right) + I'(t) \tag{95}$$

$$\nu \cdot Tv|_{t=1} \sim \cos\gamma_1 + \frac{I(1)}{(A^2+4(g'(1))^2)^{3/2}}(f_1'(s)-f_2'(s)) \tag{96}$$

$$\nu \cdot Tv|_{t=-1} \sim \cos\gamma_2 - \frac{I(-1)}{(A^2+4(g'(-1))^2)^{3/2}}(f_1'(s)-f_2'(s)) \tag{97}$$

$$\begin{aligned}
\nabla \cdot Tv \sim\ & \frac{A}{f_1(s)-f_2(s)} - \frac{3}{2}A\frac{\left[2A\left\{(1-a)g(t)\right\}+8g'(t)\left(\frac{1-a}{A}g'(t)g(t)+I(t)\right)\right]}{A^2+4(g'(t))^2}\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)} \\
&+\frac{\left[8A(1-a)g(t)g''(t)+4(g'(t))^2A(a-1)+4A^2\left(\frac{1-a}{A}\left(g''(t)g(t)+(g'(t))^2\right)+I'(t)\right)\right]}{\left(A^2+4(g'(t))^2\right)^{3/2}}\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)} \\
&-\frac{A}{\sqrt{A^2+4(g'(t))^2}}\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)} && (98)\\
=\ & \frac{A}{f_1(s)-f_2(s)} - \frac{3}{2}\frac{\left[2(1-a)g(t)\left\{A^2+4(g'(t))^2\right\}+8g'(t)I(t)\right]}{A^2+4(g'(t))^2}\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)} \\
&+\frac{\left[12A(1-a)g(t)g''(t)+4A^2I'(t)\right]}{\left(A^2+4(g'(t))^2\right)^{3/2}}\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)} \\
&-\frac{A}{\sqrt{A^2+4(g'(t))^2}}\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)} && (99)\\
=\ & \frac{A}{f_1(s)-f_2(s)} \\
&+\left(-3(1-a)g(t)-\frac{3}{2}\frac{8g'(t)I(t)}{A^2+4(g'(t))^2}+3A(1-a)g(t)+\frac{4A^2I'(t)}{\left(A^2+4(g'(t))^2\right)^{3/2}}\right)\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)} \\
&-\frac{A}{\sqrt{A^2+4(g'(t))^2}}\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)} && (100)\\
=\ & \frac{A}{f_1(s)-f_2(s)} \\
&+\left(-\frac{3}{2}\frac{8g'(t)I(t)}{A^2+4(g'(t))^2}+\frac{4A^2I'(t)}{\left(A^2+4(g'(t))^2\right)^{3/2}}\right)\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)} \\
&-\frac{A}{\sqrt{A^2+4(g'(t))^2}}\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)} && (101)
\end{aligned}$$

$$\nabla \cdot Tv - v \sim \left(-12\frac{g'(t)I(t)}{A^2+4(g'(t))^2}+\frac{4A^2I'(t)}{\left(A^2+4(g'(t))^2\right)^{3/2}}-k\right)\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)} \tag{102}$$

We can show that v asymptotically satisfy the PDE and the BCs:

$$\vec{\nu}_1 \cdot \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} = \cos\gamma_1 + o(f_1'(x) - f_2'(x)) \qquad \text{on } \partial\Omega_1$$

$$\vec{\nu}_2 \cdot \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} = \cos\gamma_2 + o(f_1'(x) - f_2'(x)) \qquad \text{on } \partial\Omega_2$$

$$\nabla \cdot \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} - v = o\left(\frac{f_1'(x) - f_2'(x)}{f_1(x) - f_2(x)}\right) \qquad \text{in } \Omega \qquad \text{as } x \to 0$$

If $f_1(x)$ and $f_2(x)$ satisfies the following asymptotic relationships

$$f_1(x) - f_2(x) = o(f_1'(x) - f_2'(x)) \qquad f_1'(x) + f_2'(x) = \delta(f_1'(x) - f_2'(x)) + o(f_1'(x) - f_2'(x))$$

$$\frac{f_1''(x) - f_2''(x)}{f_1(x) - f_2(x)} = \alpha\frac{(f_1'(x) - f_2'(x))^2}{(f_1(x) - f_2(x))^2} + o\left(\frac{(f_1'(x) - f_2'(x))^2}{(f_1(x) - f_2(x))^2}\right)$$

$$\frac{f_1'''(x) - f_2'''(x)}{f_1'(x) - f_2'(x)} = O\left(\frac{(f_1'(s) - f_2'(x))^2}{(f_1(x) - f_2(x))^2}\right) \qquad f_1''(x) + f_2''(x) = O(f_1''(x) - f_2''(x))$$

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For example, an osculatory cusp is not allowed...

$$\frac{f_1''(x) - f_2''(x)}{f_1(x) - f_2(x)} = \alpha\frac{(f_1'(x) - f_2'(x))^2}{(f_1(x) - f_2(x))^2} + o\left(\frac{(f_1'(x) - f_2'(x))^2}{(f_1(x) - f_2(x))^2}\right)$$

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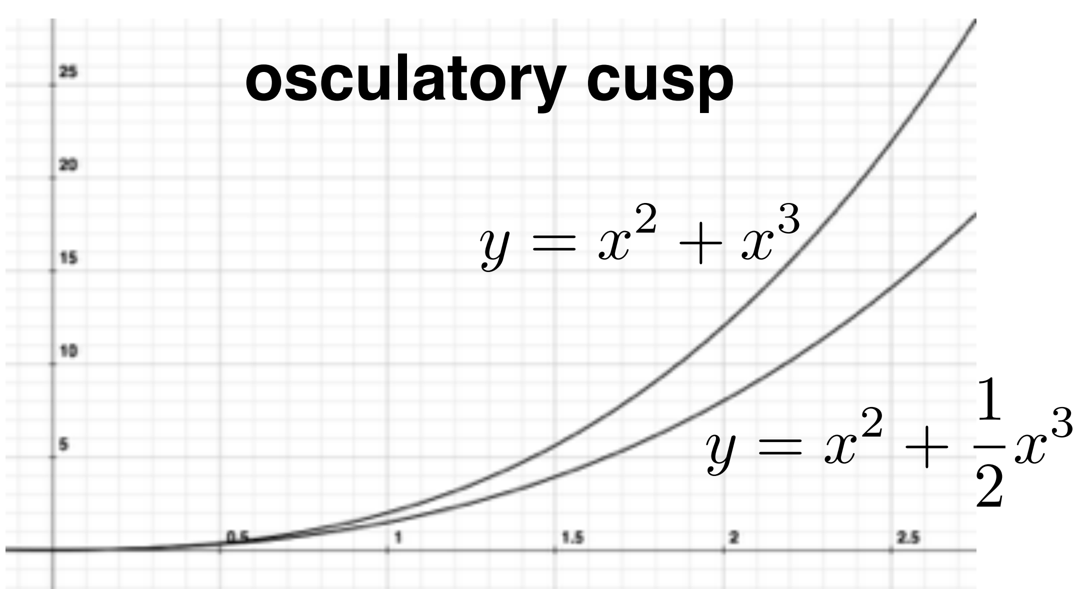
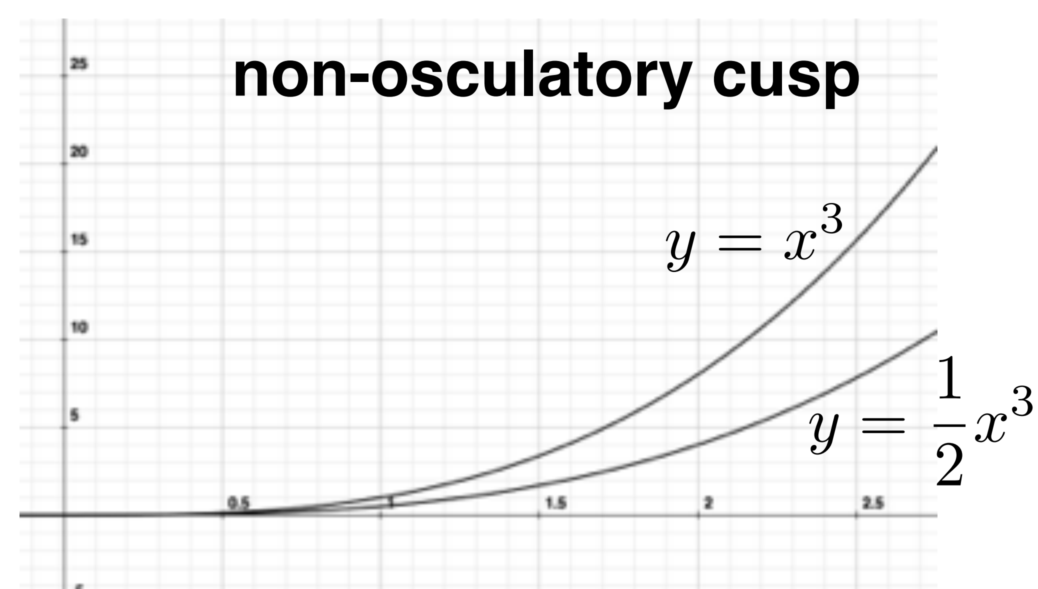




Secondly perturb the formal asymptotic series to construct super-solution

$$v = \frac{\cos\gamma_1 + \cos\gamma_2}{f_1(x) - f_2(x)} + g(x,y)\frac{f_1'(x) - f_2'(x)}{f_1(x) - f_2(x)} + h(x,y)\frac{(f_1'(x) - f_2'(x))^2}{f_1(x) - f_2(x)}$$

Secondly perturb the formal asymptotic series to construct super-solution

$$v^+ = \frac{\cos\gamma_1 + \cos\gamma_2}{f_1(x) - f_2(x)} + g(x,y)\frac{f_1'(x) - f_2'(x)}{f_1(x) - f_2(x)} + h(x,y)\frac{(f_1'(x) - f_2'(x))^2}{f_1(x) - f_2(x)}$$

$$+K_3\frac{f_1'(x) - f_2'(x)}{f_1(x) - f_2(x)} \quad + \frac{K_4}{2}t^2\frac{(f_1'(x) - f_2'(x))^2}{f_1(x) - f_2(x)} \; +K_5$$

By the Concus Finn comparison principle we can prove the following:

$$u < v^+ \qquad \text{for } 0 < x < x_o$$

Theorem 2: $$u = \frac{\cos\gamma_1 + \cos\gamma_2}{f_1(x) - f_2(x)} + O\left(\frac{f_1'(x) - f_2'(x)}{f_1(x) - f_2(x)}\right)$$

in a non-osculatory cusp domain if $\cos\gamma_1 + \cos\gamma_2 \neq 0$.

⇕

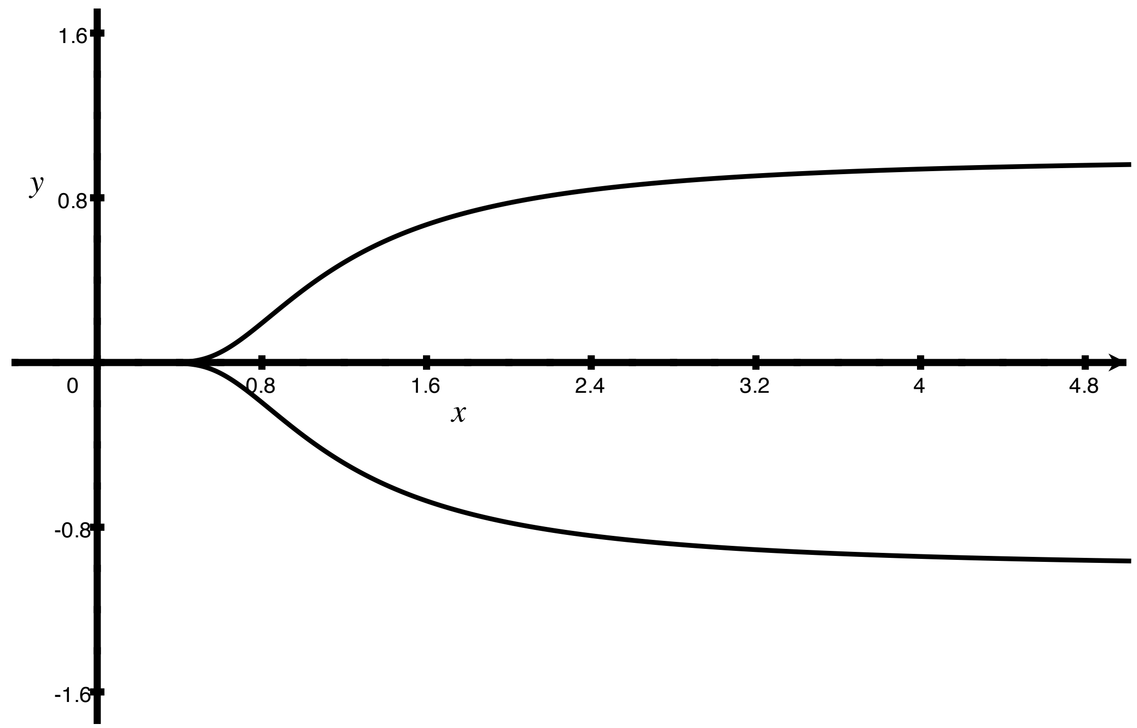
There exist positive constants M and x_o, s.t.

$$\frac{\cos\gamma_1 + \cos\gamma_2}{f_1(x) - f_2(x)} - M\left|\frac{f_1'(x) - f_2'(x)}{f_1(x) - f_2(x)}\right| \le u \le \boxed{\frac{\cos\gamma_1 + \cos\gamma_2}{f_1(x) - f_2(x)} + M\left|\frac{f_1'(x) - f_2'(x)}{f_1(x) - f_2(x)}\right|}$$

Super Solution v^+

for all $0 < x < x_o$

Example 1: $f_1(x) = p\, e^{-1/x^2}$ $f_2(x) = q\, e^{-1/x^2}$



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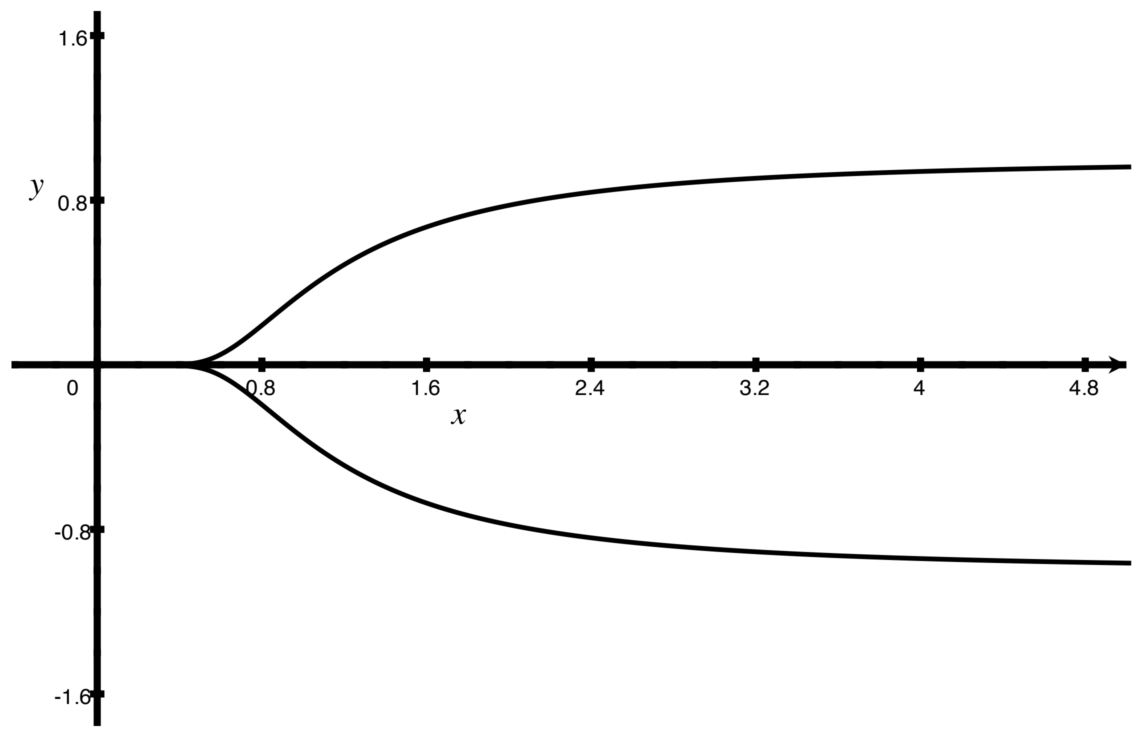

$$u(x,y) = \frac{\cos\gamma_1 + \cos\gamma_2}{p-q} e^{1/x^2} + O(x^{-3})$$

Example 2: $f_1(x) = p\left(x^{5/2} + x^3\right)$ $f_2(x) = q\left(x^{5/2} + x^3\right)$

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$$u(x, y) = \frac{\cos\gamma_1 + \cos\gamma_2}{(p-q)(x^{5/2} + x^3)} + O(x^{-1})$$

Example 2: $f_1(x) = p\left(x^{5/2} + x^3\right) \qquad f_2(x) = q\left(x^{5/2} + x^3\right)$

$$u(x, y) = \frac{\cos\gamma_1 + \cos\gamma_2}{p - q}\left(\frac{1}{x^{5/2}} - \frac{1}{x^2} + \frac{1}{x^{3/2}}\right) + O(x^{-1})$$

Unbounded Capillary Surface

$\cos\gamma_1 + \cos\gamma_2 \neq 0$

Power Series Cusp

(Scholz 2004)

Bounded Capillary Surface

$\cos\gamma_1 + \cos\gamma_2 = 0$

Finite Curvature Cusp

(Aoki and Siegel 2012)

Open Problems

What if $\cos\gamma_1 + \cos\gamma_2 = 0$?

Infinite Curvature Cusp

Non-power Series Cusp?

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Open Problems

What if $\cos\gamma_1 + \cos\gamma_2 = 0$?

Infinite Curvature Cusp

Non-power Series Cusp?

Osculatory Cusp

Open Problem 1:

Is the capillary surface bounded if $\cos\gamma_1 + \cos\gamma_2 = 0$,
and a boundary has an infinite curvature?

For example $f_1(x) = \frac{1}{6}x^{\frac{3}{2}}$ $f_2(x) = \frac{1}{8}x^{\frac{3}{2}}$

Open Problem 2:

What is the formal asymptotic series for the osculatory cusp?

For example $f_1(x) = \frac{1}{6}x^2 + \frac{1}{6}x^3 \quad f_2(x) = \frac{1}{6}x^2 - \frac{1}{8}x^3$

Open Problem 1:

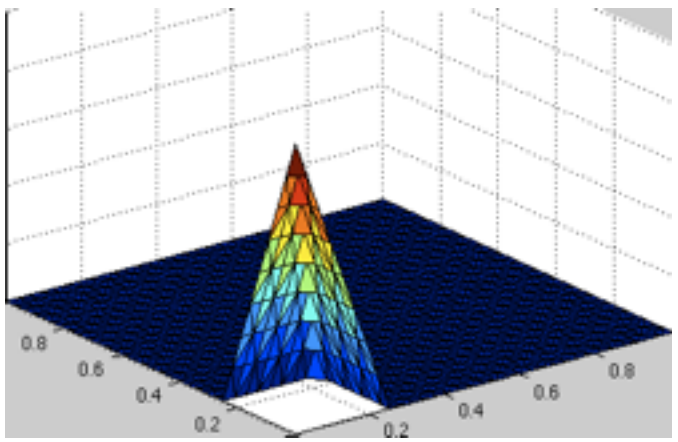
Is the capillary surface bounded if $\cos\gamma_1 + \cos\gamma_2 = 0\,,$
and a boundary has an infinite curvature?

