OPEN PROBLEMS

GEOMETRIC ASPECTS ON CAPILLARY PROBLEMS AND RELATED TOPICS
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A capillary surface, a liquid surface at equilibrium, can be modelled mathematically by a nonlinear elliptic PDE called Laplace-Young equation:

$$\nabla \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \kappa u$$

where u is the height of the capillary surface and κ is a physical constant. Consider a domain and the boundary conditions as depicted in Figure 1. The known solution characteristics can be summarised as follows (as appear in [1,2]):

Theorem 1: u is unbounded if $\cos\gamma_1 + \cos\gamma_2 \neq 0$.
Theorem 2: u is bounded if $\cos\gamma_1 + \cos\gamma_2 = 0$ and $f_{1,2}$ have finite curvatures.
Theorem 3: $u = \frac{\cos\gamma_1+\cos\gamma_2}{f_1(x)-f_2(x)} + O\left(\frac{f_1'(x)-f_2'(x)}{f_1(x)-f_2(x)}\right)$ in a non-osculatory cusp domain if $\cos\gamma_1 + \cos\gamma_2 \neq 0$.
I propose the following open problems, conjectures for Open Problems 1 and 2 can be found in [3].
Open Problem 1: Is u bounded or unbounded if $\cos\gamma_1 + \cos\gamma_2 = 0$ and $f_{1,2}$ have infinite curvature?
Open Problem 2: Does u follow the asymptotic relationship: $u = \frac{\cos\gamma_1+\cos\gamma_2}{f_1(x)-f_2(x)} + O\left(\frac{f_1'(x)-f_2'(x)}{f_1(x)-f_2(x)}\right)$ if $\cos\gamma_1 + \cos\gamma_2 \neq 0$ and the domain is an osculatory cusp?
Open Problem 3: If the contact angles $\gamma_{1,2}$ are not constants and $\lim_{x\to 0}\cos\gamma_1(x)+$

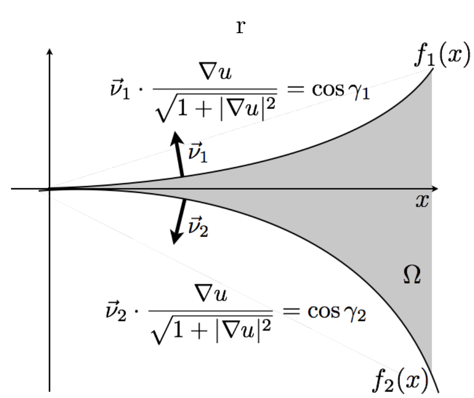

FIGURE 1. Domain and Boundary conditions, where $\gamma_{1,2}$ are physical constants.

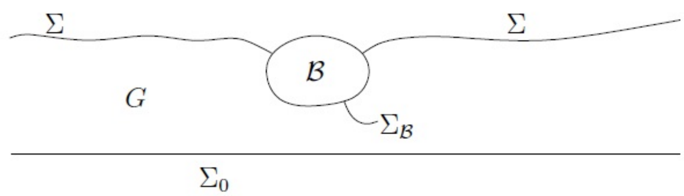

FIGURE 2. The floating problem

$\cos\gamma_2(x) = 0$. What is the asymptotic conditions required for $\gamma_{1,2}(x)$ for the solution u to be bounded? (based on a discussion with Kirk Lancaster and others in June 2015)

References
[1] Scholz M., "Über das Verhalten von Kapillarflächen in Spitzen", Ph.D. thesis, Universität Leipzig, 2001, available at http://www.people.imise.uni-leipzig.de/markus.scholz/pdf/p1.pdf.
[2] Aoki Y., Siegel D., "Bounded and Unbounded Capillary Surfaces in a cusp domain" *Pacific Journal of Mathematics*, **257**, Issue 1, 143–165 (2012).
[3] Aoki Y., De Sterck H., "Numerical Study of Unbounded Capillary Surfaces" *Pacific Journal of Mathematics*, **267**, Issue 1, 1–34 (2014).

