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Boundary value problems on differential equations with singularities

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Boundary value problems on differential equations with singularities

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Basic Notation

$\mathbb{N}$ is the set of all natural numbers;

$\mathbb{R}$ is the set of all real numbers, $\mathbb{R}^+ = (0, \infty)$, $\mathbb{R}_+ = [0, +\infty)$;

$C\big([0,\omega];\mathbb{R}\big)$ is the Banach space of continuous functions $u : [0,\omega] \to \mathbb{R}$ with the norm

$$\|u\|_\infty = \max\big\{\,|u(t)| : t \in [0,\omega]\big\};$$

$C\big([0,\omega];\mathbb{R}_+\big) = \big\{u \in C\big([0,\omega];\mathbb{R}\big) : u(t) \geq 0 \text{ para } t \in [0,\omega]\big\}$;

$C\big(D_1;D_2\big)$, where $D_1, D_2 \subseteq \mathbb{R}$, is the set of continuous functions $u : D_1 \to D_2$;

$C^1\big([0,\omega];\mathbb{R}\big)$ is the Banach space of continuous functions $u : [0,\omega] \to \mathbb{R}$ with continuous derivative, with the norm

$$\|u\|_{C^1} = \|u\|_\infty + \|u'\|_\infty\,;$$

$C^1\big([0,\omega];\mathbb{R}_+\big) = \big\{u \in C^1\big([0,\omega];\mathbb{R}\big) : u(t) \geq 0 \text{ for } t \in [0,\omega]\big\}$;

$AC\big([0,\omega];\mathbb{R}\big)$ is the set of all absolutely continuous functions $u : [0,\omega] \to \mathbb{R}$;

$AC^1\big([0,\omega];\mathbb{R}\big)$ is the set of all functions $u : [0,\omega] \to \mathbb{R}$ such that u and u' are absolutely continuous;

For a number $p \in [1,+\infty)$, $L^p\big([0,\omega];\mathbb{R}\big)$ is the Banach space of Lebesgue integrable functions $h:[0,\omega]\to\mathbb{R}$ in the power p, endowed with the norm

$$\|h\|_p = \left(\int_0^\omega |h(t)|\,dt\right)^{1/p};$$

$L\big([0,\omega];\mathbb{R}\big) = L^1([0,\omega];\mathbb{R})$;

For a $h \in L\big([0,\omega];\mathbb{R}\big)$, its mean value is defined by

$$\overline{h} = \frac{1}{\omega}\int_0^\omega h(s)ds.$$

$L\big([0,\omega];\mathbb{R}_+\big) = \big\{p \in L\big([0,\omega];\mathbb{R}\big) : p(t) \geq 0 \text{ for almost every } t \in [0,\omega]\big\}$;

$f:[0,\omega]\times D_1 \to D_2$ belongs to the set of Carathéodory functions Car $\big([0,\omega]\times D_1; D_2\big)$ if and only if $f(\cdot,x):[0,\omega]\to D_2$ is measurable for all $x \in D_1$, $f(t,\cdot): D_1 \to D_2$ is continuous for almost every $t \in [0,\omega]$, and for each compact set $D_0 \subseteq D_1$ it verifies

$$\sup\big\{\,|f(\cdot,x)| : x \in D_0\big\} \in L\big([0,\omega];\mathbb{R}_+\big);$$

$[x]_+ = \max\{x,0\}$, $[x]_- = \max\{-x,0\}$;

Unless otherwise stated whenever $u \in C\big([0,\omega];\mathbb{R}\big)$ we define the numbers

$$M = \max\big\{u(t) : t \in [0,\omega]\big\}, \qquad m = \min\big\{u(t) : t \in [0,\omega]\big\}.$$

Given $\varphi,\psi \in L\big([0,\omega];\mathbb{R}\big)$, then

$$\Phi_+ = \int_0^\omega [\varphi(s)]_+ ds, \qquad \Phi_- = \int_0^\omega [\varphi(s)]_- ds,$$
$$\Psi_+ = \int_0^\omega [\psi(s)]_+ ds, \qquad \Psi_- = \int_0^\omega [\psi(s)]_- ds.$$

Introduction

The topic of singular boundary value problems has been of substantial and rapidly growing interest for many scientists and engineers. The importance of such investigation is emphasized by the fact that numerical simulations of solutions to such problems usually break down near singular points.