Open Problem 2:

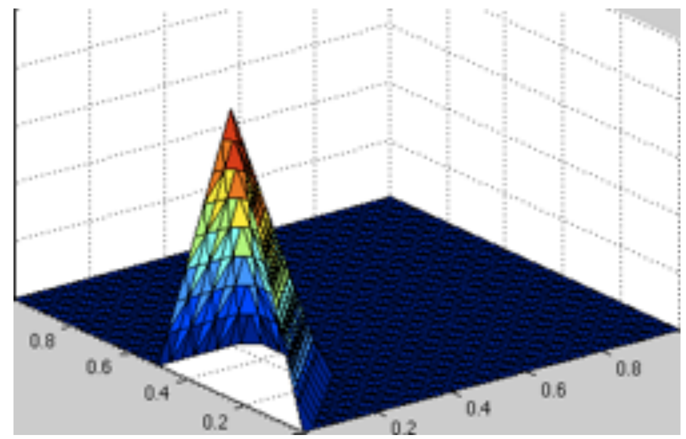
What is the formal asymptotic series for the osculatory cusp?

Finite Element Approximation

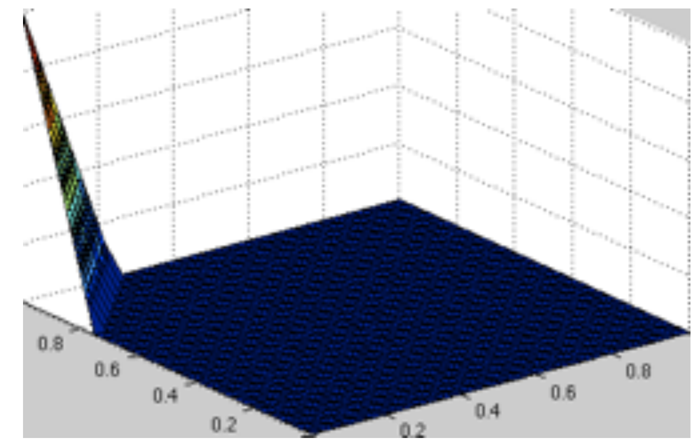
Finite Element Approximation



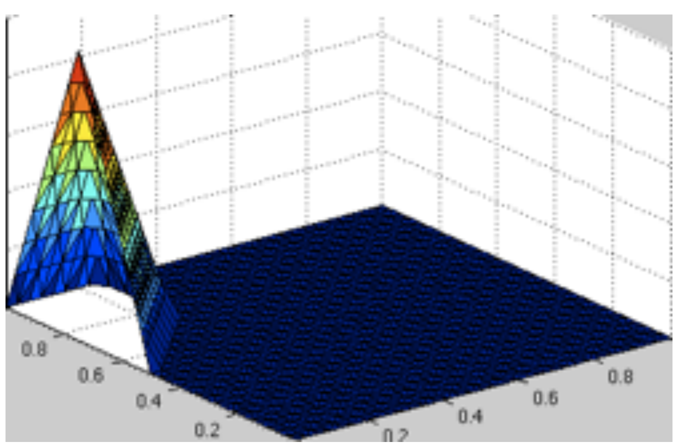
ϕ_1



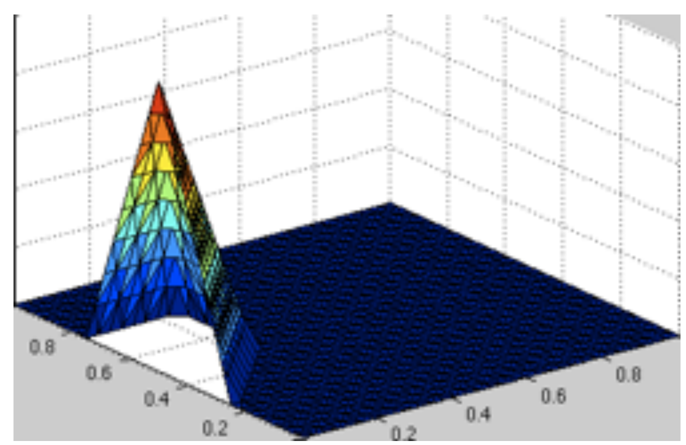
ϕ_2



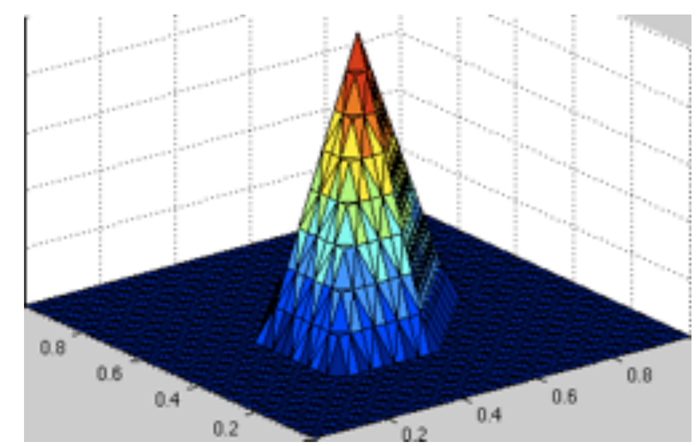
ϕ_5



ϕ_4



ϕ_3

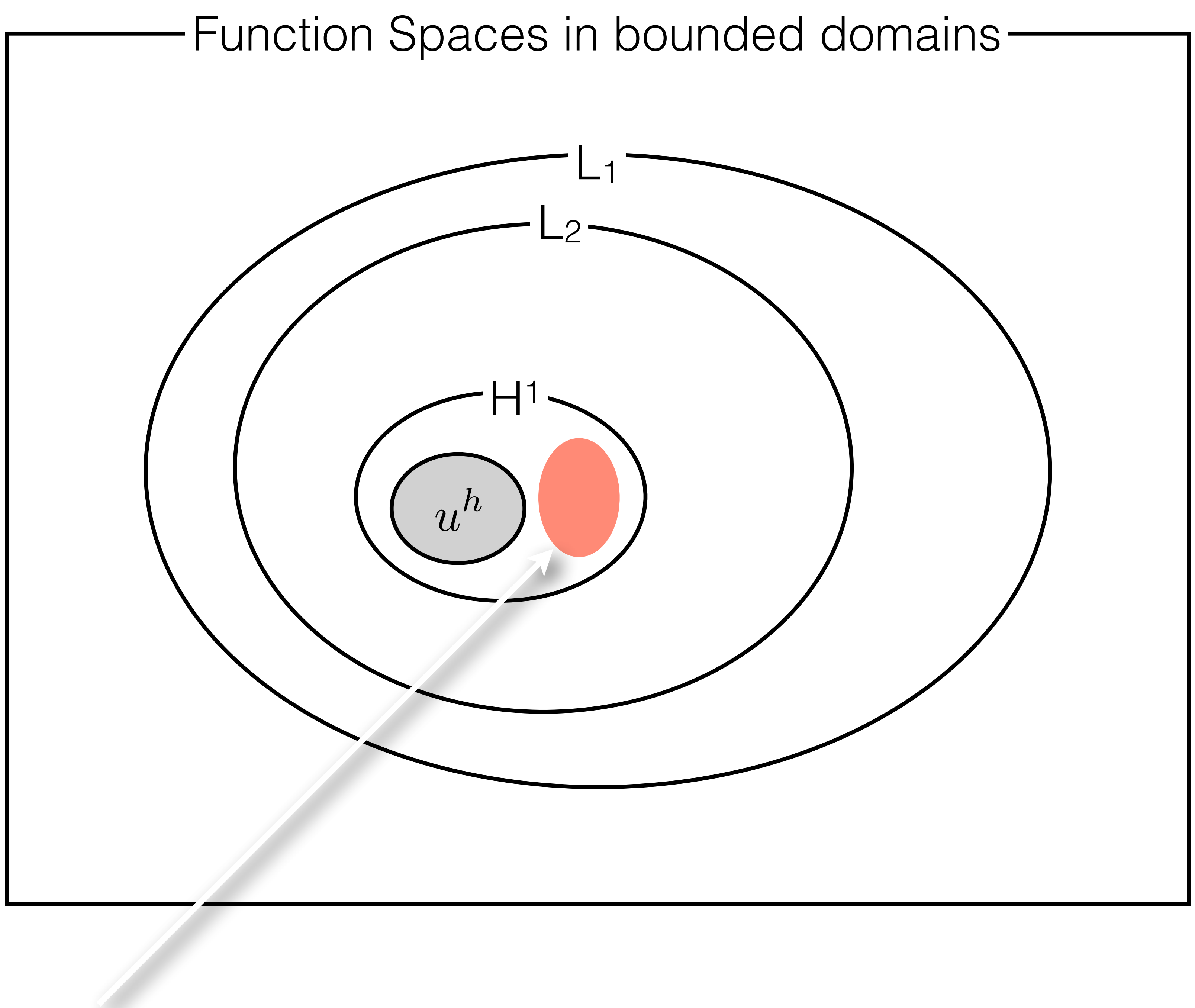


ϕ_i

Standard Trial Function

$$u \approx u^h := \sum_{i=1}^{N_{\text{node}}} c_i \phi_i$$

Finite Element Approximation



Bounded Solutions of the Laplace-Young Equation

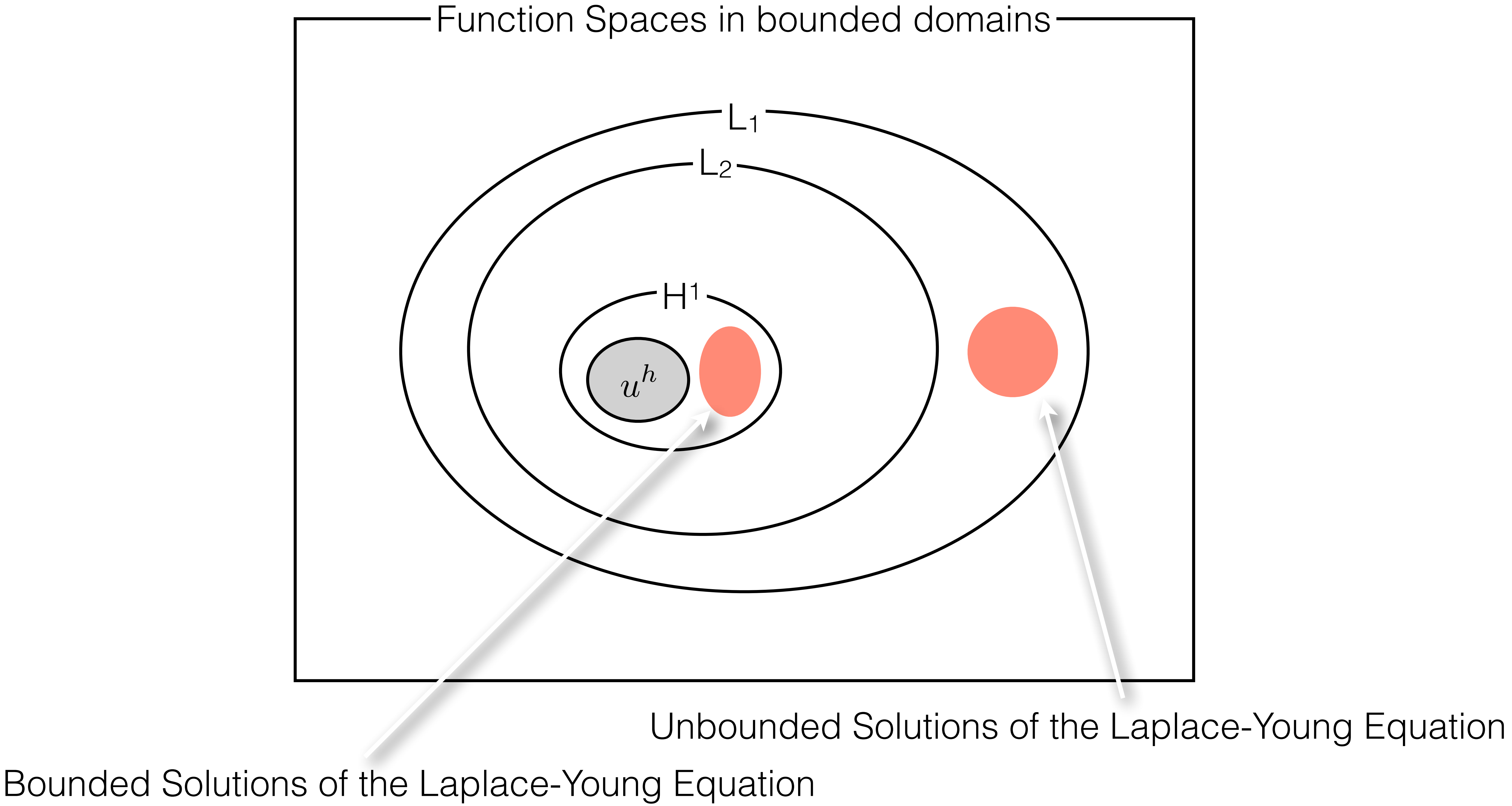
Finite Element Approximation

Proposition 1: If $\cos\gamma_1 + \cos\gamma_2 \neq 0$,
then $u \in L_p(\Omega)$ if and only if the following integral is finite for $\epsilon > 0$.

$$\int_0^{\epsilon} \frac{1}{(f_1(x) - f_2(x))^{p-1}} \mathrm{d}x$$

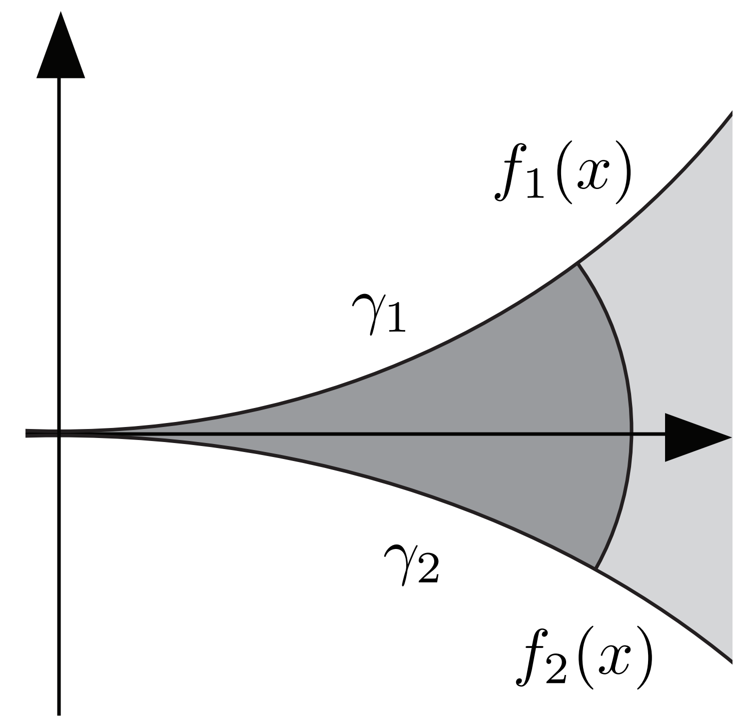
Yasunori Aoki and Hans De Sterck:
of Mathematics, Volume 267, Issue 1, 2014, Pages 1-34.

Finite Element Approximation



Finite Element Approximation

$$u = O\left(\frac{1}{f_1(x) - f_2(x)}\right)$$

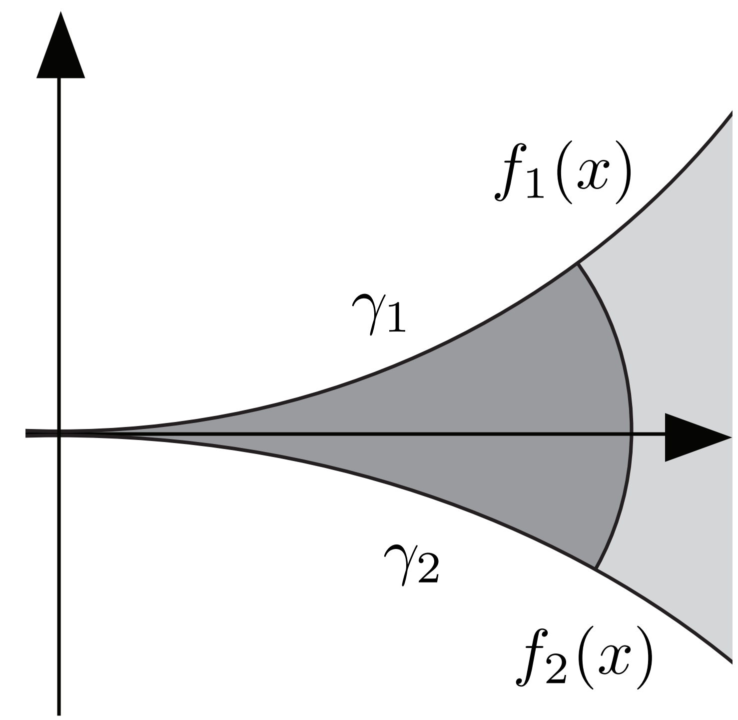


$$\gamma_1 + \gamma_2 \neq \pi$$

Finite Element Approximation

$$u = O\left(\frac{1}{f_1(x) - f_2(x)}\right)$$

$$= \frac{O(1)}{f_1(x) - f_2(x)}$$



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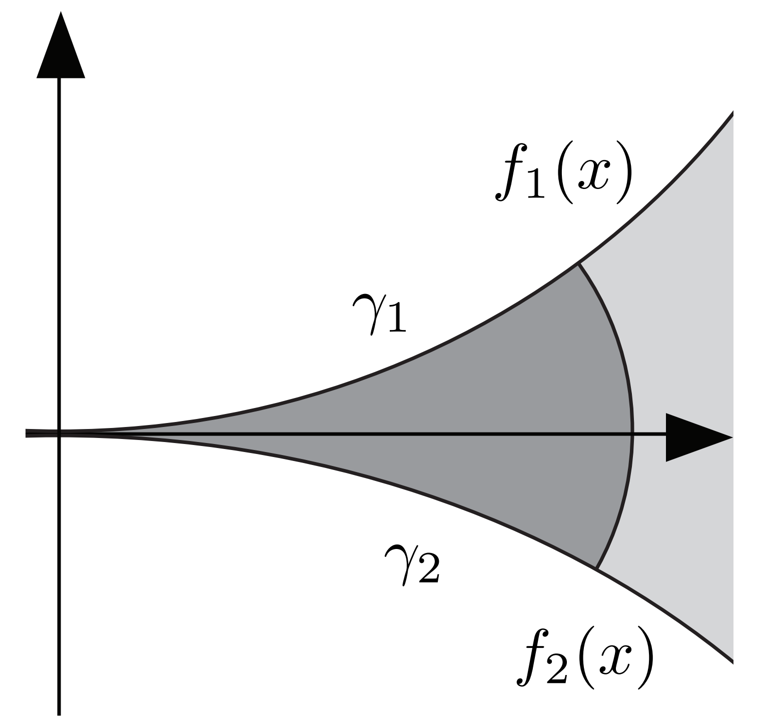
Finite Element Approximation