(2) By Josef Bemelmans (RWTH Aachen University)
Capillary Surfaces and Floating Bodies - Problems.
A rigid body $\mathcal{B}$ is floating on a layer of a fluid; the upper surface Σ of the fluid domain (which is an unknown of the problem) is governed by surface tension. We show existence of a solution to the equations that describe this configuration. The surface Σ meets the floating body in the contact line Γ. Question 1: Let γ be the projection of Γ onto $\mathbb{R}^2$; is γ then the boundary of a convex region, if the floating body $\mathcal{B}$ is strictly convex and if the contact angle is 90^o?
Question 2: Is the solution stable with respect to small variations of thedata, e.g. of the density of the floating body?

(3) By Paolo Caldiroli (Dipartimento di Matematica, Università di Torino)

Spherical surfaces with prescribed mean curvature

Given a smooth mapping $H\colon \mathbb{R}^3 \to \mathbb{R}$, we call H-*sphere* a closed surface S homeomorphic to $\mathbb{S}^2$, whose mean curvature at every point $p \in S$ equals $H(p)$.

Let $H\colon \mathbb{R}^3 \to (0,\infty)$ be a smooth, radially symmetric function with $\lim_{|p|\to\infty} H(p) = H_\infty \in (0,\infty)$. Hence there exists $R > 0$ such that $H(p) = R^{-1}$ if $|p| = R$, and the round sphere S_R of radius R centered at the origin is an H-sphere.

Question: assuming that H is non increasing and non constant, is the round sphere S_R the unique H-sphere?

Remarks: The answer is positive in low dimension, i.e., considering closed curves in the plane rather than spheres in $\mathbb{R}^3$ (see: Musina, R.:

Planar loops with prescribed curvature: existence, multiplicity and uniqueness, Proc. Amer. Math. Soc. **139** (2011) 4445–4459). In low dimension one can also provide an example of a positive, radially symmetric and increasing mapping k for which a suitable ellipse has curvature k at every point. Hence the kind of monotonicity seems to have a role.

Closed immersed surfaces with prescribed mean curvature

Let $H\colon \mathbb{R}^3 \to \mathbb{R}$ be a Lipschitz continuous function which is strictly monotone in some direction, i.e., there is a versor $\vec{e} \in \mathbb{R}^3$ such that $\nabla H(p) \cdot \vec{e} > 0$ for almost every $p \in \mathbb{R}^3$. It is known that there exists no C^1-closed embedded surface in $\mathbb{R}^3$ of mean curvature H. For a proof, just use the formula for the first variation which gives

$$\int_S \langle H\vec{e}, \vec{N}\rangle \, d\Sigma = 0$$

where $\vec{N}$ is the Gauss map for the surface S and $d\Sigma$ denotes its surface element. Since S is embedded, there exists an open domain Ω in $\mathbb{R}^3$ such that $S = \partial\Omega$. Then use the Stokes formula

$$\int_S \langle H\vec{e}, \vec{N}\rangle \, d\Sigma = \int_\Omega \nabla H(p) \cdot \vec{e} \, dp$$

and conclude.

Question: does the same nonexistence result hold true also for closed immersed surfaces?

(4) By Jaigyoung Choe (Korea Institute for Advanced Study)

1. (well known problem) If Σ is a minimal annulus in a ball $B \subset \mathbb{R}^3$ and orthogonal to ∂B^3, is Σ the catenoidal waist?

2. Let $\Sigma \subset \mathbb{R}^3$ be a capillary surface in a solid cylinder C. Show that if the boundaries of Σ are not null homotopic in the surface cylinder ∂C then Σ is part of the Delaunay surface. Do we need to assume that Σ is stable?

3. Let $\Sigma \subset \mathbb{R}^3$ be a capillary surface **outside** C whose boundaries are not null homotopic on ∂C. Prove that Σ is also part of the Delaunay surface with (or without) the stability assumption.