On the other hand numerous questions in physics, chemistry, biology and economics lead to this type of problems; for example: deformations of rods, plates and shells, behaviour of plastic materials, surface waves of fluids, flows around objects in fluids or gases, shock waves in gases, movements of viscous fluids, behaviour of magnetic fields of astrophysical objects, existence and stability of periodic and quasiperiodic orbits in celestial mechanics, etc.

As a rule, there arise nonlinear differential and integral equations, variational problems and more general optimization problems. In the Thesis we use the main techniques coming from Nonlinear Analysis to obtain important existence results of periodic solutions to a specific nonlinear differential equations with singularities: *Rayleigh-Plesset, Lazer and Solimini and Brillouin beam equations*, all of them being particular cases of a more general type of equations known as *Liénard equations.*

If we ignore the particular form of the problem, we can usually reduce the question to find a fixed points of a compact operator $T : X \to X$ defined on a Banach space X. In this way, formulating particular problems abstractly in the framework has the advantages of distilling the essential and their relationships, of allowing a uniform treatment of differing practical problems, and of enabling the use of deep and powerful mathematical methods, without which the problems could not be solved. Nevertheless it must be emphasized that the typical approach to a specific problem generally consists of two steps:

(a) the use of precise analytical methods to obtain estimates on the solutions;

(b) the use of general methods of functional analysis.

Based on this idea, one of the most fruitful techniques in Nonlinear Analysis – *the lower and upper functions method* – was introduced by G. Scorza Dragoni in 1931. This method was originally applied to a Dirichlet problem, but since then a large number of contributions made the theory more complete, allowing the aforementioned method to be extended to any type of boundary value problem. The construction of upper and lower functions can be regarded as a numerical approximation of solution that satisfies the equation up to an error term with a constant sign. Then the existence of a solution, together with its localization, is deduced from two of such approximations with error terms of opposite signs.

This manuscript is based on the papers [35, 34, 32, 31, 25], the main used tools are the lower and upper functions method (Chapters 1, 2, 3) and a well known fixed point theorems for compact operators (Schaefer and Poincaré-Bohl fixed point theorems). As it was mentioned above, our aim is to contribute to solving three important periodic problems in a certain field of Applied Mathematics. Thus the presented work is composed by selected papers of our recent investigation making the content as concise as it is possible. We begin with a preliminary chapter (Chapter 1) devoted to the study of a very classical equation, from mathematical point of view, well-known as a Liénard type equation (see [7, 8, 11, 13, 16, 17, 19, 27, 28, 30, 41, 45, 46, 49, 48, 61]). In spite of the fact that this type of equation was investigated by many mathematicians during the last decades, the most of their works deal with the repulsive case and/or the case when the friction-like term has no singularity. However, the physical model studied in Chapter 3 (Rayleigh-Plesset equation) justifies to consider also equations with singularities in the friction-like term. In Chapter 2, we establish a genuine relationship between the order of the singularity and the regularity of the coefficients in the classical equation proposed by Lazer and Solimini in [41]. This relation allows us to disprove an intuitive conjecture on this equation. Moreover, we expect that the same relation can be extended to a general class of the Liénard type equations, even with singular friction-like term, whenever the singularity is attractive. In the last chapter, we consider other type of singular equation with relevance in Electronics. At the beginning, we show the necessity of developing new methods to obtain something new in this field. After this, quite unexpectedly with respect to numerical and analytical results found in the literature, we establish a new range for the existence of 2π-periodic solutions of Brillouin focusing beam equation. This is possible due to suitable non-resonance conditions acting on the rotation number of the solutions in the phase plane.

Finally, we would like to emphisize that every chapter begins with a small introduction to the problem and the actual state of art, then the used tools and the structure of the chapter are described.