$$u = O\left(\frac{1}{f_1(x) - f_2(x)}\right)$$

$$= \frac{O(1)}{f_1(x) - f_2(x)}$$

$$u^h = \sum_{i=1}^{N_{\mathrm{node}}} c_i \frac{\phi_i}{f_1(x) - f_2(x)}$$



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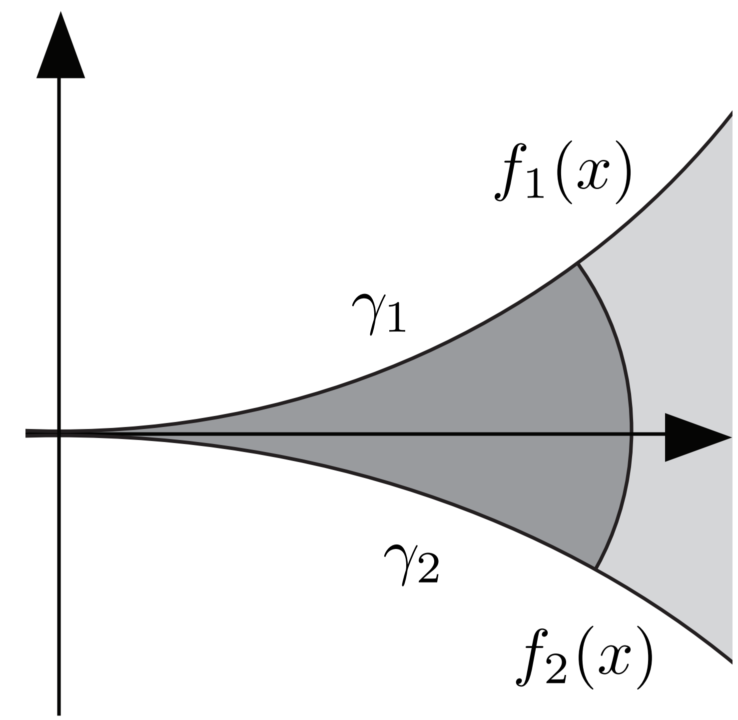
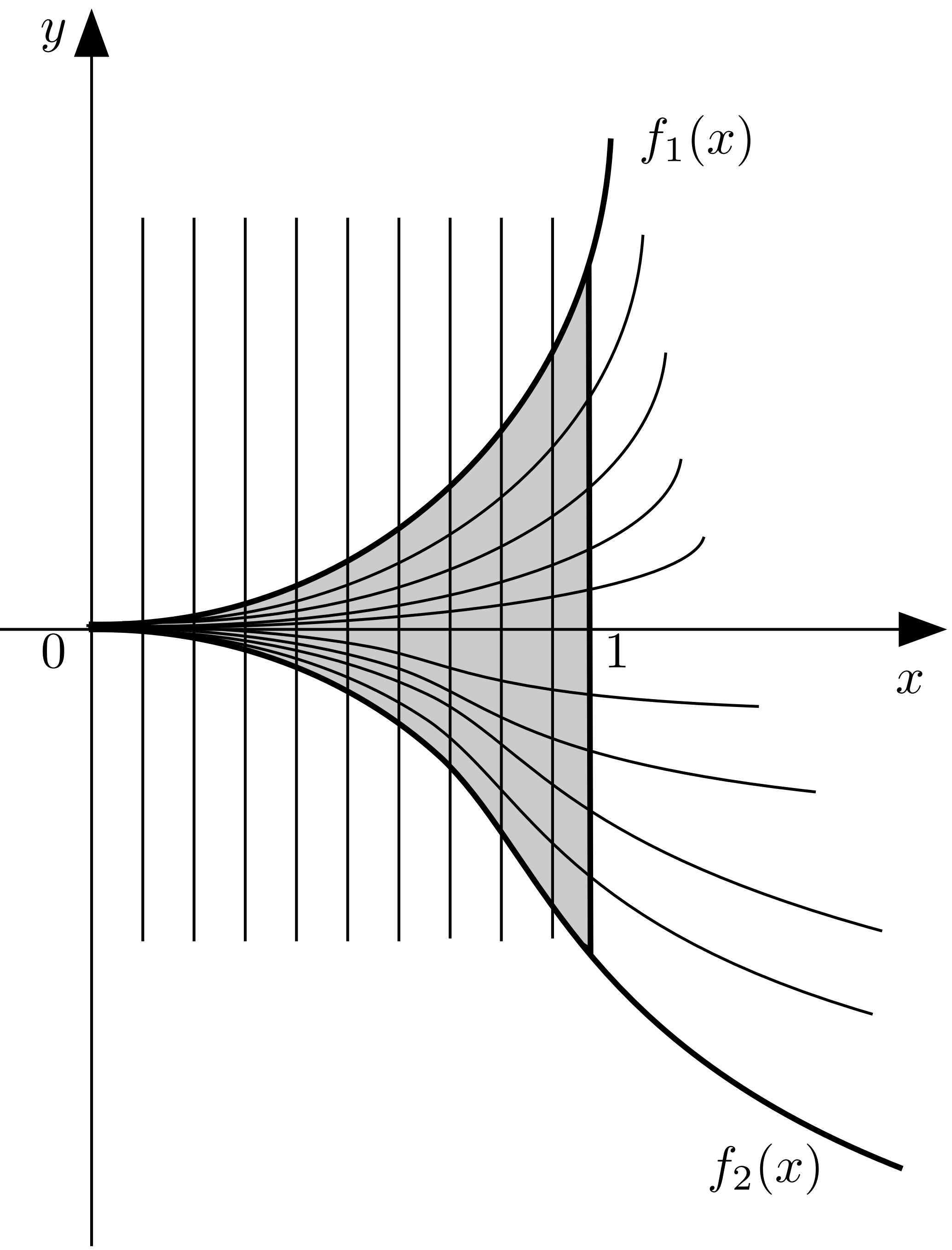


$$\gamma_1 + \gamma_2 \neq \pi$$

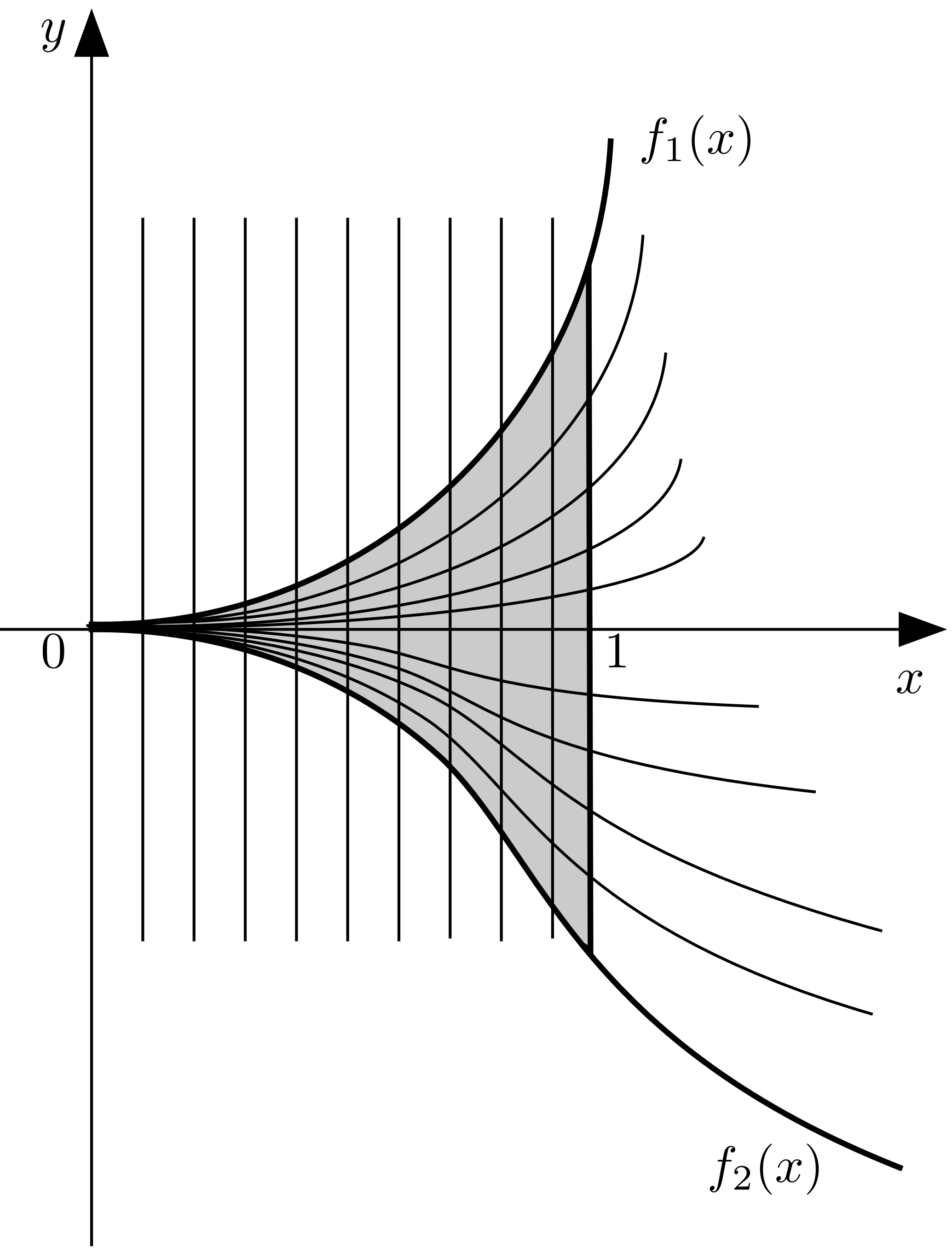
$$u^h = \sum_{i=1}^{N_{\text{node}}} c_i \frac{\phi_i}{f_1(x) - f_2(x)}$$

Use Finite Element Method to find c_i .

Finite Element Approximation



Finite Element Approximation



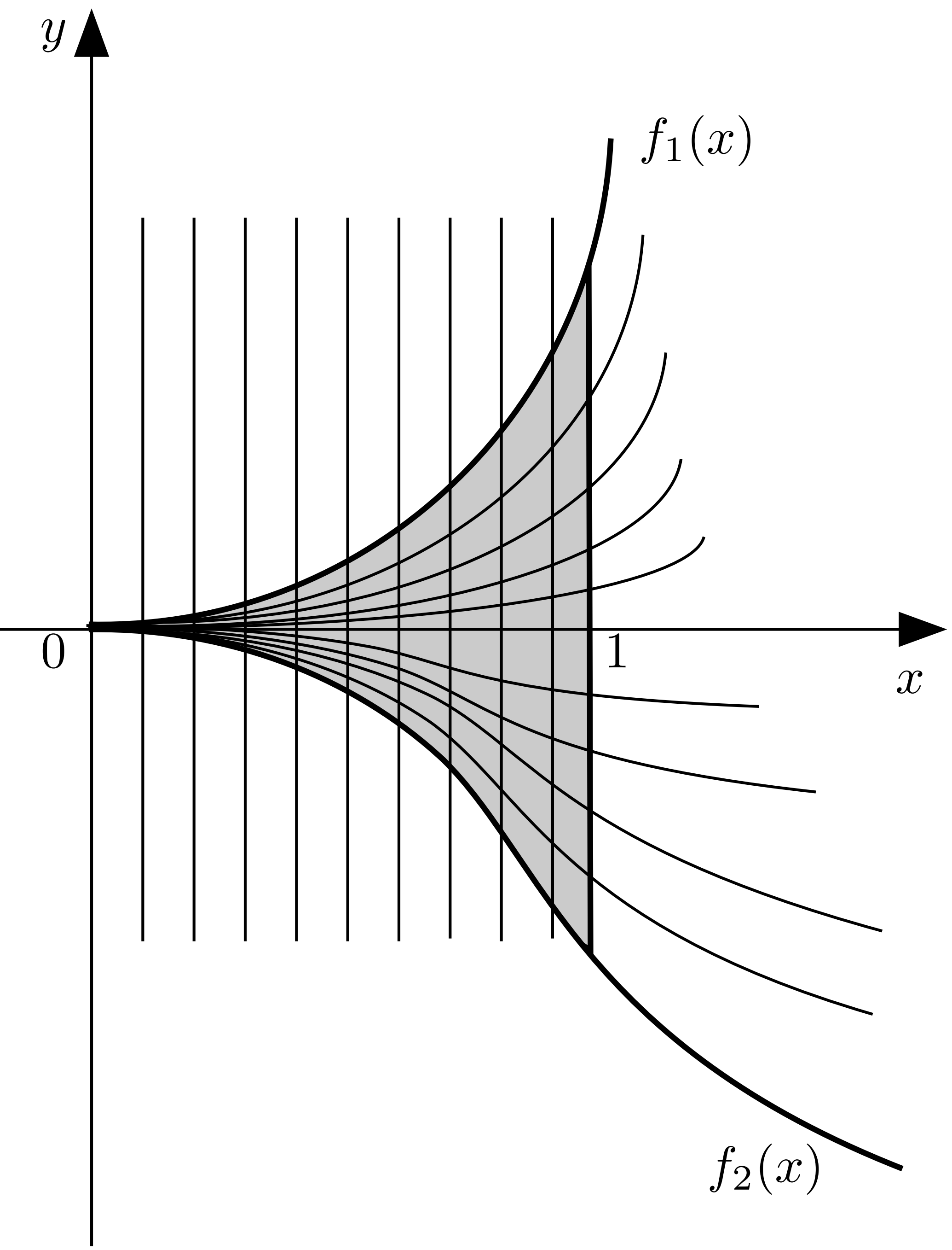
$$t = \frac{2y - (f_1 + f_2)}{f_1 - f_2}$$

$$s = x$$

$$x = s$$

$$y = \frac{1+t}{2} f_1(s) + \frac{1-t}{2} f_2(s)$$

Finite Element Approximation



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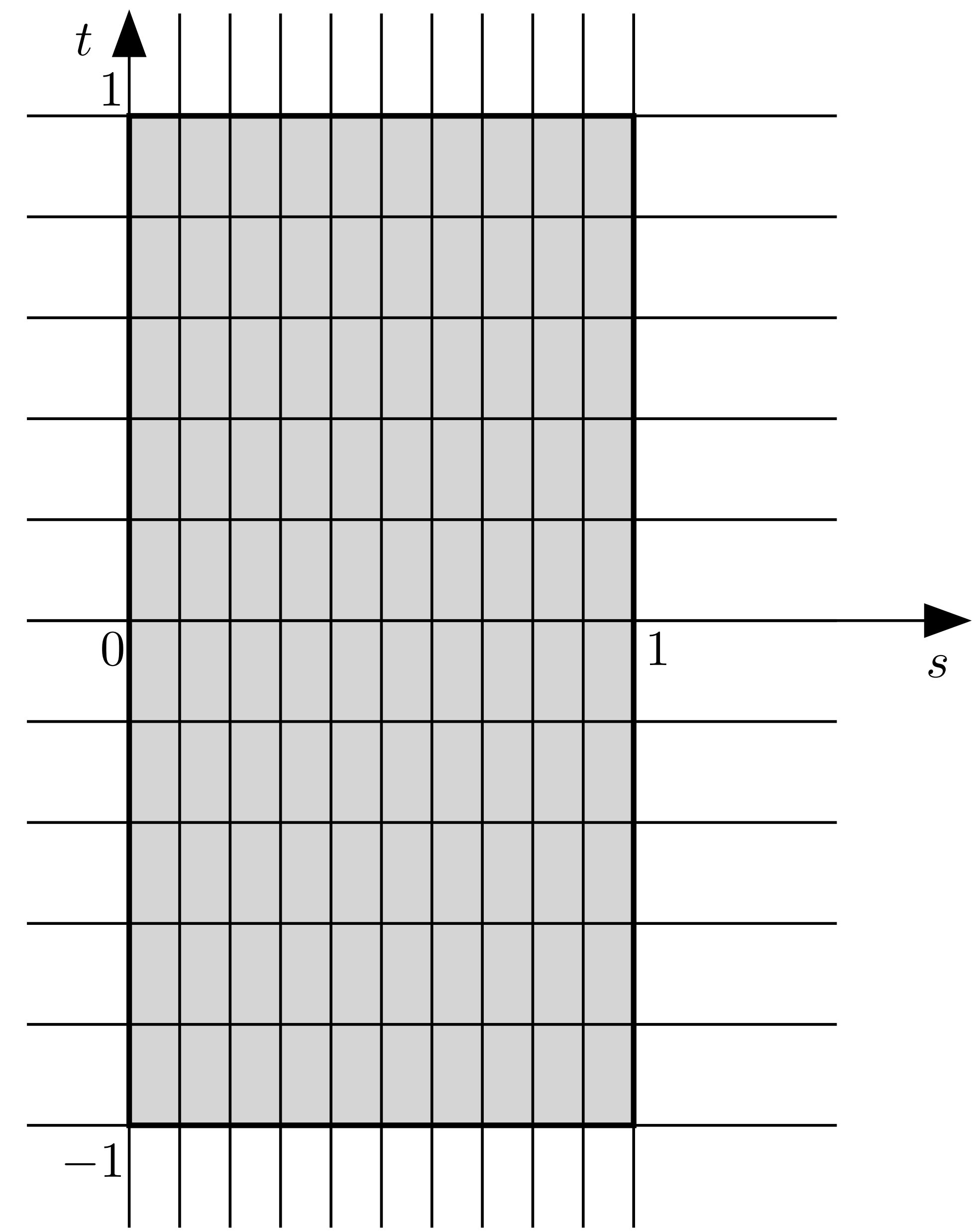

Asymptotic Laplace-Young Equation

Yasunori Aoki:
University of Waterloo, 2007

Asymptotic Laplace-Young Equation

$$\nabla \cdot \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = u$$

$$\nu \cdot \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = \cos \gamma$$

Yasunori Aoki:
University of Waterloo, 2007

Asymptotic Laplace-Young Equation

$$\nabla \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = u \qquad \sim \qquad \nabla \cdot \frac{\nabla v}{|\nabla v|} = v \qquad \text{in } \Omega$$

$$\nu \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \cos\gamma \qquad \qquad \nu \cdot \frac{\nabla v}{|\nabla v|} = \cos\gamma \qquad \text{on } \partial\Omega$$

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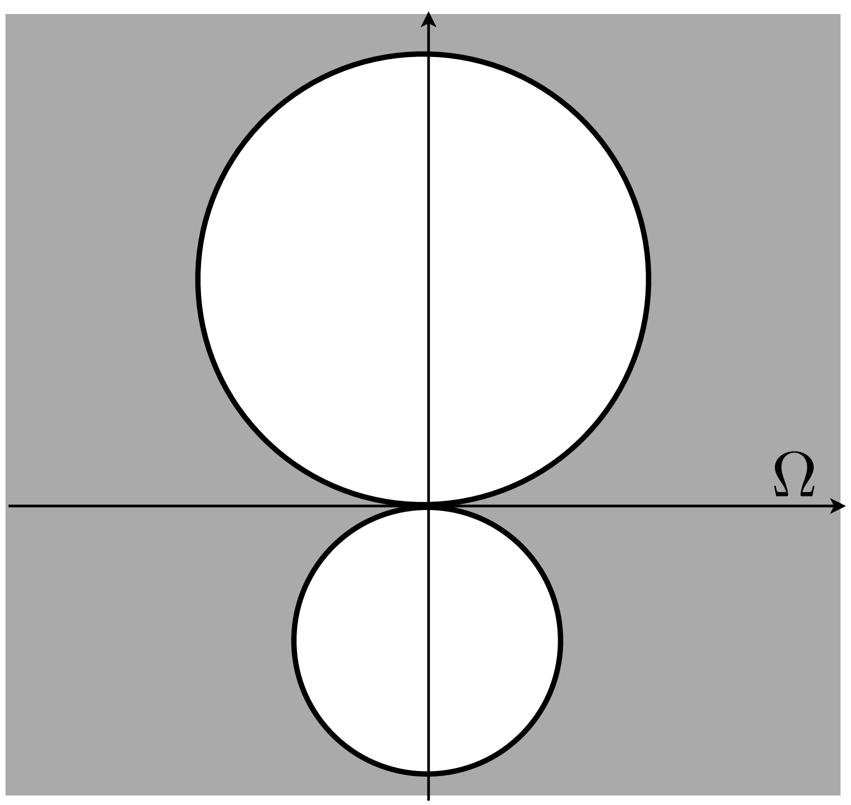