4. Let Σ be a CMC surface in a half space $\mathbb{H} \subset \mathbb{R}^3$ making a constant contact angle $\neq \pi/2$ with the plane $\partial\mathbb{H}$. Can one extend Σ across $\partial\mathbb{H}$ analytically? In case Σ is minimal, an affirmative answer is obtained in [C]. The Schwarz reflection principle and the Weierstrass representation formula are used in the proof of [C].

[C] J. Choe, On the analytic reflection of a minimal surface, Pacific J. Math. 157 (1993), 29-35.

5. Let Σ_1, Σ_2 be compact minimal surfaces in $\mathbb{S}^3$. Suppose they intersect at a constant angle $\neq \pi/2$. Is it true that they are both great spheres or both Clifford tori?

(5) By Ailana Fraser (Columbia University)

Question 1. Is the critical catenoid the only embedded free boundary minimal annulus in B^3?

This is a free boundary analog of Lawson's conjecture that the Clifford torus is the only compact embedded minimal surface in S^3 of genus 1 ([L], [B]).

Question 2. Let Σ be an embedded free boundary minimal surface in B^3. Is it true that the first Steklov eigenvalue $\sigma_1(\Sigma) = 1$?

This is a free boundary analog of Yau's conjecture [Y] that for any embedded minimal surface Σ in S^3, $\lambda_1(\Sigma) = 2$. If Question 2 is true, then [2] Theorem 6.6 implies Question 1 is true; cf. [MR].

References

[B] S. Brendle, Embedded minimal tori in S^3 and the Lawson conjecture, *Acta Math.* **211** (2013), no. 2, 171–190.

[FS] A. Fraser, R. Schoen, Sharp eigenvalue bounds and minimal surfaces in the ball, to appear in *Invent. Math.*

[L] H.B. Lawson, The unknottedness of minimal embeddings, *Invent. Math.* **11** (1970), 183-187.

[MR] S. Montiel, A. Ros, Minimal immersions of surfaces by the first eigenfunctions and conformal area, *Invent. Math.* **83** (1986), no. 1, 153–166.

[Y] S. T. Yau, *Seminar on differential geometry*, Annals of Math. Studies, No. 102, Princeton Univ. Press, Princeton, N.J., 1982.

(6) By Brien Freidin and Peter McGrath (Brown University).
   (a) Suppose $\Sigma^k \subset B^n(r) \subset \mathbb{S}^n_+(1)$ is a free boundary minimal surface, where $B^n(r)$ is a geodesic ball and $\S^n_+$ is the upper hemisphere with the round metric. Is it true that $|\Sigma^k| \geq |B^k(r)|$?
   (b) Suppose $\Sigma \subset \mathbb{R}^3$ is a CMC topological disk and $\partial\Sigma$ is a round circle in a plane in $\mathbb{R}^3$. Is Σ necessarily a spherical cap?

(7) By David Hartley (ICMAT).
   (a) Unduloids with Maximum Volume. When considering a family of Delaunay hypersurfaces with a fixed period in $\mathbb{R}^{n+1}$, the enclosed volume exhibits an intriguing feature in certain dimensions. For hypersurfaces of dimension $n \leq 8$ the maximum enclosed volume (over a one period length) occurs at spheres, while for dimension $n \geq 11$ it occurs at the cylinders. Interestingly in dimensions $n = 9, 10$ the enclosed volume is maximised by an unduloid (in fact for $n = 8$ there is also an unduloid that obtains a local maximum enclosed volume). It seems strange that such an optimisation would occur at an unduloid instead of a sphere or cylinder, an analysis and explanation of this attribute would be worthwhile.
   (b) Topology of Level Sets of Minimal Graphs. Suppose that $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ satisfies the minimal surface equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0,$$

   what forms can the nodal set $\{x \in \Omega : u(x) = 0\}$ take? Recently it was proved by A. Enciso and D. Peralta that given any compact hypersurface $\Omega \subset \mathbb{R}^n$ there is a solution to the Allen-Cahn equation in $\mathbb{R}^n$ such that the nodal set contains Ω as a component. It would be interesting to see if this is also true for the minimal surface equation, a difficulty occurs in using their methods due to the minimal surface equation being quasi-linear while the Allen-Cahn equation is semi-linear.