For other results obtained during my PhD studies see [3, 4, 33, 57, 62, 63].

Chapter 1

Singular second order differential equations

In this chapter we are going to study from mathematical point of view the periodic problem for the second-order equation

$$u''(t) + f(u(t))u'(t) + g(u(t)) = h(t, u(t)) \qquad \text{for a. e. } t \in [0, \omega] \tag{1.1}$$

where $f, g \in C\big(\mathbb{R}^+; \mathbb{R}\big)$ may have singularities at zero, and $h \in \text{Car}\,\big([0, \omega] \times \mathbb{R}_+; \mathbb{R}\big)$; and some consequences to the particular Liénard equation

$$u''(t) + f(u(t))u'(t) + g(u(t)) = h_0(t) \qquad \text{for a. e. } t \in [0, \omega] \tag{1.2}$$

where $f \in C\big(\mathbb{R}_+; \mathbb{R}\big)$, $h_0 \in L\big([0, \omega]; \mathbb{R}\big)$ and g as before. In this way, we will compare our results with some classical ones.

In the related literature, g is said to present an attractive (resp. repulsive) singularity if $\lim\limits_{x \to 0^+} g(x) = +\infty$ (resp. $\lim\limits_{x \to 0^+} g(x) = -\infty$). By periodic solution to (1.1) we understand a function $u : [0, \omega] \to \mathbb{R}^+$ which is absolutely continuous together with its first derivative, satisfies (1.1) almost everywhere on $[0, \omega]$, and verifies

$$u(0) = u(\omega), \qquad u'(0) = u'(\omega). \tag{1.3}$$

Next we investigate some general criteria to guarantee existence of periodic solution to

(1.1), i.e., existence of positive solution to the boundary problem (1.1), (1.3); taking into account the type of singularity presented by (1.1).

1.1 Repulsive singularities

Roughly speaking, the singularity being of repulsive type, one can expect that periodic solutions exit, provided that the force is attractive at some distance from the origin. However, some care must be taken in order to avoid what seems to be a kind of *resonance* at infinity. This fact is put in evidence in the following result by Del Pino, Manásevich and Montero proved in [13] for the particular non-restoring forced equation (1.1) when $f \equiv 0$, $h(t,x) = -a(t)x + h_0(t)$, where $a, h_0 \in L\big([0,\omega];\mathbb{R}\big)$.

Theorem 1.1.1 *Let the following two assumptions hold.*

1. *There are an integer M and two constants α, β for which*

$$\left(\frac{M\pi}{\omega}\right)^2 < \alpha \leq a(t) + \lim_{x\to+\infty}\frac{g(x)}{x} \leq \beta < \left(\frac{(M+1)\pi}{\omega}\right)^2,$$

   uniformly for almost every $t \in [0,\omega]$.

2. *there are positive constants c', c'' and $\nu \geq 1$, such that*

$$\frac{c'}{x^\nu} \leq g(x) \leq \frac{c''}{x^\nu},$$

   for every $x \in (0,\delta]$.

Then, (1.1) *has a periodic solution (i.e., the problem* (1.1), (1.3) *has at least one positive solution).*

The result in [13] is somehow related to a paper by Fabry and Habets [15], where a periodic problem without singularities is treated. Indeed, in [15] an asymmetric oscillator is considered, assuming, roughly speaking, that the nonlinearity at $+\infty$ asymptotically lies between the asymptotes of two consecutive curves of Dancer-Fučik spectrum; cf. [24]. It is readily seen that the constants appearing in *1.* correspond to theses asymptotes. The analogy between these results can be explained, see, i.e. [16], by the fact that the singularity provides for the solutions a similar behavior as when a superlinear assumption on the nonlinearity at $-\infty$ is made. However, the methods of proof followed in [13] and [15] differ considerably, despite the fact they both use topological degree theory.