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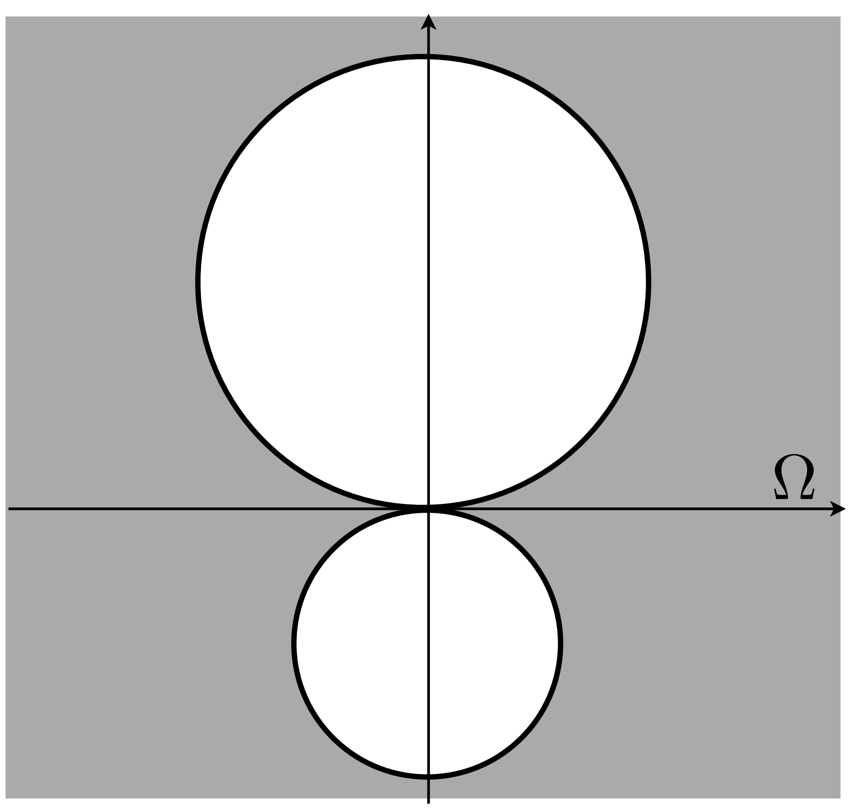


$$v(p,q) = Ap^2 - 2\sqrt{1 - A^2(q-q_0)^2}\; p - A(q-q_0)^2 + Aq_0^2$$

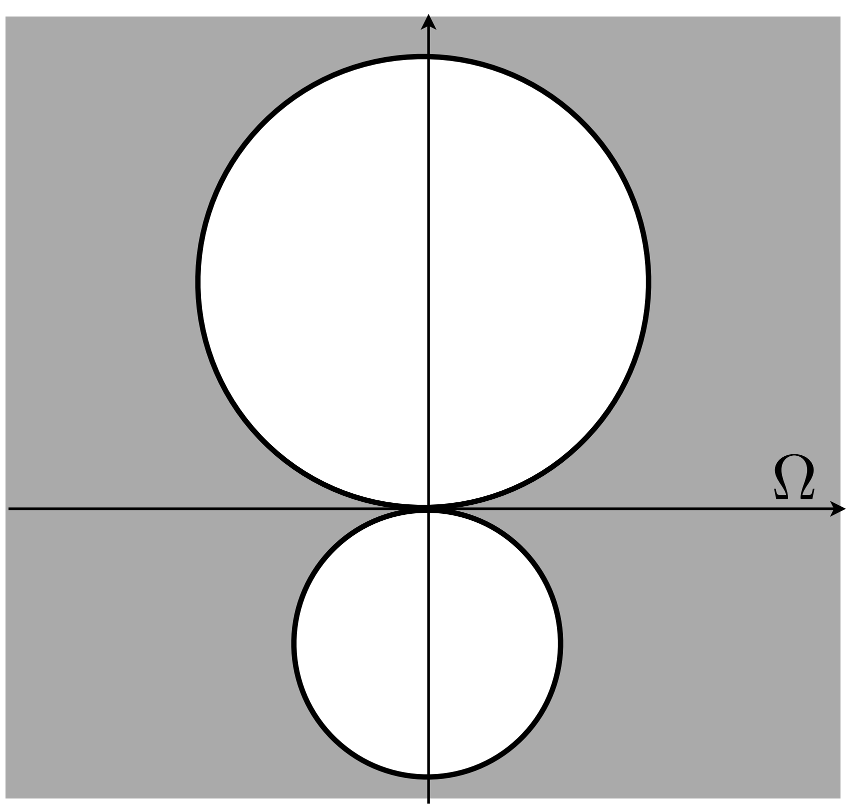
$$p := \frac{x}{x^2+y^2}\,, \qquad q := \frac{y}{x^2+y^2}$$

Yasunori Aoki:
University of Waterloo, 2007

Asymptotic Laplace-Young Equation

$$\nabla \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = u \qquad \sim \qquad \nabla \cdot \frac{\nabla v}{|\nabla v|} = v \qquad \text{in } \Omega$$

$$\nu \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \cos\gamma \qquad \qquad \nu \cdot \frac{\nabla v}{|\nabla v|} = \cos\gamma \qquad \text{on } \partial\Omega$$



$$v(p,q) = Ap^2 - 2\sqrt{1-A^2(q-q_0)^2}\, p - A(q-q_0)^2 + Aq_0^2$$

$$p := \frac{x}{x^2+y^2}\,, \qquad q := \frac{y}{x^2+y^2}$$

$$u(p,q) = v(p,q) + O(p^{-5}) \qquad \text{as } p \to \infty$$

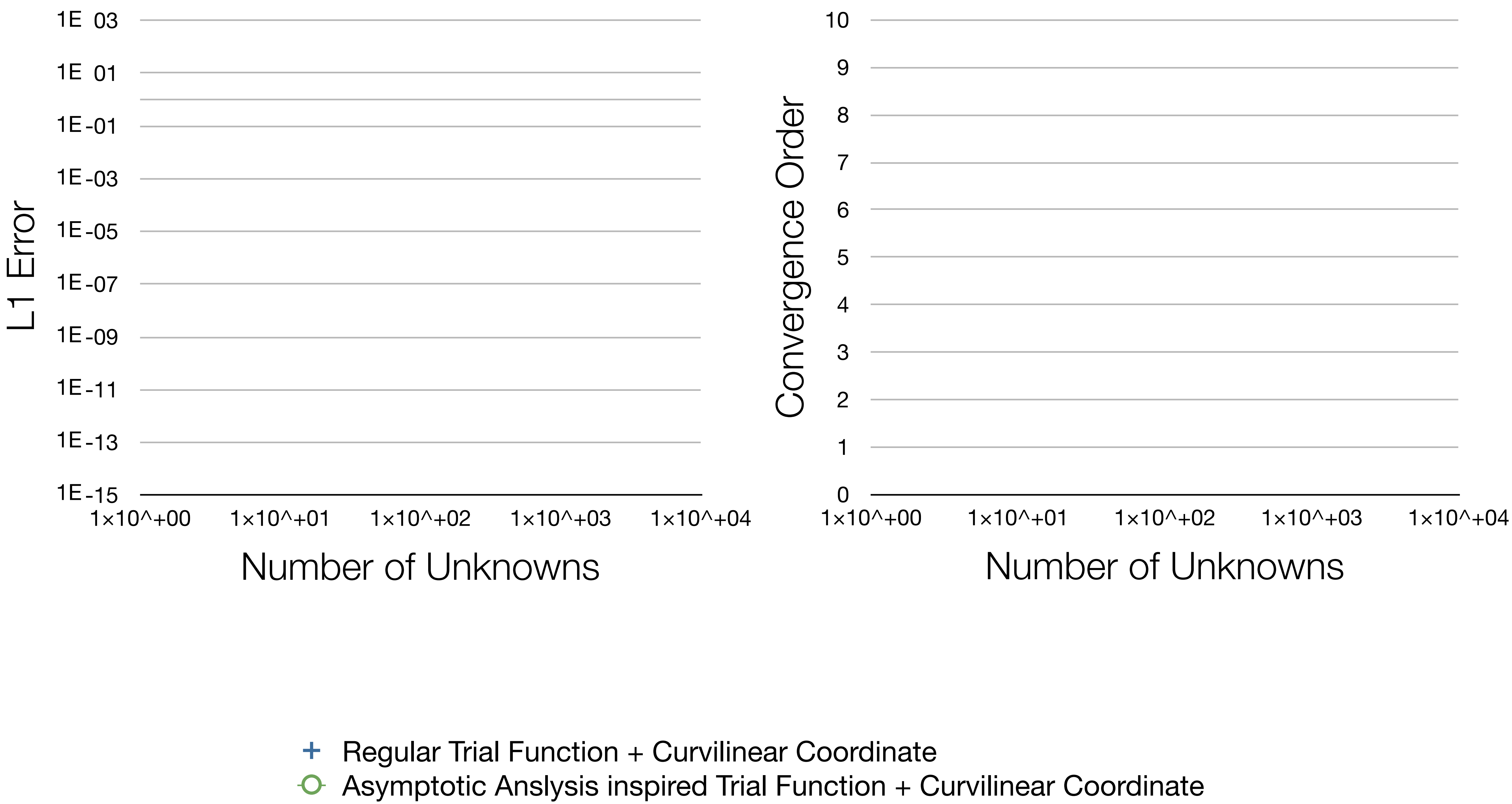
Yasunori Aoki:
University of Waterloo, 2007