(8) By Miyuki Koiso, Kyushu University

The following result was proved by J. Choe and M. Koiso (to appear in Pacific Journal of Mathematics).

Let Σ be a compact immersed stable capillary hypersurface in a wedge bounded by two hyperplanes in $\mathbb{R}^{n+1}$, $n \geq 2$. Suppose that Σ meets those two hyperplanes in constant contact angles and does not hit the edge of the wedge. We also assume that each contact angle is greater than or equal to $\pi/2$, and that $\partial\Sigma$ consists of two smooth embedded $(n-1)$-dimensional manifolds, one in each hyperplane of the wedge, and that each component of $\partial\Sigma$ is convex when $n \geq 3$. Then Σ is part of the sphere. Also, the same conclusion holds if Σ is in the half-space of $\mathbb{R}^{n+1}$ and $\partial\Sigma$ is connected.

Question 1. Weaken the assumption that "each support hyperplane includes at most one boundary component of the capillary hypersurface".

Question 2. Weaken the assumption that "each boundary component of the capillary hypersurface is convex" for $n \geq 3$.

(9) By Bennett Palmer (Idaho State University)

1. Consider a liquid drop contained in the unit ball $B \subset \mathbf{R}^3$ and having free boundary on the sphere S^2. We consider an energy functional E made up of three terms: the free surface of the drop is assigned its area $A[\Sigma]$, the part Ω of the sphere in contact with the drop is assigned a multiple of its area $\omega A[\Omega]$ and the boundary curve $C = \partial\Sigma$ is assigned a multiple of its length $\beta L[C]$. The total energy is thus

$$E = A[\Sigma] + \omega A[\Omega] + \beta L[C]\,.$$

The critical points of this functional, constrained by the requirements that the volume is preserved and that $\partial\Sigma$ always lies on S^2, are characterized by the conditions that the mean curvature H of Σ is constant in the interior of Σ and along C, $X \cdot N = X \cdot N = -\beta\bar{k}_g + \omega$ holds. Here X is the position vector, N is the unit normal to Σ which points out of the drop, and $\bar{k}_g$ is the geodesic curvature of C in S^2.

Must every critical point of disc type must be either a flat disc or spherical cap? With the additional hypothesis that the drop is stable, the answer is in the affirmative.

In the case that $\beta = 0$, the answer is also in the affirmative as was shown by Nitsche in the case $\omega = 0$ and by Ros-Souam in the case of general ω.

2. The surface of a rotating liquid drop satisfies an equation

$$H = -\frac{\alpha r^2}{4} + H_0 \,, \tag{1}$$

where α and H_0 are constants and r is the distance from the rotation axis. The problem is to determine which axially symmetric disc type critical points are stable for the prescribed boundary (Dirichlet) problem.

The surfaces can be explicitly determined. Noether's Theorem implies that each drop has a first integral

$$2r\cos\theta + 2H_0 r^2 - \frac{\alpha r^4}{4} \equiv \text{constant} =: A \,. \tag{2}$$

Here $\cos^2\theta =: \sqrt{1-N_3^2}$ where $N = (N_1, N_2, N_3)$ is the surface normal. If (2) is solved for $r = r(\theta)$, then the vertical coordinate z of the surface can be found by

$$z(\theta) = -\int_{\theta_0} r_\theta \cot(\theta)\, d\theta$$

and the surface is given by $X = (r(\theta)e^{i\theta}, z(\theta))$. With the assumption that the surface is of disc type, we can take $r = 0$ in (2) and get that $A = 0$ holds so (2) becomes a cubic which can be explicitly solved.