Condition *2.* in Theorem 1.1.1 has been improved in [59, 60], obtaining instead of it the condition

$$\lim_{x\to 0} g(x) = -\infty, \qquad \int_0^1 g(x)dx = -\infty,$$

which is commonly called *strong force condition*. Since then, the previous condition has been frequently considered for the existence of periodic solutions of (1.1) (see [19, 26, 67, 65, 66, 49, 41, 27]). However, although less frequent, there are papers detailing with weak singularities, i.e., they are not fulfil the strong force condition (see [53, 55, 54, 49]). But it will not be treated in this manuscript.

The strong force condition has as a consequence that the energy of a solution passing near the origin is arbitrarily large, allowing to have a priori estimates from below of the periodic solutions of (1.1).

In this section we want to prove a general theorem on the existence of periodic solutions to (1.1) and discuss some consequences in order to compare it with related in literature. For that we will use *Schaefer's fixed point theorem* which required to introduce a convenient functional analysis framework for our problem.

1.1.1 Compact operators and Schaefer's theorem

We consider the Banach space $X = C^1\big([0,\omega];\mathbb{R}\big)\times\mathbb{R}$ with the norm $\|(u,a)\| = \|u\|_{C^1} + |a|$. The following result is known as Schaefer fixed point theorem and it is a direct consequence of the Schauder fixed point theorem (see [51], or more recent books [52], [64]).

Theorem 1.1.2 (Schaefer[64]) *Let $F : X \to X$ a continuous and compact operator. If there exists $r > 0$ such that every solution of*

$$(u,a) = \lambda F(u,a) \tag{1.4}$$

for $\lambda \in (0,1)$ verifies

$$\|(u,a)\| \le r, \tag{1.5}$$

then (1.4) has at least a solution for $\lambda = 1$.

Our aim is to apply this result to a given operator whose fixed points correspond to periodic solutions of our differential equation. In order to define such operator and prove its compactness the following definition is needed.

Definition 1.1.1 *An operator $H : X \to L\big([0,\omega];\mathbb{R}\big)$, resp. $A : X \to \mathbb{R}$ is called* Carathéodory *if it is continuous and it fulfills that for every $r > 0$ there exists a function $q_r \in L\big([0,\omega];\mathbb{R}_+\big)$, resp. a constant $M_r \in \mathbb{R}_+$ such that*

$$\big|H(u,a)(t)\big| \le q_r(t) \qquad \text{for a. e. } t\in[0,\omega], \quad \|(u,a)\|\le r,$$

resp.

$$\big|A(u,a)\big| \le M_r \qquad \textit{for } \|(u,a)\| \le r.$$

Lemma 1.1.1 *Let $H : X \to L\big([0,\omega];\mathbb{R}\big)$ and $A : X \to \mathbb{R}$ be Caratheodory operators. Let us define the operator $\Omega : X \to C^1\big([0,\omega];\mathbb{R}\big)$ by*

$$\Omega(u,a)(t) = -\frac{1}{\omega}\left[(\omega - t)\int_0^t sH(u,a)(s)ds + t\int_t^\omega (\omega - s)H(u,a)(s)ds\right] \quad \textit{for } t \in [0,\omega].$$

Then, the operator $F : X \to X$ given by $F = (\Omega, A)$ is compact.

Proof 1 Is sufficient to prove that both Ω and A transform each bounded set of X into a precompact set. First, note that the image of each bounded set of X by A is in fact a precompact set since $\mathbb{R}$ is a finite dimensional space and A is a Carathéodory operator. On the other hand, the definition of Ω involves

$$|\Omega(u,a)(t)| \le \frac{\omega}{4}\int_0^\omega |H(u,a)(s)|\,ds \qquad \text{for } t \in [0,\omega], \tag{1.6}$$

$$\left|\frac{d}{dt}\Omega(u,a)(t)\right| \le \int_0^\omega |H(u,a)(s)|\,ds \qquad \text{for } t \in [0,\omega], \tag{1.7}$$

$$\left|\frac{d^2}{dt^2}\Omega(u,a)(t)\right| \le |H(u,a)(t)| \qquad \text{for a. e. } t \in [0,\omega]. \tag{1.8}$$