Asymptotic Laplace-Young Equation

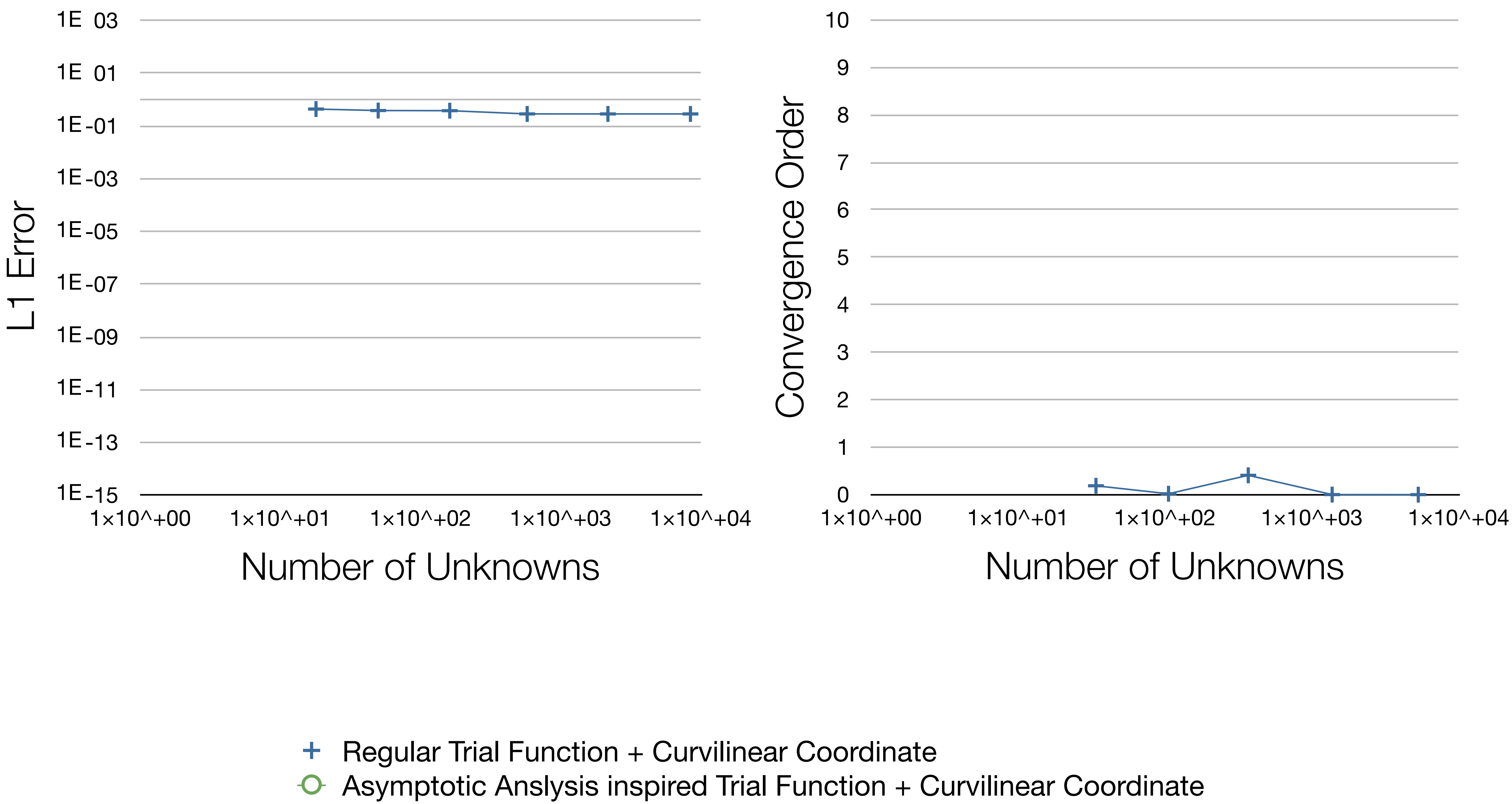
L1 Error

1E
1E
1E
1E
1E
1E
1E
1E
1E
1E

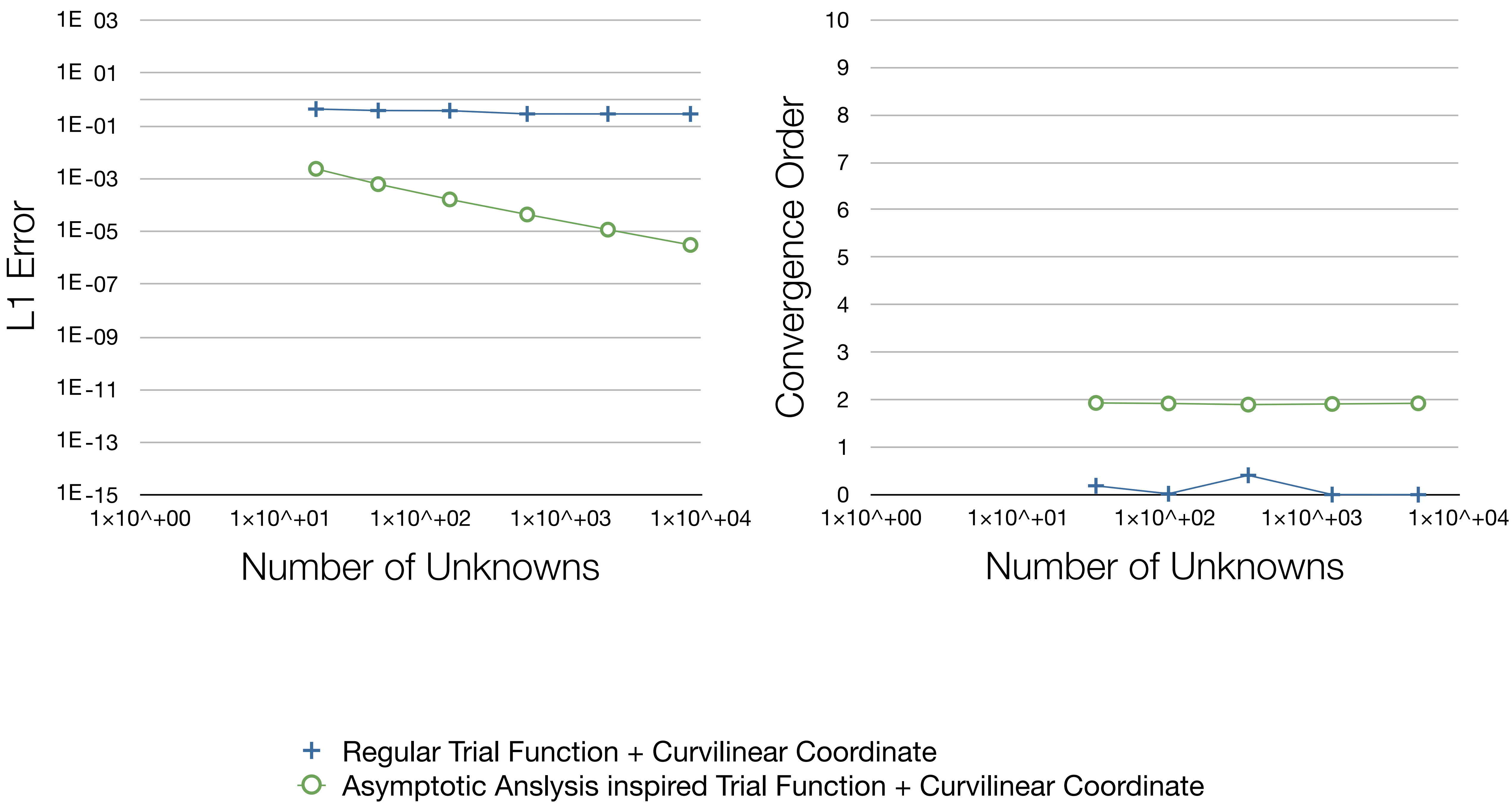
Asymptotic Laplace-Young Equation



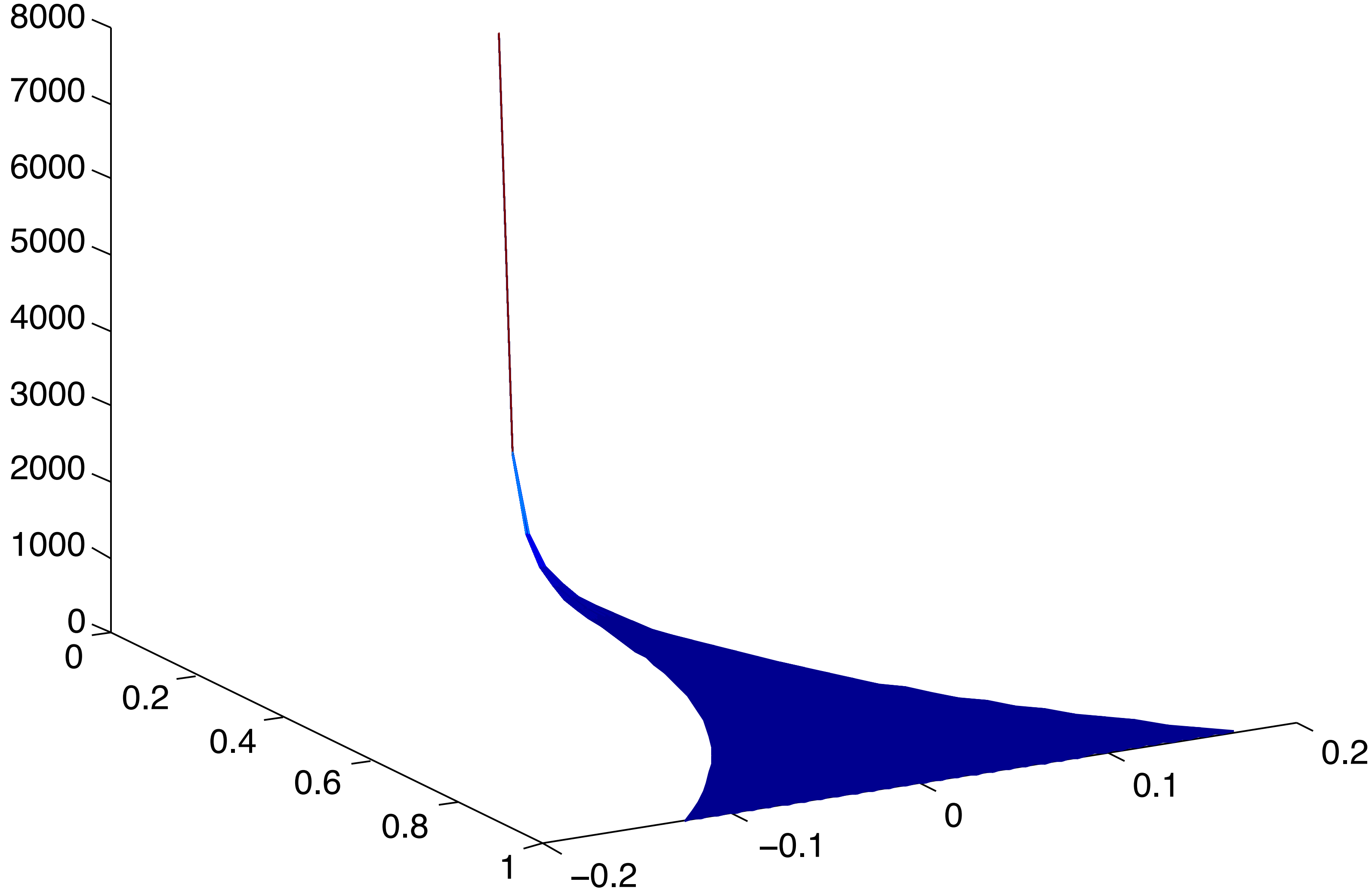
Asymptotic Laplace-Young Equation



Asymptotic Laplace-Young Equation



Capillary Surface in a Cusp Domain



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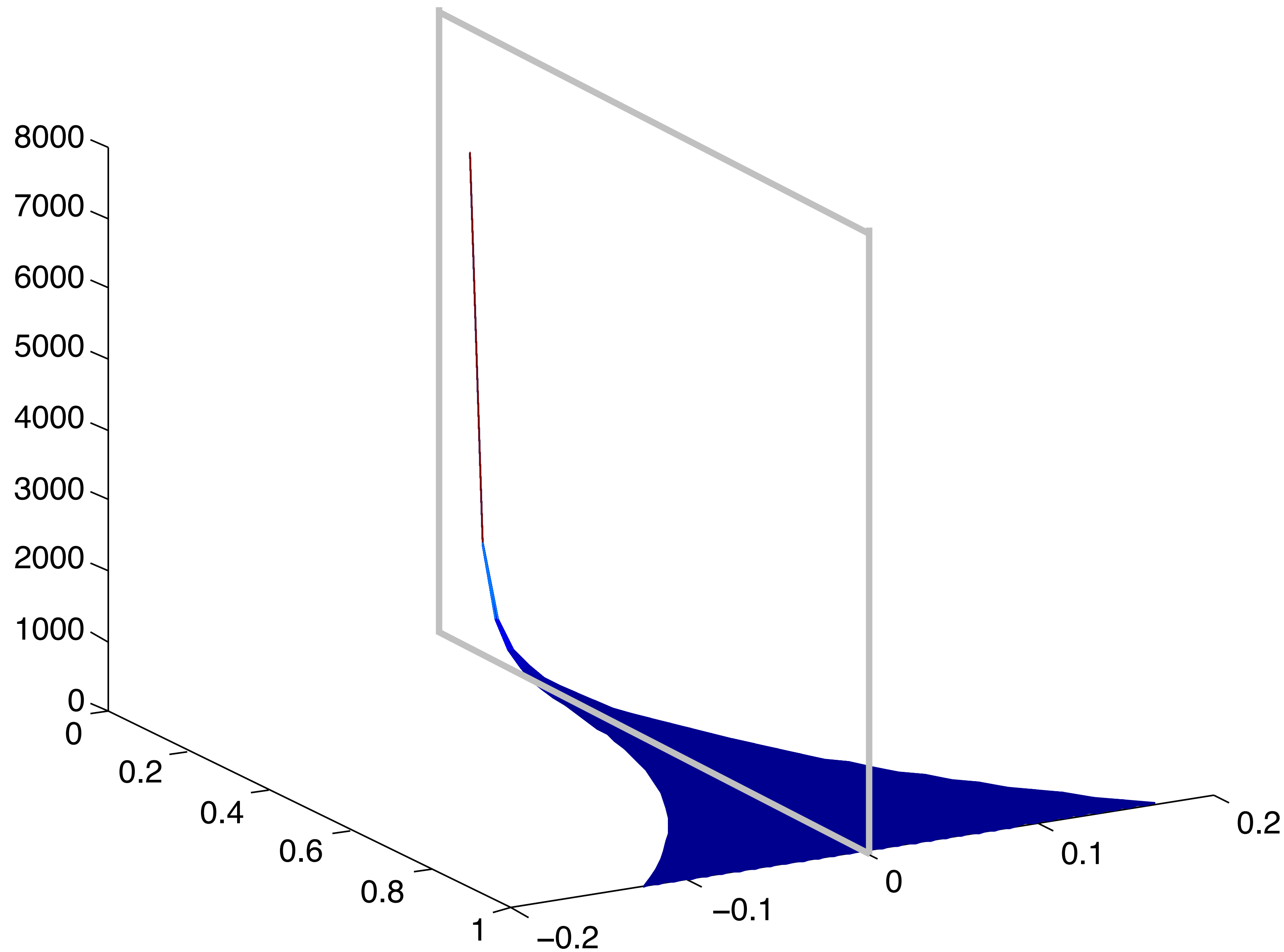

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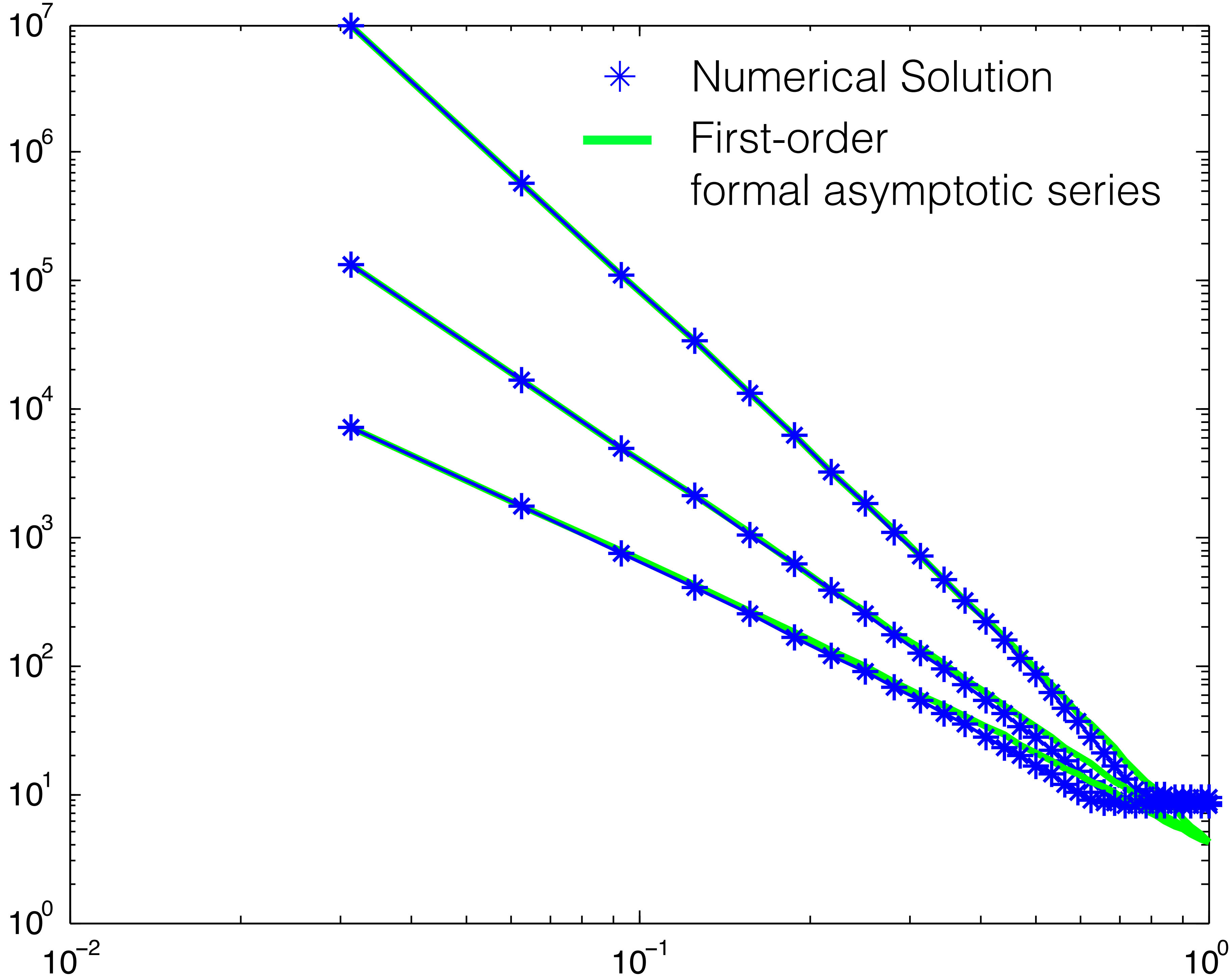

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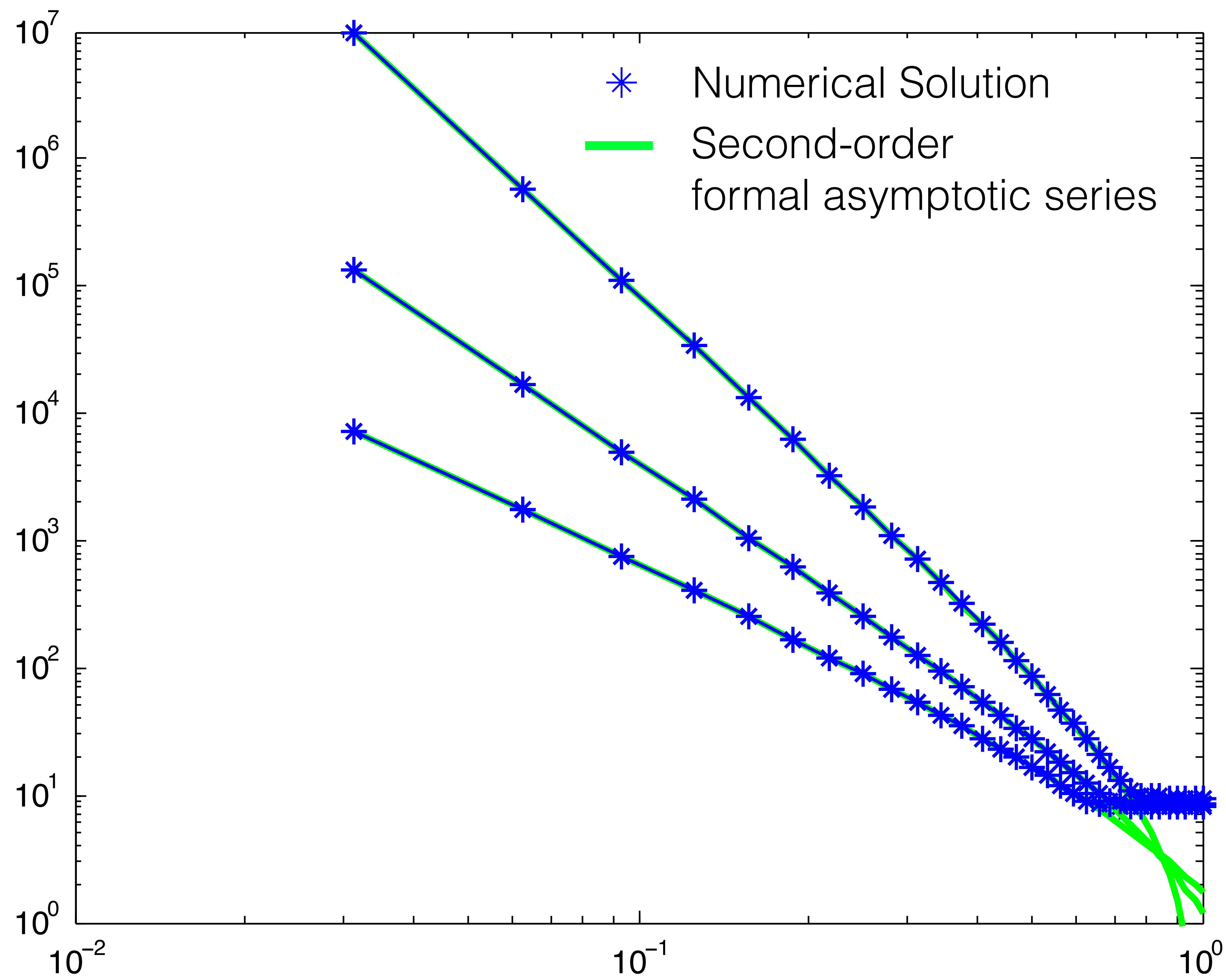
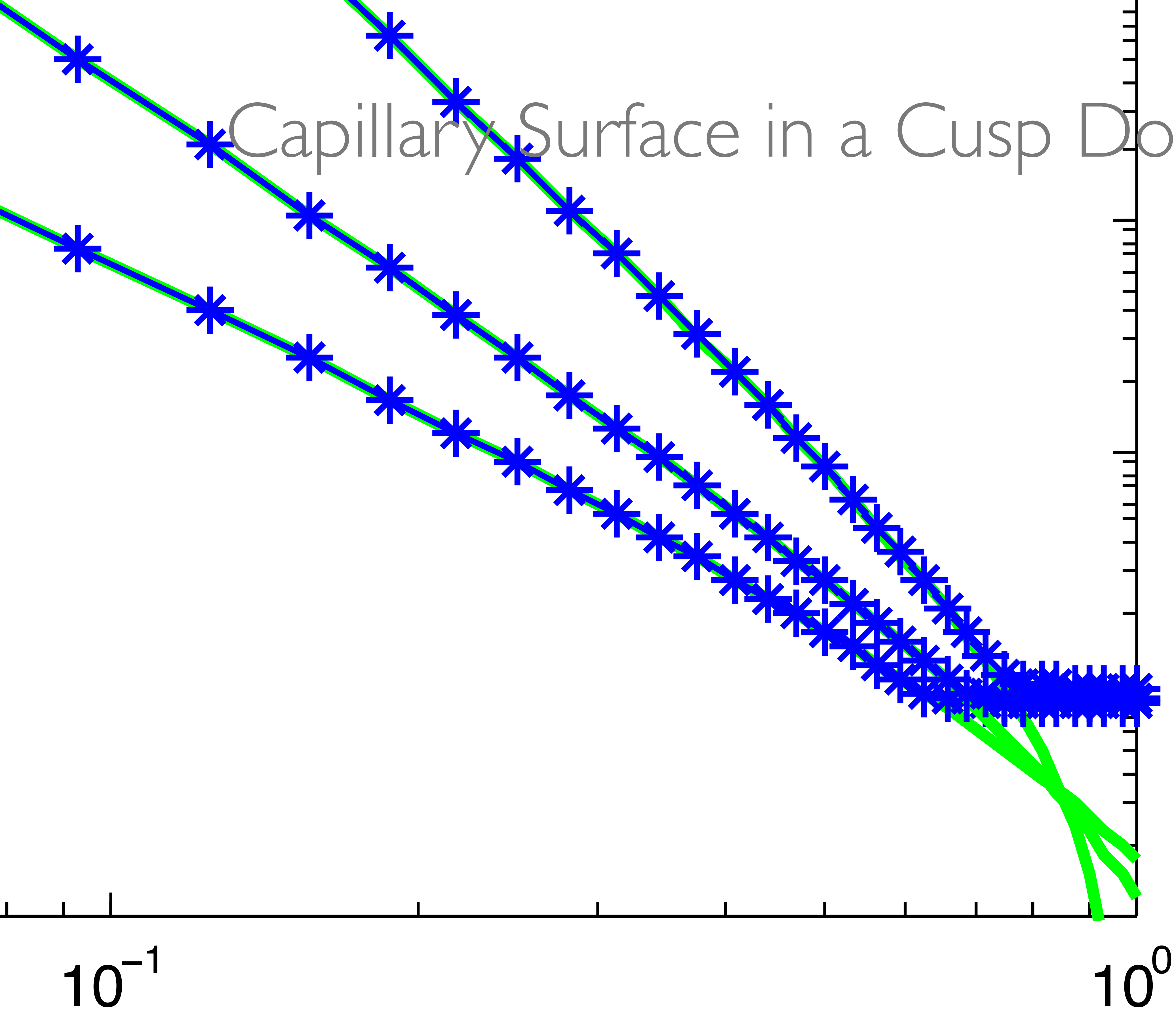


Capillary Surface in a Cusp Domain



Open Problem 1:

Is the capillary surface bounded if $\cos\gamma_1 + \cos\gamma_2 = 0$,
and a boundary has an infinite curvature?

$$f_1(x) = \frac{1}{6}x^{\frac{3}{2}} \qquad\qquad f_2(x) = \frac{1}{8}x^{\frac{3}{2}}$$