The Jacobi operator for this problem is

$$L[\psi] = \Delta\psi + (|dN|^2 + \alpha(X_1N_1 + X_2N_2))\psi \,.$$

Write the position vector on S^2 as $N = (ue^{i\theta}, v)$ and define the triple branched covering $h : S^2 \to S^2$ by $(ue^{i\theta}, v) \mapsto (ue^{i3\theta}, v)$. In the special case when $H_0 = 0$, the map $h \circ N : \Sigma \to S^2$ is a branched conformal map under which the Jacobi operator L is transformed into a multiple of the operator $\Delta_{S^2} + 2$. Using this, we can establish that a maximal stable domain in Σ is given by the inequality $N_3 \geq -1/3$.

(10) By Julian Scheuer (Freiburg Unviersity).

In [1] it is shown that the smooth inverse mean curvature flow (IMCF) in $\mathbb{R}^{n+1}$,

$$\dot{x} = \frac{1}{H}\nu, \tag{3}$$

with perpendicular free boundary on the unit sphere drives strictly convex hypersurfaces to the embedding of a flat unit disk. The proof makes tremendous use of the convexity. Hence we would be really interested in the

Question 1: Does the same result hold for mean-convex hypersurfaces?

There seem to be canonical solutions to this flow: Let $C \subset \mathbb{R}^{n+1}$ be a rotationally symmetric convex cone pointed at the origin, whose intersection with $\mathbb{S}^n$ is a geodesic hypersphere of $\mathbb{S}^n$. Let $M_0 = C \cap \bar{B}_1(0)$ be the intersection of this cone with the unit ball. This is a hypersurface of $\mathbb{R}^{n+1}$ with singularity at the origin and perpendicular to $\mathbb{S}^n$.

Question 2: Can the weak notion of IMCF be used to show that the flow preserves the conical shape of M_0 and converges to a flat disk? If yes, can this flow be used as barrier to control the original flow and to get better control of the speed of convergence?

References

[1] B. Lambert and J. Scheuer: *The inverse mean curvature flow perpendicular to the sphere*, Math. Ann. (2015), doi:10.1007/s00208-015-1248-2

(11) By Keomkyo Seo (Sookmyung Women's University)

Problem(1)
In [2], Fraser and Schoen proved that if Σ is a free boundary minimal surface in a unit ball $\mathbb{B}^3$, then

$$\text{area}(\Sigma) \geq \text{area}(flat\ disk) = \pi,$$

Equality holds if and only if Σ is a flat disk. (See also [1] for higher dimensional cases.) From this, one can see that the lowest area among free boundary minimal surfaces is achieved by a flat disk. It is natural to ask what is the next possible lower bound. The conjecture is the following: Conjecture: Let Σ be a free boundary minimal surface which is nonflat. Then

$$\text{area}(\Sigma) \geq \text{area}(critical\ catenoid).$$

It seems that this problem might be closely related with Marques-Neves's min-max method [4].

Problem(2)
Nitsche's Theorem about capillary surface says that if Σ is a capillary disk in a ball in space forms, then Σ is totally umbilic. (See [3, 5, 6] and the references therein.)
Prove this theorem without using Hopf differential. From this, one might be able to prove the higher-dimensional analogue.

References

[1] S. Brendle, *A sharp bound for the area of minimal surfaces in the unit ball*, Geometric and Functional Analysis **22** (2012), 621-626.
[2] A. Fraser, R. Schoen, *The first Steklov eigenvalue, conformal geometry, and minimal surfaces*, Adv. Math. **226** (2011), no. 5, 4011-4030.
[3] A. Fraser, R. Schoen, *Uniqueness Theorems for Free Boundary Minimal Disks in Space Forms*, IMRN, **17** (2015), 8268-8274.
[4] F.C. Marques, A. Neves, *Min-max theory and the Willmore conjecture*, Ann. of Math. (2) **179** (2014), no. 2, 683-782.
[5] J. C. C. Nitsche, *Stationary partitioning of convex bodies*, Arch. Rational Mech. Anal. **89** (1985), no. 1, 1-19.
[6] A. Ros, R. Souam, *On stability of capillary surfaces in a ball*, Pacific J. Math. **178** (1997), no. 2, 345-361.