Furthermore, since H is a Carathéodory operator, for every $r > 0$ there exists a function $q_r \in L\big([0,\omega];\mathbb{R}_+\big)$ such that

$$|H(u,a)(t)| \le q_r(t) \qquad \text{for a. e. } t \in [0,\omega], \quad \|(u,a)\| \le r. \tag{1.9}$$

Now let $M \subset X$ be a bounded set. Obviously, there exists $r > 0$ such that $\|(u,a)\| \le r$

for every $(u,a) \in M$. Then, from (1.6)–(1.9), for $(u,a) \in M$, we obtain

$$\|\Omega(u,a)\|_\infty \le \frac{\omega}{4}\|q_r\|_1\,,$$
$$\left\|\frac{d}{dt}\Omega(u,a)\right\|_\infty \le \|q_r\|_1\,,$$
$$\left|\frac{d^2}{dt^2}\Omega(u,a)(t)\right| \le q_r(t) \qquad \text{for a. e. } t \in [0,\omega].$$

By Arzelà–Ascoli's theorem, the set $\Omega(M)$ is precompact.

The following corollary is an immediate consequence of Theorem 1.1.2 and Lemma 1.1.1.

Corollary 1.1.1 *Let $H : X \to L\big([0,\omega];\mathbb{R}\big)$ and $A : X \to \mathbb{R}$ be Carathédory operators. If there exists $r > 0$ (not depending on λ) such that every solution of the problem*

$$u''(t) = \lambda H(u,a)(t) \qquad \text{for a. e. } t \in [0,\omega], \tag{1.10}$$

$$u(0) = 0, \qquad u(\omega) = 0, \tag{1.11}$$

$$a = \lambda A(u,a) \tag{1.12}$$

for $\lambda \in (0,1)$ verifies (1.5), then (1.10)–(1.12) admits a solution for $\lambda = 1$.

1.1.2 Auxiliary results

In this subsection we will develop some preliminaries in order to prove the main theorem. The first aim is to rewrite the problem (1.1), (1.3) as a fixed point problem.

Let us define the continuous operator $T : X \to C^1\big([0,\omega];\mathbb{R}\big)$ by

$$T(u,a)(t) = e^a + u(t) - \min\big\{u(s) : s \in [0,\omega]\big\}.$$

For $\lambda \in (0,1)$ we consider the problem

$$u''(t) + \lambda f(T(u,a)(t))u'(t) + \lambda g(T(u,a)(t)) = \lambda h(t, T(u,a)(t)) + \\ + \frac{\lambda}{\omega}\left[\int_0^\omega g(T(u,a)(s))ds - \int_0^\omega h(s, T(u,a)(s))ds\right] \quad \text{for a. e. } t \in [0,\omega], \quad (1.13)$$

$$u(0) = 0, \qquad u(\omega) = 0, \tag{1.14}$$

$$a = \lambda a - \frac{\lambda}{\omega}\left[\int_0^\omega g(T(u,a)(s))ds - \int_0^\omega h(s, T(u,a)(s))ds\right]. \tag{1.15}$$

Remark 1.1.1 It can be easily seen that if $(u,a) \in X$ is a solution to (1.13)–(1.15), then the function u is periodic.

Lemma 1.1.2 *If there exists $r > 0$ such that for each solution of (1.13)–(1.15) with $\lambda \in (0,1)$ is fulfilled*

$$\|(u,a)\| \leq r,$$

then, there exists a solution of (1.1), (1.3).

Proof 2 We define the operators $H : X \to L\big([0,\omega];\mathbb{R}\big)$ y $A : X \to \mathbb{R}$ as

$$H(u,a)(t) = -f(T(u,a)(t))u'(t) - g(T(u,a)(t)) + h(t, T(u,a)(t)) \\ + \frac{1}{\omega}\left[\int_0^\omega g(T(u,a)(s))ds - \int_0^\omega h(s, T(u,a)(s))ds\right] \qquad \text{for a. e. } t \in [0,\omega],$$

$$A(u,a) = a - \frac{1}{\omega}\left[\int_0^\omega g(T(u,a)(s))ds - \int_0^\omega h(s, T(u,a)(s))ds\right].$$