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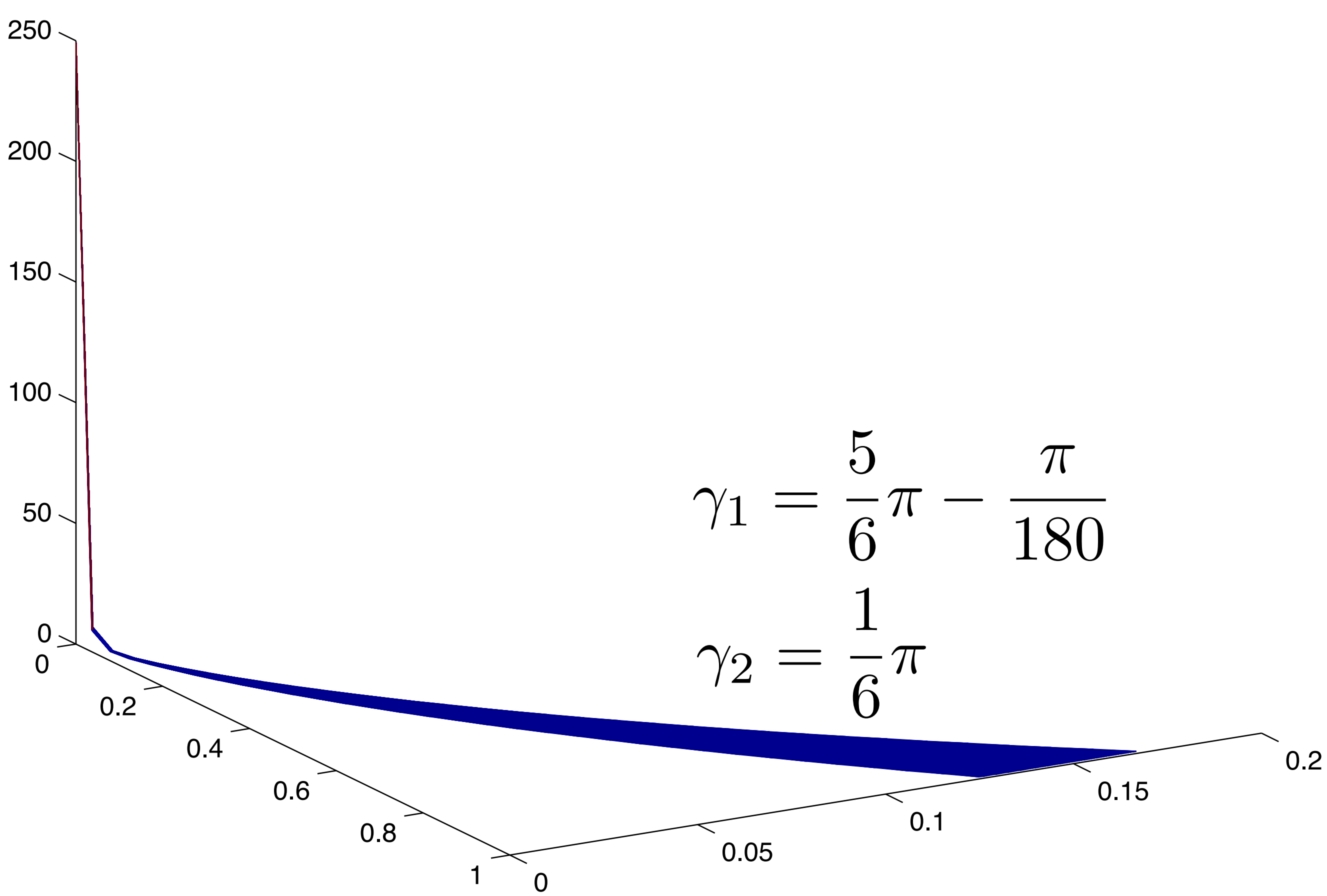


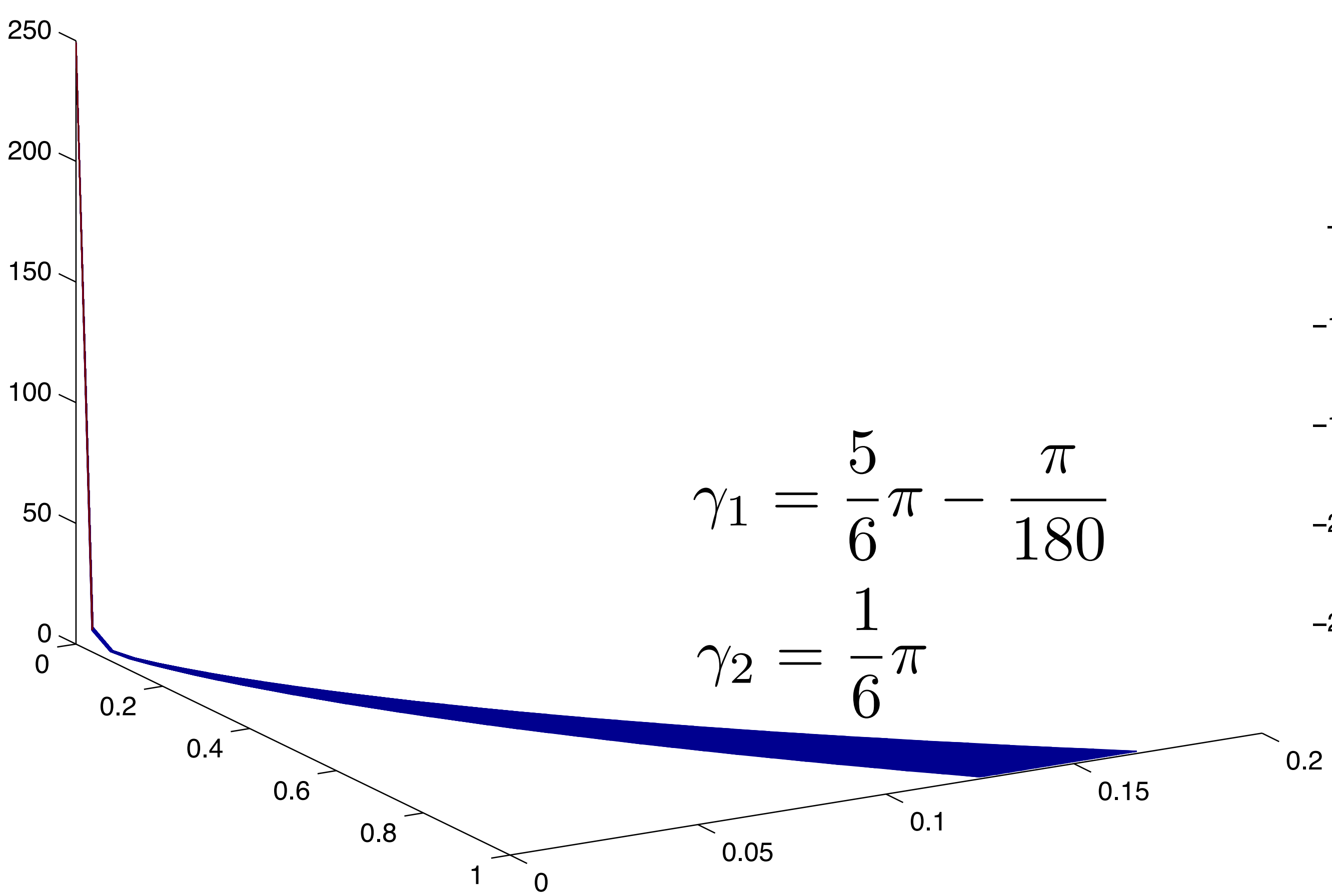
$$\gamma_1 = \frac{5}{6}\pi - \frac{\pi}{180}$$

$$\gamma_2 = \frac{1}{6}\pi$$

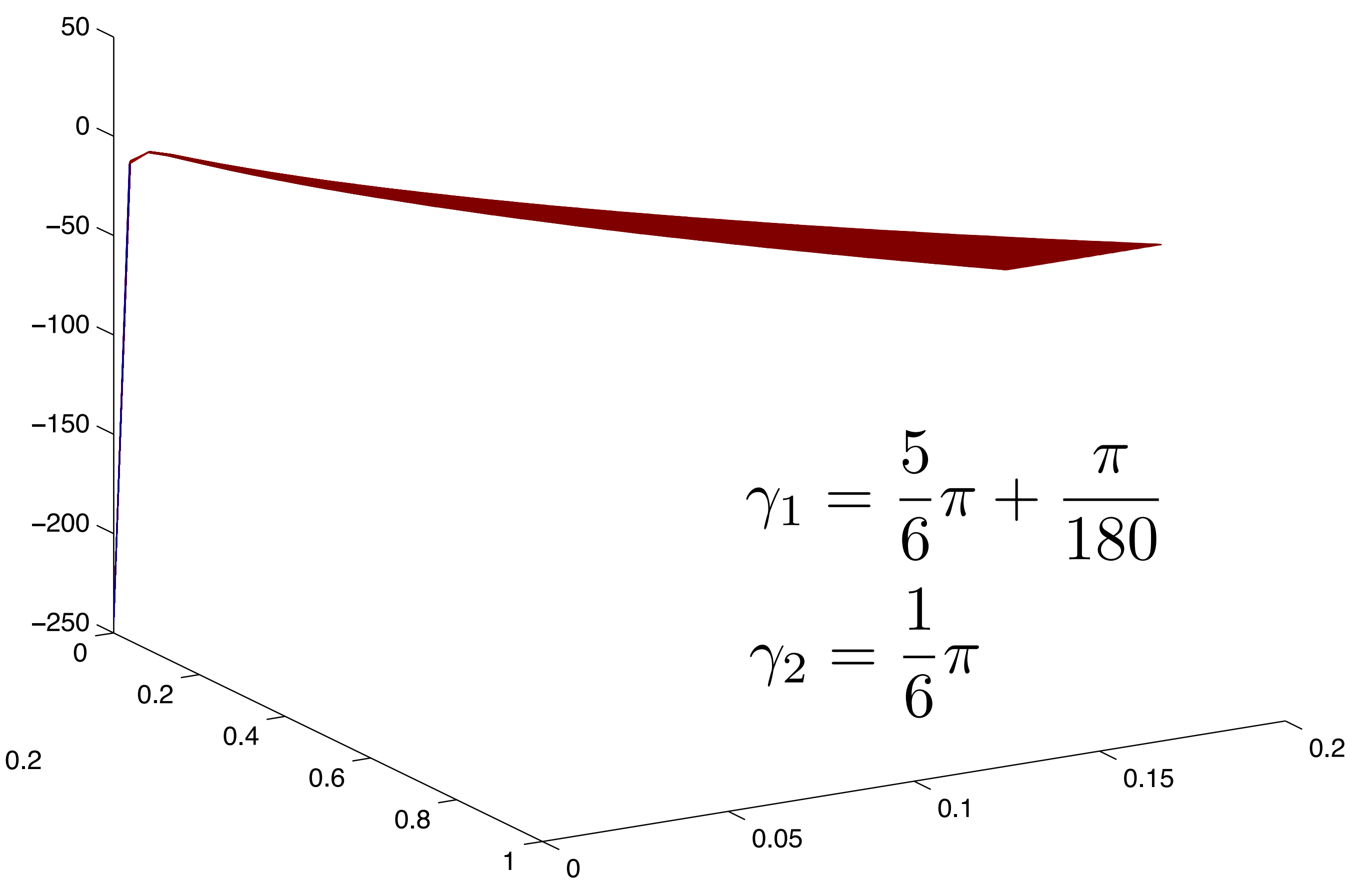
Open Problem I:

Is the capillary surface bounded if $\cos\gamma_1 + \cos\gamma_2 = 0$, and a boundary has an infinite curvature?

$$f_1(x) = \frac{1}{6}x^{\frac{3}{2}} \qquad\qquad f_2(x) = \frac{1}{8}x^{\frac{3}{2}}$$



$$\gamma_1 = \frac{5}{6}\pi - \frac{\pi}{180}$$
$$\gamma_2 = \frac{1}{6}\pi$$



$$\gamma_1 = \frac{5}{6}\pi + \frac{\pi}{180}$$
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Open Problem 1:

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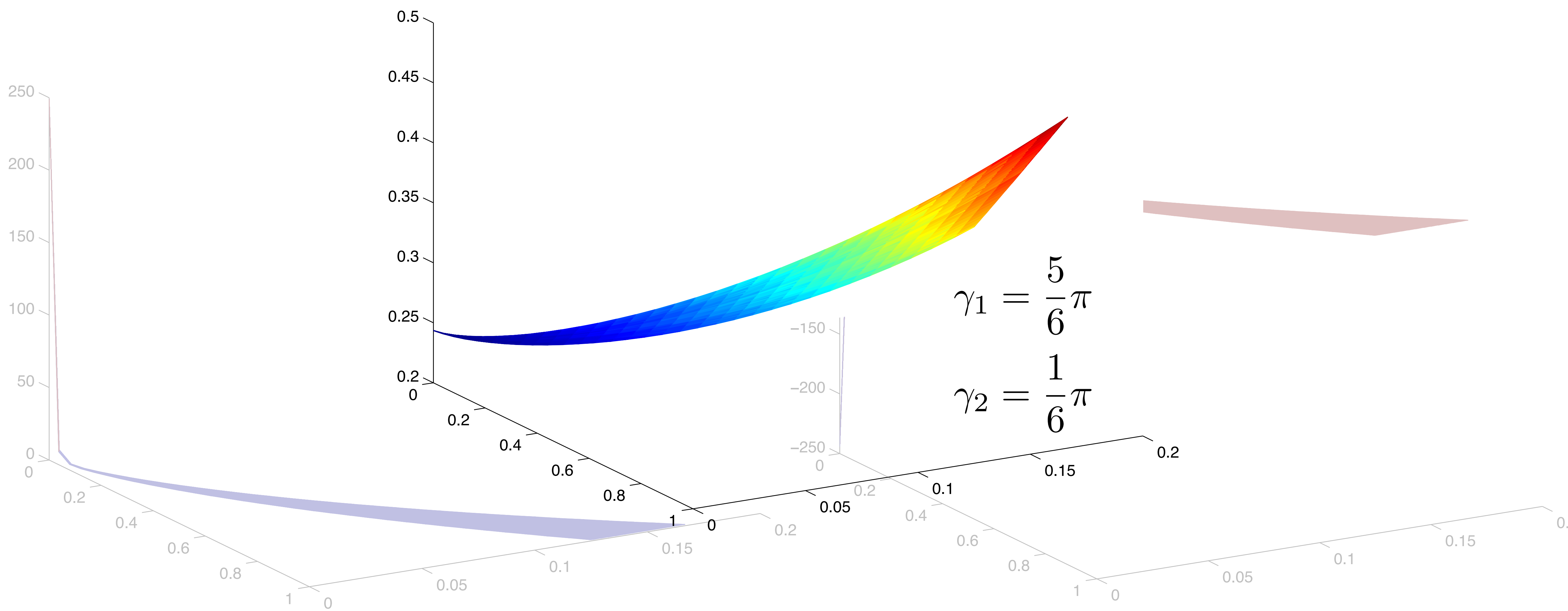

$$\gamma_1 = \frac{5}{6}\pi$$

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Open Problem 1:
Is the capillary surface bounded if $\cos\gamma_1 + \cos\gamma_2 = 0$,
and a boundary has an infinite curvature?

Conjecture 1:
The capillary surface is bounded if $\cos\gamma_1 + \cos\gamma_2 = 0$.

Open Problem 2:

What is the formal asymptotic series for the osculatory cusp?

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The formal asymptotic series solution for the **non-osculatory** cusp domain:

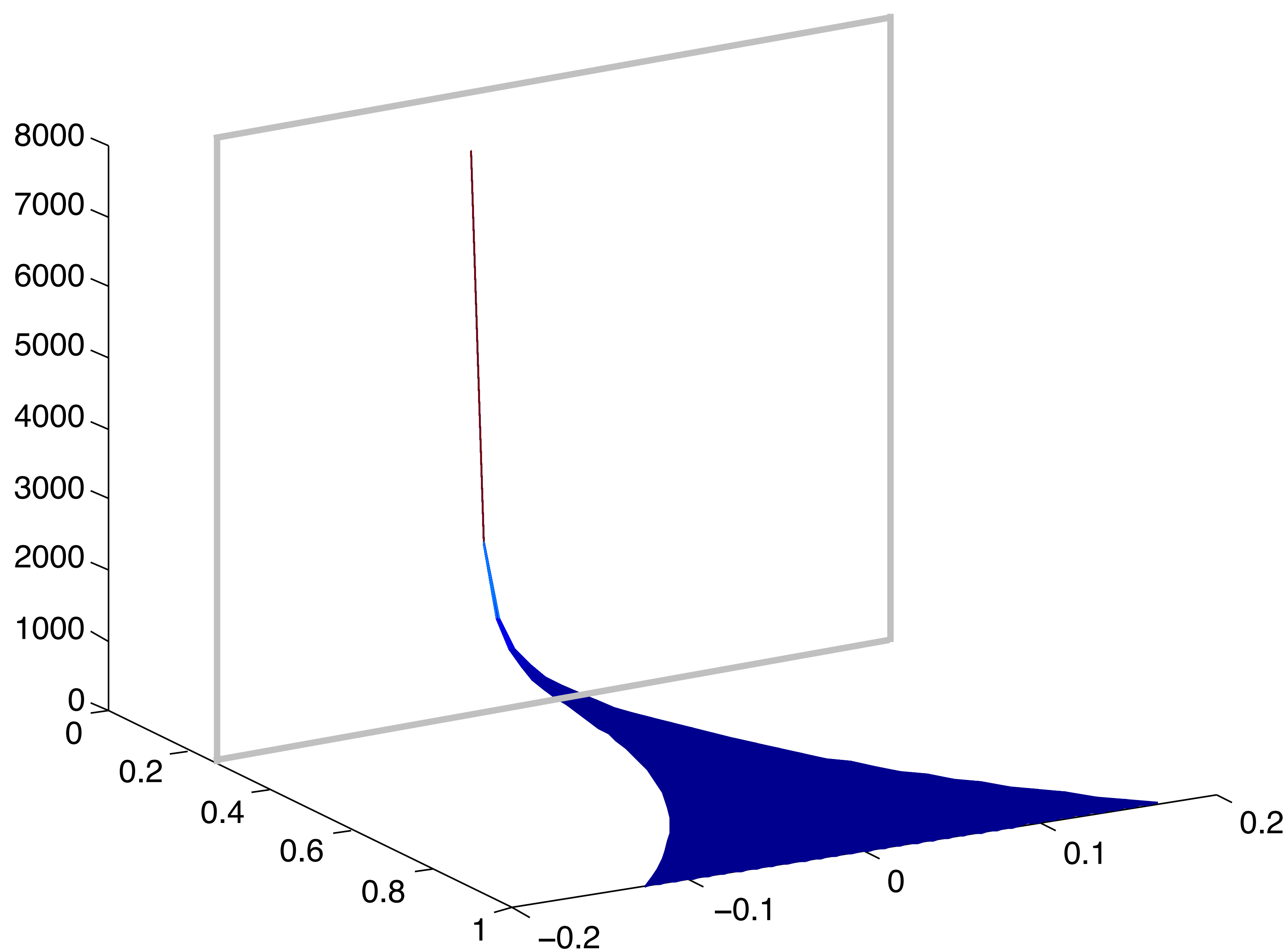
$$v = \frac{\cos\gamma_1 + \cos\gamma_2}{f_1(x) - f_2(x)} + g(x,y)\frac{f_1'(x) - f_2'(x)}{f_1(x) - f_2(x)} + h(x,y)\frac{(f_1'(x) - f_2'(x))^2}{f_1(x) - f_2(x)}$$

$$g(x,y) = -\sqrt{1 - \left(\frac{\cos\gamma_1(t+1) + \cos\gamma_2(t-1)}{2}\right)^2}$$

$$h(x,y) = -\frac{\cos\gamma_1 + \cos\gamma_2}{4}\left(\delta t + \frac{t^2}{2}\right) + \frac{1-\alpha}{2(\cos\gamma_1 + \cos\gamma_2}g(x,y)^2$$