Is clear that both H and A are Carathéodory operators. By Corollary 1.1.1, the problem (1.13)–(1.15) with $\lambda = 1$ has got at least one solution. Furthermore, from (1.15) (with

$\lambda = 1$) we obtain that

$$\int_0^\omega g(T(u,a)(s))ds = \int_0^\omega h(s, T(u,a)(s))ds, \tag{1.16}$$

and, consequently, from (1.13) with $\lambda = 1$, (1.14) and (1.16) we prove that u is a periodic solution satisfying

$$u''(t) + f(T(u,a)(t))u'(t) + g(T(u,a)(t)) = h(t, T(u,a)(t)) \qquad \text{for a. e. } t \in [0,\omega].$$

Now we define v by

$$v(t) = T(u,a)(t) \qquad \text{for } t \in [0,\omega].$$

Then v is a solution of (1.1), (1.3).

The section is concluded by lemmas presenting some useful inequalities.

Lemma 1.1.3 *Let be $u \in AC\big([0,\omega];\mathbb{R}\big)$ such that*

$$u(0) = u(\omega). \tag{1.17}$$

Then the inequality

$$(M-m)^2 \le \frac{\omega}{4}\int_0^\omega {u'}^2(s)ds \tag{1.18}$$

holds.

Proof 3 Let us define $\widetilde{u} : [0, 2\omega] \to \mathbb{R}$ by

$$\widetilde{u}(t) = \begin{cases} u(t) & \text{if } t \in [0,\omega], \\ u(t-\omega) & \text{if } t \in (\omega, 2\omega]. \end{cases} \tag{1.19}$$

Evidently, (1.17) implies that $\widetilde{u} \in AC\big([0,2\omega];\mathbb{R}\big)$ and also there exist $t_0 \in [0,\omega]$ and $t_1 \in (t_0, t_0+\omega)$ such that

$$\widetilde{u}(t_0) = m, \qquad \widetilde{u}(t_1) = M, \qquad \widetilde{u}(t_0+\omega) = m.$$

Then

$$M - m = \int_{t_0}^{t_1} \widetilde{u}'(s)ds, \qquad m - M = \int_{t_1}^{t_0+\omega} \widetilde{u}'(s)ds.$$

Using the Cauchy–Bunyakovskii–Schwarz inequality we prove that

$$M - m \le \sqrt{(t_1 - t_0)\left(\int_{t_0}^{t_1} \widetilde{u}'^2(s)ds\right)},$$
$$M - m \le \sqrt{(t_0 + \omega - t_1)\left(\int_{t_1}^{t_0+\omega} \widetilde{u}'^2(s)ds\right)}.$$

Multiplying both inequalities and using that $AB \le \frac{1}{4}(A+B)^2$ for each $A, B \in \mathbb{R}_+$ we can prove

$$(M-m)^2 \le \frac{\omega}{4}\int_{t_0}^{t_0+\omega} \widetilde{u}'^2(s)ds.$$

Finally, from the last inequality, in virtue of (1.19), we obtain (1.18).

Lemma 1.1.4 *Let $\rho \in C\big(\mathbb{R}^+;\mathbb{R}^+\big)$ a non–decreasing function and let $v \in AC^1\big([0,\omega];\mathbb{R}\big)$ be a positive function such that $v(0) = v(\omega)$, $\quad v'(0) = v'(\omega)$. Then*

$$\int_0^\omega \frac{v''(t)}{\rho(v(t))}\,dt \ge 0. \tag{1.20}$$

Proof 4 There exists a sequence $\rho_n \in C\big(\mathbb{R}^+;\mathbb{R}^+\big)$ of non–decreasing functions which are absolutely continuous such that

$$\lim_{n\to+\infty}\|\rho_n\circ v-\rho\circ v\|_\infty=0, \tag{1.21}$$

$$\rho_n(m_v)=\rho(m_v)\qquad\text{where}\qquad m_v=\min\big\{v(s):s\in[0,\omega]\big\}.$$