$$t = \frac{2y - (f_1(x) + f_2(x))}{f_1(x) - f_2(x)}$$

Capillary Surface in a Cusp Domain



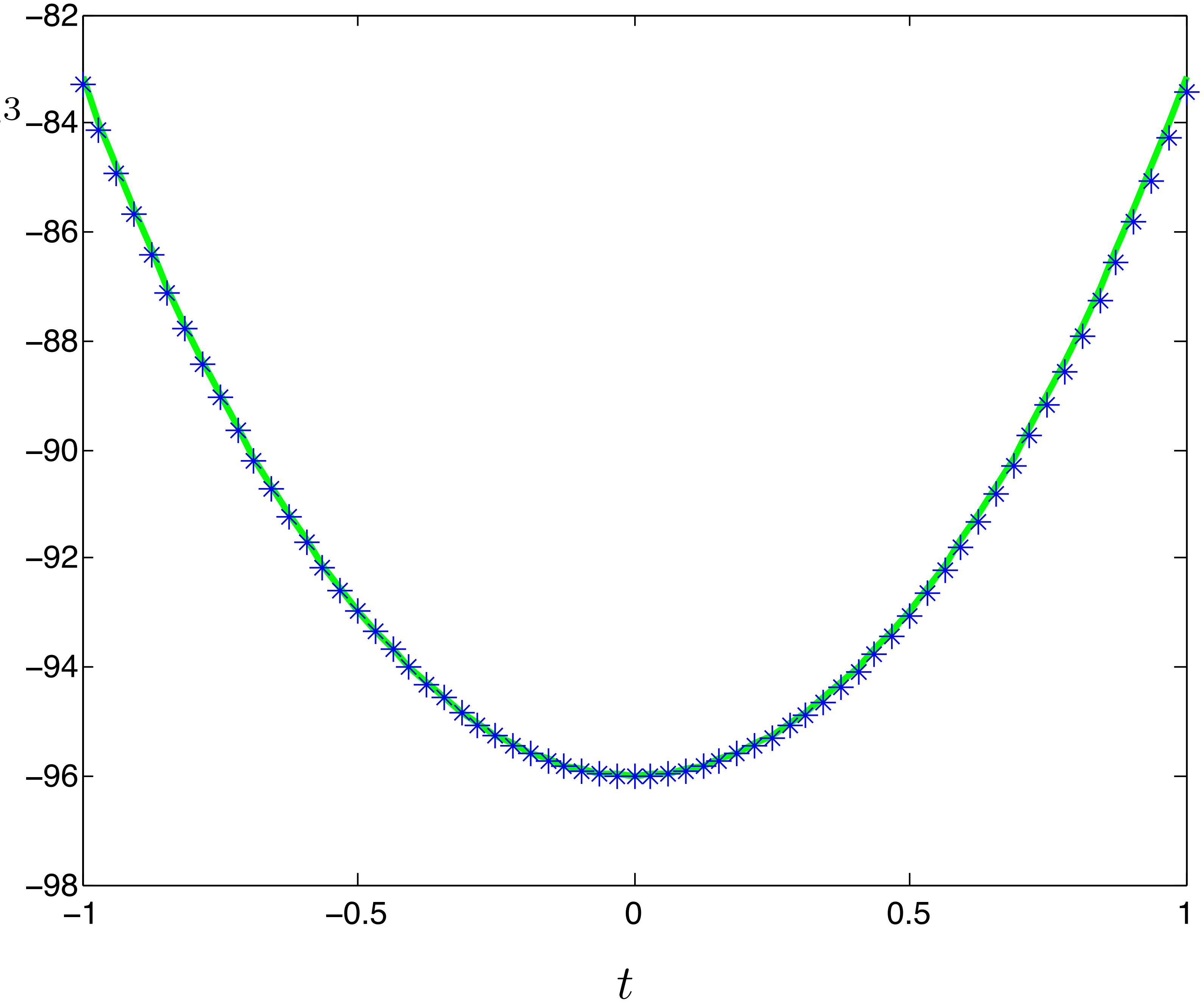
non-osculatory case

$\ast$ Numerical Solution of the second order term

— $-\sqrt{1-\left(\dfrac{\cos\gamma_1(t+1)+\cos\gamma_2(t-1)}{2}\right)^2}\,\dfrac{f_1'(x)-f_2'(x)}{f_1(x)-f_2(x)}$

$f_2(x) = -\dfrac{1}{8}x^3$

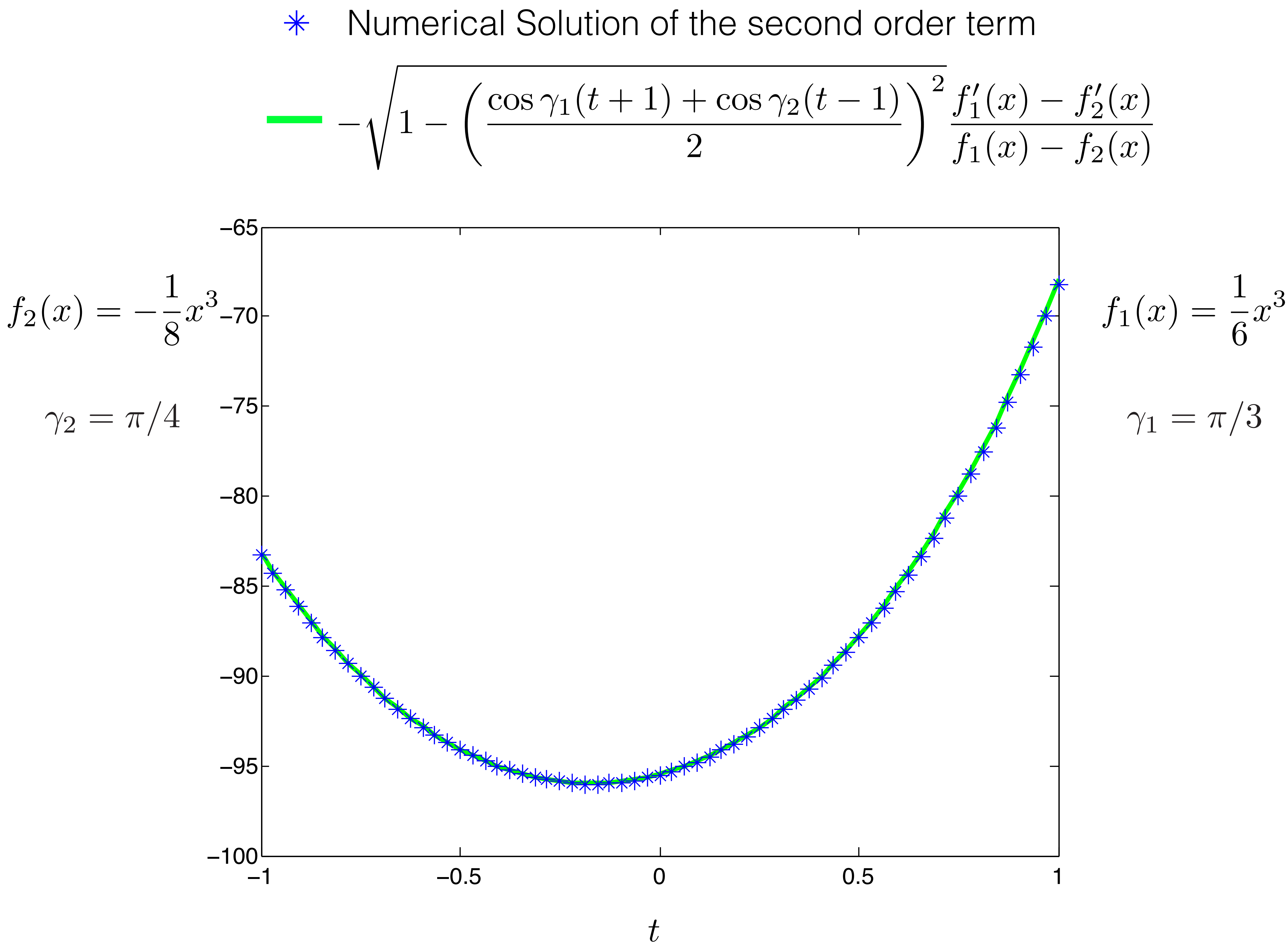
$\gamma_2 = \pi/3$



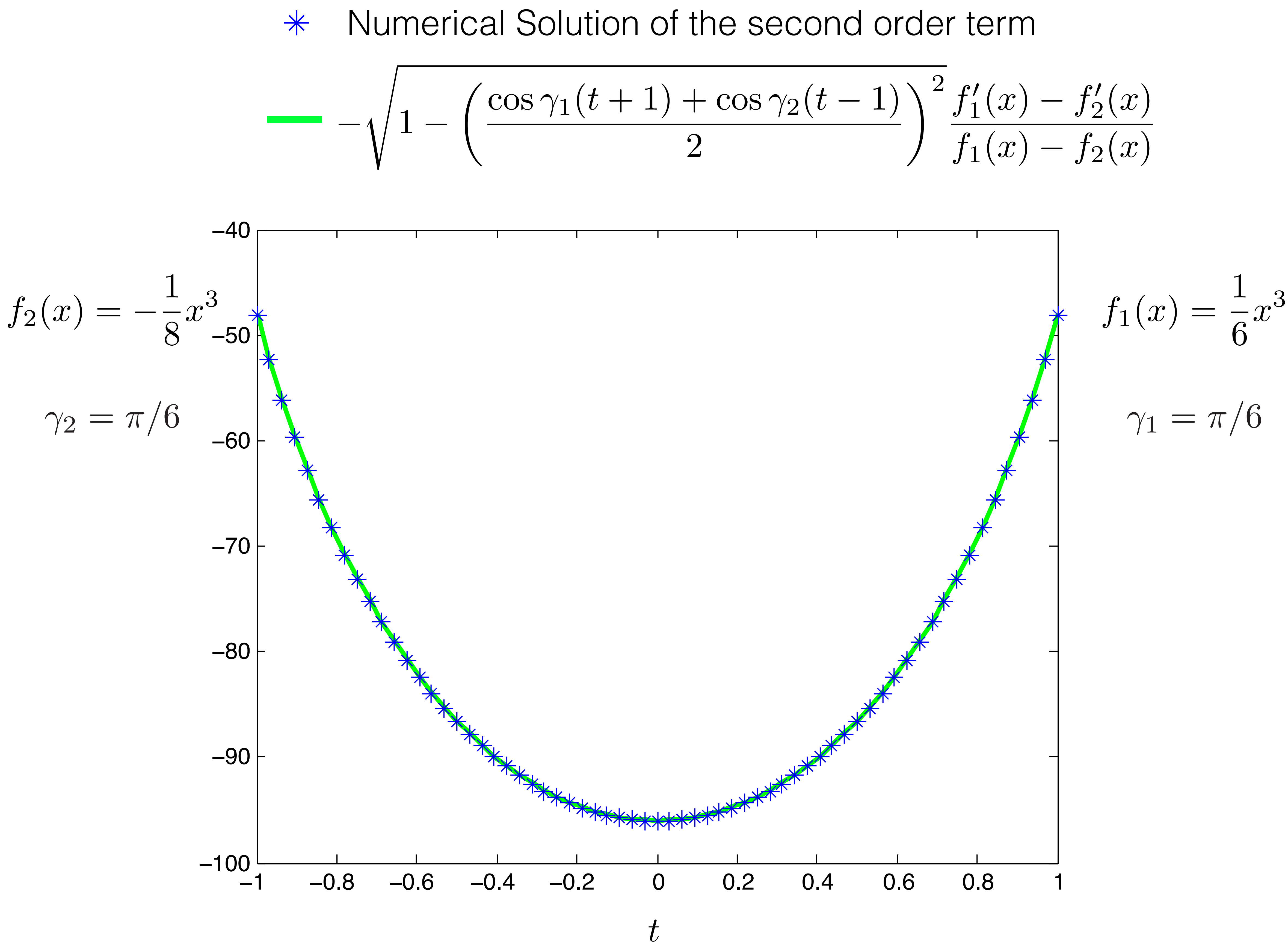
$f_1(x) = \dfrac{1}{6}x^3$

$\gamma_1 = \pi/3$

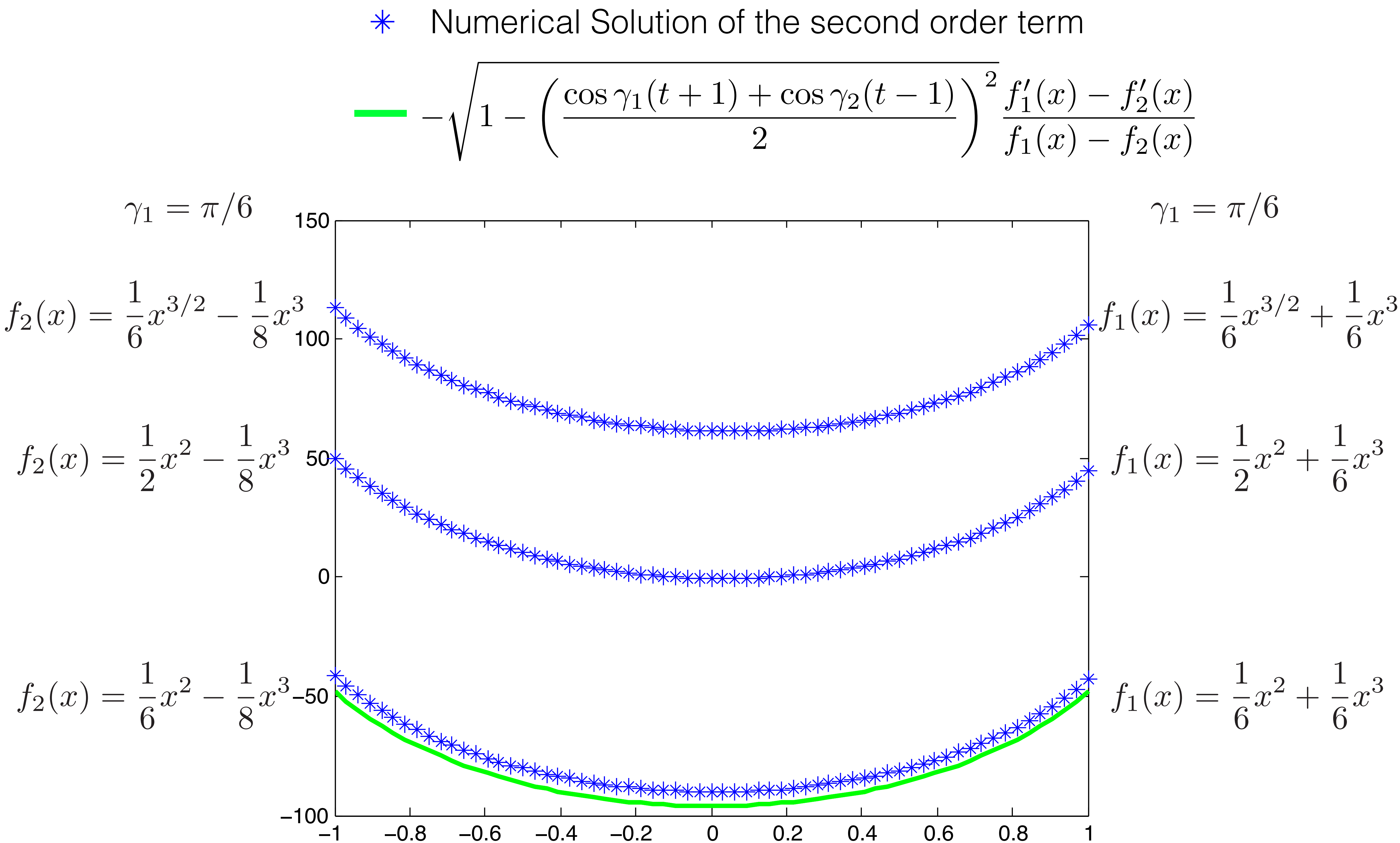
non-osculatory case



non-osculatory case



osculatory case



Open Problem 2:

What is the formal asymptotic series for the osculatory cusp?

The formal asymptotic series solution for the **osculatory cusp** domain:

$$v = \frac{\cos\gamma_1 + \cos\gamma_2}{f_1(x) - f_2(x)} + g(x,y)\frac{f_1'(x) - f_2'(x)}{f_1(x) - f_2(x)} + h(x,y)\frac{(f_1'(x) - f_2'(x))^2}{f_1(x) - f_2(x)}$$

$$g(x,y) = -\sqrt{1 - \left(\frac{\cos\gamma_1(t+1) + \cos\gamma_2(t-1)}{2}\right)^2} + C$$

$$h(x,y) = ?$$

$$t = \frac{2y - (f_1(x) + f_2(x))}{f_1(x) - f_2(x)}$$

Open Problem 2:

What is the formal asymptotic series for the osculatory cusp?

Conjecture 2:

The second order term of the formal asymptotic series has an additional constant.

$$g(x,y) = -\sqrt{1 - \left(\frac{\cos\gamma_1(t+1) + \cos\gamma_2(t-1)}{2}\right)^2} \boxed{+C}$$

Unbounded Capillary Surface

$$\cos\gamma_1 + \cos\gamma_2 \neq 0$$

Power Series Cusp

(Scholz 2004)

Unbounded Capillary Surface

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Open Problems

What if $\cos\gamma_1 + \cos\gamma_2 = 0$?

Non-power Series Cusp?

Unbounded Capillary Surface

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Power Series Cusp

(Scholz 2004)

Non-osculatory Cusp

(Aoki and Siegel 2012)

Bounded Capillary Surface

$\cos\gamma_1 + \cos\gamma_2 = 0$

Finite Curvature Cusp

(Aoki and Siegel 2012)

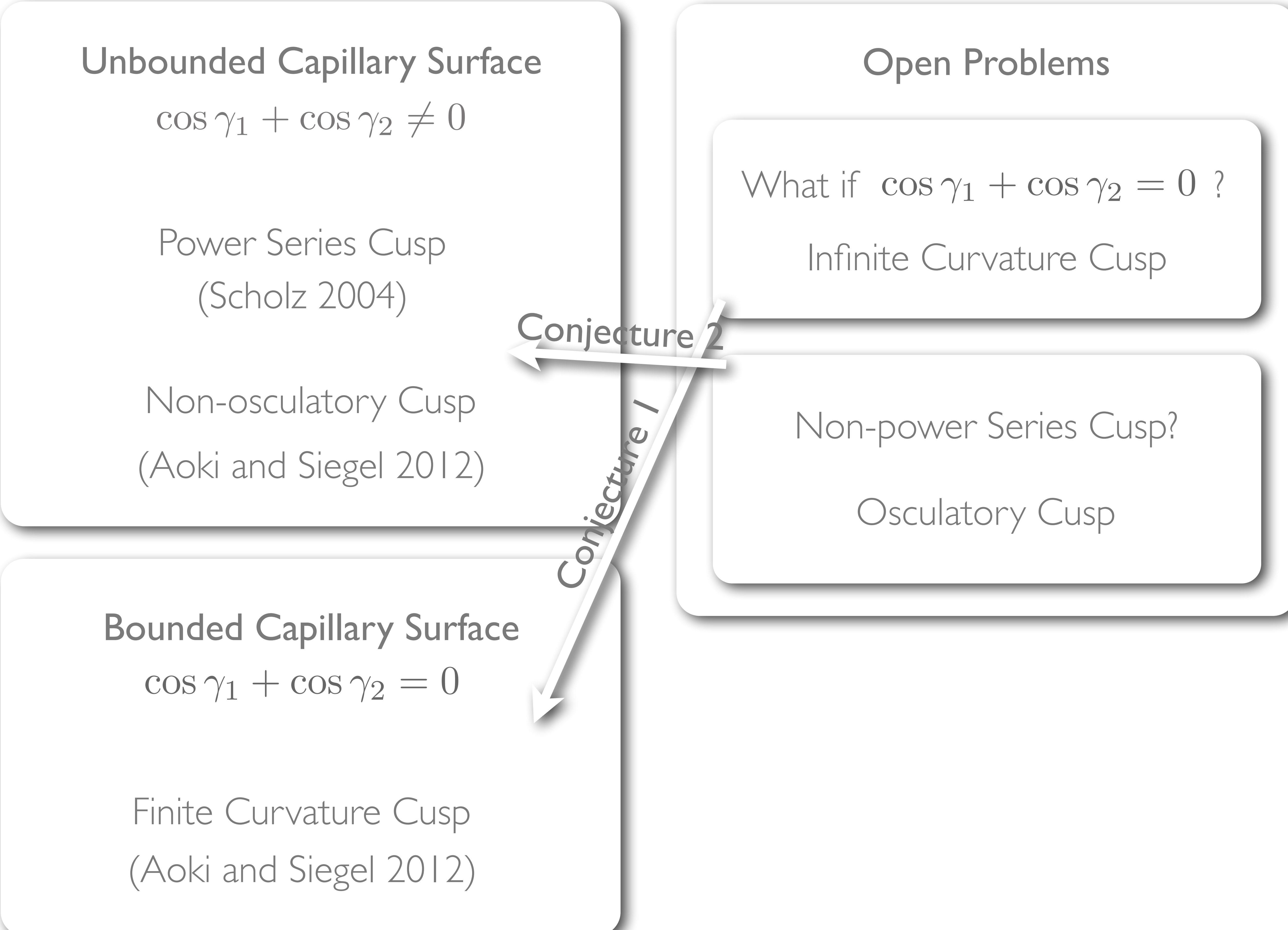
Open Problems

What if $\cos\gamma_1 + \cos\gamma_2 = 0$?