After,

$$\int_0^\omega\frac{v''(t)}{\rho_n(v(t))}\,dt=\int_0^\omega\frac{\rho_n'(v(t)){v'}^2(t)}{\rho_n^2(v(t))}\,dt\geq 0 \tag{1.22}$$

and

$$\left|\int_0^\omega\left[\frac{v''(t)}{\rho_n(v(t))}-\frac{v''(t)}{\rho(v(t))}\right]dt\right|\leq\frac{\|\rho_n\circ v-\rho\circ v\|_\infty}{\rho^2(m_v)}\int_0^\omega|v''(t)|\,dt. \tag{1.23}$$

Now from (1.21)–(1.23) we have (1.20).

Lemma 1.1.5 *Let be $v\in AC^1\big([0,\omega];\mathbb{R}\big)$ such that*

$$v(0)=v(\omega),\qquad v'(0)=v'(\omega). \tag{1.24}$$

Then,

$$\int_0^\omega v^2(t)dt\leq\left(\frac{\omega}{\pi}\right)^2\int_0^\omega {v'}^2(t)dt+2m\int_0^\omega v(t)dt \tag{1.25}$$

where

$$m=\min\big\{v(t):t\in[0,\omega]\big\}.$$

Proof 5 Let $t_m\in[0,\omega]$ be a point such that

$$v(t_m)=m, \tag{1.26}$$

and define

$$w(t)=\begin{cases}v(t)-m & \text{for } t\in[0,\omega],\\ v(t-\omega)-m & \text{for } t\in(\omega,2\omega].\end{cases} \tag{1.27}$$

Obviously, in accordance with (1.24) and (1.26) we have

$$w \in AC^1\big([0, 2\omega]; \mathbb{R}\big), \tag{1.28}$$

$$w(t_m) = 0, \qquad w(t_m + \omega) = 0. \tag{1.29}$$

Using Wirtinger inequality, by virtue of (1.27)–(1.29), we obtain

$$\int_{t_m}^{t_m+\omega} w^2(t)dt \leq \left(\frac{\omega}{\pi}\right)^2 \int_0^\omega {v'}^2(t)dt. \tag{1.30}$$

On the other hand,

$$\int_{t_m}^{t_m+\omega} w^2(t)dt = \int_0^\omega (v(t) - m)^2 dt \geq \int_0^\omega v^2(t)dt - 2m \int_0^\omega v(t)dt. \tag{1.31}$$

From (1.30) and (1.31) we get (1.25).

1.1.3 A priori estimates

A priori estimates of possible solutions to the problem (1.13)–(1.15) with $\lambda \in (0, 1)$ are established in this section. This will lead to a direct proof of expected main theorem.

Lemma 1.1.6 *Let be $h_0 \in L\big([0, \omega]; \mathbb{R}\big)$ and $\rho \in C\big(\mathbb{R}^+; \mathbb{R}^+\big)$ a non–decreasing function such that*

$$h(t, x) \leq h_0(t)\rho(x) \qquad \text{for a. e. } t \in [0, \omega], \quad x \geq r, \tag{1.32}$$

for some $r > 0$, and assume that

$$g(x) \geq \overline{h}_0 \rho(x) \qquad \text{for } x \geq r. \tag{1.33}$$

Then, for each solution (u, a) of (1.13)–(1.15), we have

$$a \leq \ln(1 + r). \tag{1.34}$$

Proof 6 Let us suppose that (1.34) is false. Then

$$a > \ln(1 + r) > 0, \tag{1.35}$$

$$T(u, a)(t) > 1 + r \qquad \text{for } t \in [0, \omega]. \tag{1.36}$$

Using (1.35) in (1.15), we get

$$\frac{\lambda}{\omega}\left[\int_0^\omega g(T(u, a)(s))ds - \int_0^\omega h(s, T(u, a)(s))ds\right] < 0. \tag{1.37}$$