Infinite Curvature Cusp

Non-power Series Cusp?

Osculatory Cusp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Osculatory Cusp
Conjecture 2
Conjecture 1
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Open Problems

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If we can prove this…

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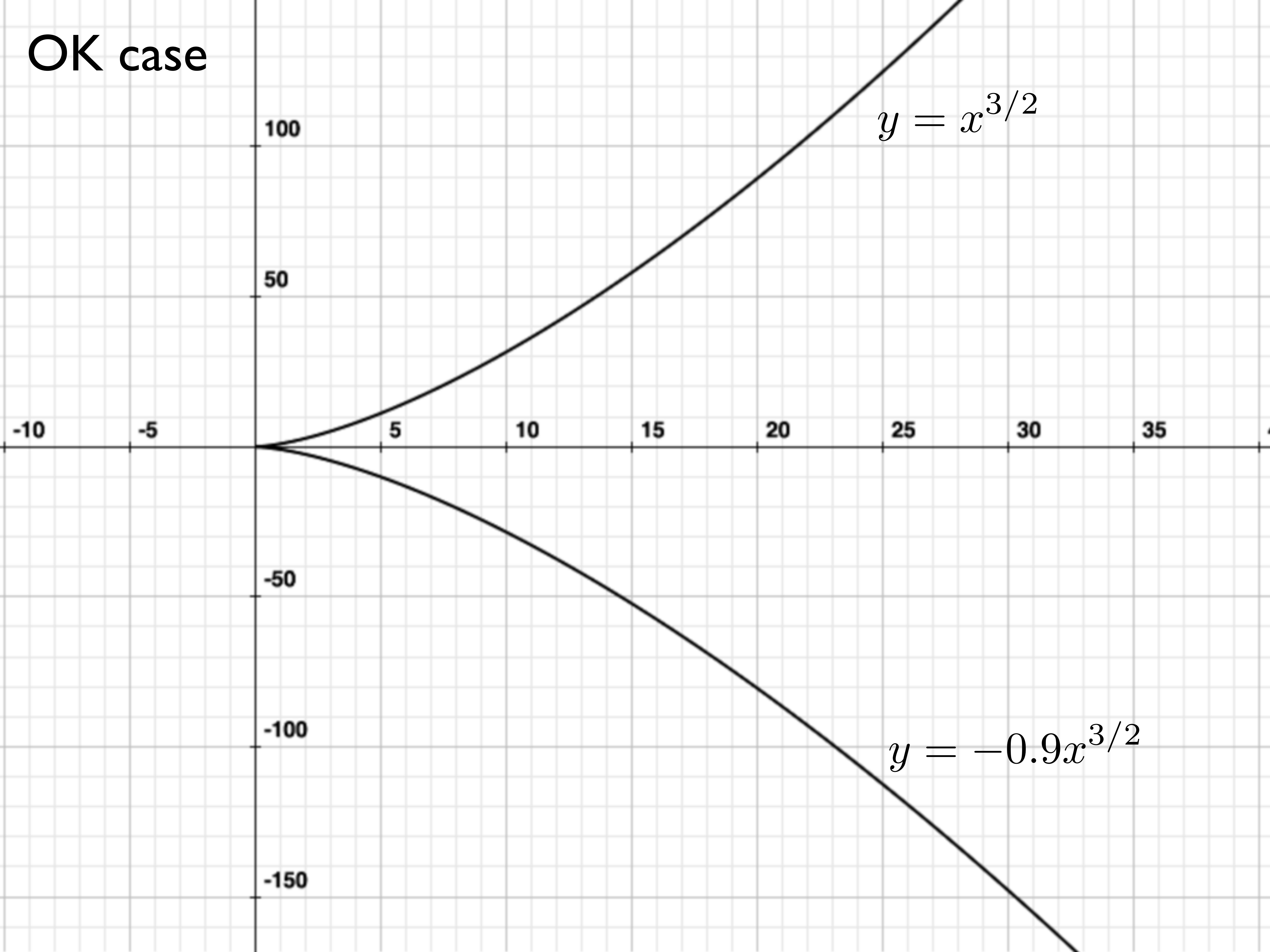
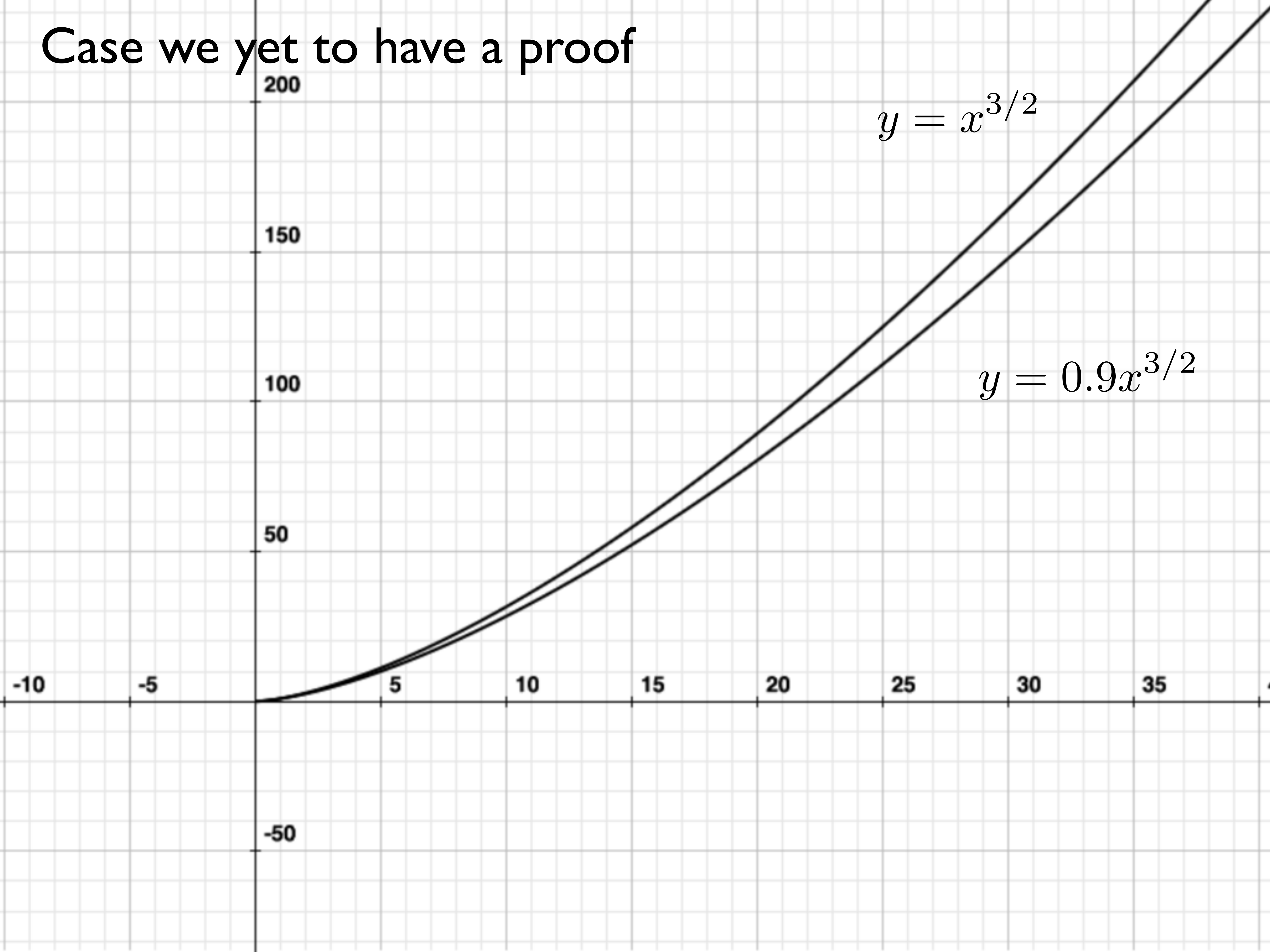
If we can prove this…

The capillary surface at a cusp is bounded **if and only if** $\cos\gamma_1 + \cos\gamma_2 = 0$

OK case



Case we yet to have a proof



We want to construct a super-surface satisfies the following conditions:

$$\nabla \cdot \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} - v < 0 \qquad ①$$

$$\vec{\nu}_1 \cdot \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} \geq \cos\gamma_1 \qquad ②$$

$$\vec{\nu}_2 \cdot \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} \geq \cos\gamma_2 \qquad ③$$

We want to construct a super-surface satisfies the following conditions:

finite mean curvature surface ①

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We want to construct a super-surface satisfies the following conditions:

finite mean curvature surface ①

$$\vec{\nu}_1 \cdot \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} = \cos\gamma_1 \quad \text{on } y = f_1(x) \quad ②$$

We consider a parametric surface:

$$s\hat{\mathbf{i}} + (f(s)t)\hat{\mathbf{j}} + g(s)h(t)\hat{\mathbf{k}}$$

$$g(s) \quad h(t) \text{ bounded}$$

$$\text{but } \lim_{s \to 0} f''(s) = \pm\infty$$

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Mean Curvature (need to be finite)

$$\nabla \cdot Tu = -\frac{EN + GL - 2FM}{EG - F^2}$$

$$E = x_s^2 + y_s^2 + z_s^2$$
$$F = x_s x_t + y_s y_t + z_s z_t$$
$$G = x_t^2 + y_t^2 + z_t^2$$

$$L = \frac{\begin{vmatrix} x_{ss} & y_{ss} & z_{ss} \\ x_s & y_s & z_s \\ x_t & y_t & z_t \end{vmatrix}}{\sqrt{EG - F^2}}, \qquad M = \frac{\begin{vmatrix} x_{st} & y_{st} & z_{st} \\ x_s & y_s & z_s \\ x_t & y_t & z_t \end{vmatrix}}{\sqrt{EG - F^2}}, \qquad N = \frac{\begin{vmatrix} x_{tt} & y_{tt} & z_{tt} \\ x_s & y_s & z_s \\ x_t & y_t & z_t \end{vmatrix}}{\sqrt{EG - F^2}}.$$

We consider a parametric surface:

$$s\hat{\mathbf{i}} + (f(s)t)\hat{\mathbf{j}} + g(s)h(t)\hat{\mathbf{k}}$$

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$$\nabla \cdot Tu = -\frac{EN + GL - 2FM}{EG - F^2}$$

Upwards normal vector of the surface

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & f'(s) & g'(s)h(t) \\ 0 & 1 & g(s)h'(t) \end{vmatrix}$$

We consider a parametric surface:

$$s\hat{\mathbf{i}} + (f(s)t)\hat{\mathbf{j}} + g(s)h(t)\hat{\mathbf{k}}$$
$$g(s) \quad h(t) \text{ bounded}$$
$$\text{but } \lim_{s\to 0} f''(s) = \pm\infty$$

Mean Curvature (need to be finite)

$$\nabla \cdot Tu = -\frac{EN + GL - 2FM}{EG - F^2}$$

Upwards unit normal vector of the surface

$$\frac{(f'(x)g(s) - g'(s)h(t))\hat{\mathbf{i}} - g(s)h'(t)\hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{(f'(x)g(s) - g'(s)h(t))^2 + (g(s)h'(t))^2 + 1}}$$

We consider a parametric surface:

$$s\hat{\mathbf{i}} + (f(s)t)\hat{\mathbf{j}} + g(s)h(t)\hat{\mathbf{k}}$$
$$g(s) \quad h(t) \text{ bounded}$$
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Mean Curvature (need to be finite)

$$\nabla \cdot Tu = -\frac{EN + GL - 2FM}{EG - F^2}$$

Contact Angle Condition

$$\cos\gamma = \frac{(f'(x)g(s) - g'(s)h(t))f'(s) + g(s)h'(1)}{\sqrt{1+f'(s)^2}\sqrt{(f'(x)g(s) - g'(s)h(1))^2 + (g(s)h'(1))^2 + 1}}$$

We consider a parametric surface:

$$s\hat{\mathbf{i}} + (f(s)t)\hat{\mathbf{j}} + g(s)h(t)\hat{\mathbf{k}}$$
$$g(s) \quad h(t) \text{ bounded}$$
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Mean Curvature (need to be finite)

$$\nabla \cdot Tu = -\frac{EN + GL - 2FM}{EG - F^2}$$

Contact Angle Condition

$$g'(s) = \frac{-bg(s) \pm \sqrt{b^2 g(s)^2 - 4a(cg(s)^2 + d)}}{2a}$$

$$a = \{((1 - \cos^2\gamma)f'(s)^2 - \cos^2\gamma)h(t)^2\}$$
$$b = \{2(1 - \cos^2\gamma)f'(s)^3 h(t) - 2h'(t)h(t)f'(s) - 2\cos^2\gamma f'(x)h(t)\}$$
$$c = \{(1 - \cos^2\gamma)(h'(t)^2 + f'(x)^4) + 2f'(x)^2 h'(t) - \cos^2\gamma(1 + h'(t)^2)f'(x)^2\}$$
$$d = -\cos^2\gamma(1 + f'(s)^2)$$

Open Problem 2:

What is the formal asymptotic series for the osculatory cusp?

The formal asymptotic series solution for the **non-osculatory** cusp domain:

$$v = \frac{\cos\gamma_1 + \cos\gamma_2}{f_1(x) - f_2(x)} + g(x,y)\frac{f_1'(x) - f_2'(x)}{f_1(x) - f_2(x)} + h(x,y)\frac{(f_1'(x) - f_2'(x))^2}{f_1(x) - f_2(x)}$$

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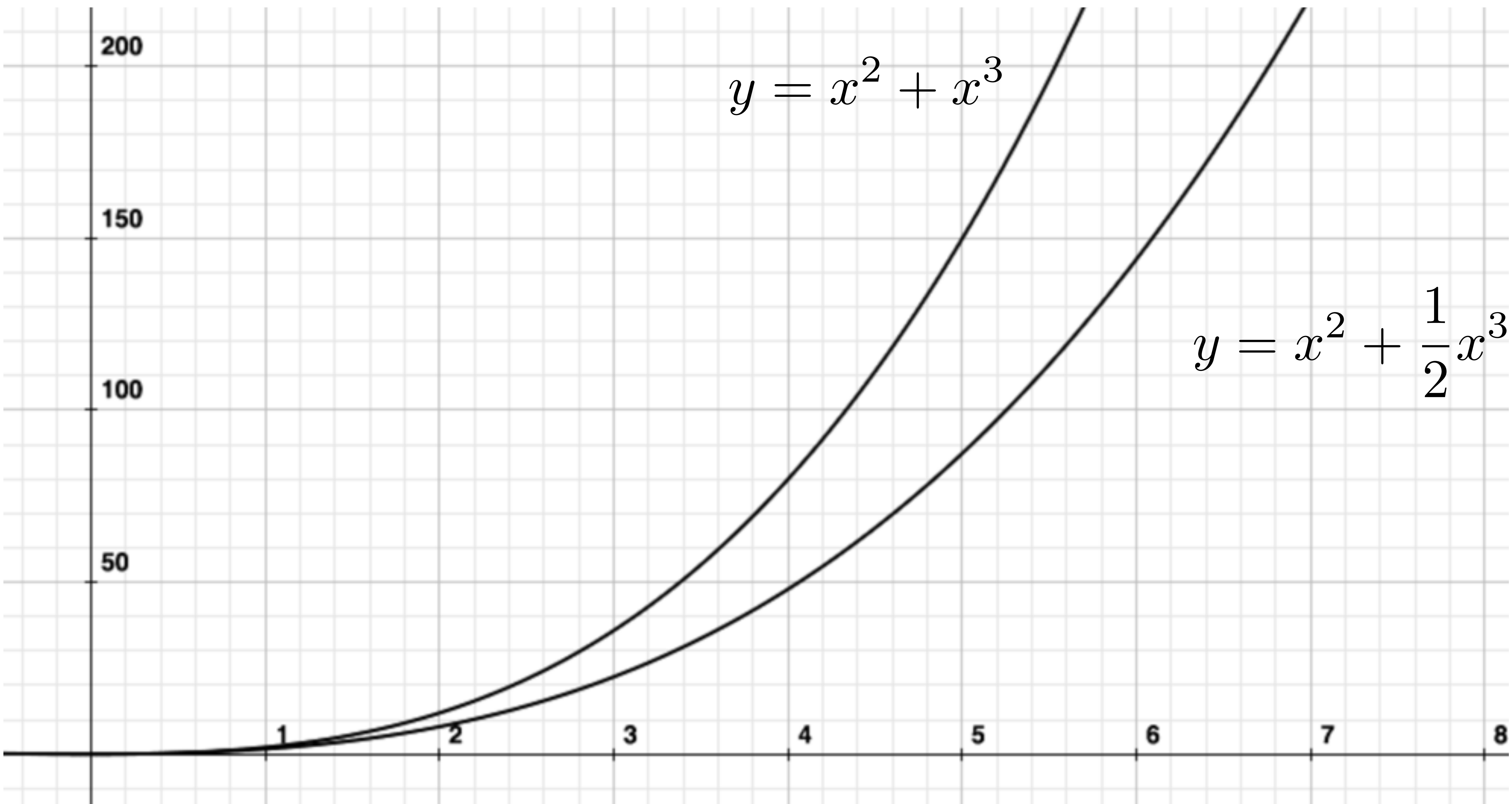
Open Problem 2:

What is the formal asymptotic series for the **osculatory cusp**?



Open Problem 3:

Consider non-constant contact angles.

What kind of asymptotic condition do we need for $\gamma_{1,2}(x)$ so that the solution is bounded when $\lim_{x\to 0} \cos\gamma_1 + \cos\gamma_2 = 0$?

Research funded and supported by

Natural Science and Engineering Research Council of Canada

Department of Applied Mathematics, University of Waterloo

The Royal Swedish Academy of Sciences

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