From (1.13) using (1.32), (1.36) and (1.37) we obtain

$$u''(t) + \lambda f(T(u, a)(t))u'(t) + \lambda g(T(u, a)(t)) < \lambda h_0(t)\rho(T(u, a)(t)) \quad \text{for a. e. } t \in [0, \omega]. \tag{1.38}$$

Dividing by $\rho(T(u, a)(t))$ the equation (1.38), integrating in $[0, \omega]$, and using (1.14), one gets

$$\int_0^\omega \frac{u''(t)}{\rho(T(u, a)(t))}dt + \lambda \int_0^\omega \frac{g(T(u, a)(t))}{\rho(T(u, a)(t))}dt < \lambda\omega\overline{h}_0.$$

In accordance with Lemma 1.1.4, Remark 1.1.1 and $\lambda > 0$, it gives

$$\int_0^\omega \frac{g(T(u, a)(t))}{\rho(T(u, a)(t))}dt < \omega\overline{h}_0. \tag{1.39}$$

On the other hand, by applying (1.36) and the hypothesis (1.33) we obtain

$$\omega\overline{h}_0 \leq \int_0^\omega \frac{g(T(u, a)(t))}{\rho(T(u, a)(t))}dt$$

which, however, it is a contradiction with (1.39).

Lemma 1.1.7 *Let* $r > 0$, $\eta \in \mathrm{Car}\big([0,\omega] \times \mathbb{R}_+; \mathbb{R}_+\big)$ *a non–decreasing function in the second variable such that*

$$-\eta(t,x) \le h(t,x) \qquad \textit{for a. e. } t \in [0,\omega], \quad x \ge r. \tag{1.40}$$

Furthermore,

$$\limsup_{x \to 0_+} g(x) < +\infty, \tag{1.41}$$

$$g^* \stackrel{def}{=} \limsup_{x \to +\infty} \frac{[g(x)]_+}{x} < \left(\frac{\pi}{\omega}\right)^2, \tag{1.42}$$

$$\limsup_{x \to +\infty} \frac{\int_0^\omega \eta(s,x)ds}{x} < \frac{4}{\omega}\left(1 - g^*\left(\frac{\omega}{\pi}\right)^2\right). \tag{1.43}$$

Then, for each $a_0 > 0$ *there exists a constant* $K > 0$ *such that any solution* (u,a) *of* (1.13)–(1.15) *with* $a \le a_0$ *verifies*

$$M - m \le K \tag{1.44}$$

where

$$M = \max\big\{u(s) : s \in [0,\omega]\big\}, \qquad m = \min\big\{u(s) : s \in [0,\omega]\big\}.$$

Proof 7 Define the truncated function

$$\widetilde{\eta}(t,x) = \begin{cases} \eta(t,x) & \text{if } \ x \ge r, \\ \eta(t,r) & \text{if } \ x < r \end{cases} \tag{1.45}$$

and

$$\xi(t,x) = \widetilde{\eta}(t,x) + \varphi_r(t) \tag{1.46}$$

where

$$\varphi_r(t) = \sup\left\{\, |h(t,x)| : 0 \le x \le r \right\} \qquad \text{for a. e. } \ t \in [0,\omega]. \tag{1.47}$$

Of course, ξ is a non–decreasing function in the second variable. Using (1.40) and (1.45)–(1.47), we obtain the inequality

$$-\,\xi(t,x) \le h(t,x) \qquad \text{for a. e. } \ t \in [0,\omega], \quad x \in \mathbb{R}_+. \tag{1.48}$$

Furthermore,

$$\limsup_{x\to+\infty} \frac{\int_0^\omega \xi(s,x)ds}{x} = \limsup_{x\to+\infty} \left(\frac{\int_0^\omega \widetilde{\eta}(s,x)ds}{x} + \frac{\|\varphi_r\|_1}{x} \right) = \limsup_{x\to+\infty} \frac{\int_0^\omega \eta(s,x)ds}{x}. \tag{1.49}$$

According to (1.15) we can re–write (1.13) as

$$u''(t) + \lambda f(T(u,a)(t))u'(t) + \lambda g(T(u,a)(t)) = \lambda h(t, T(u,a)(t)) - (1-\lambda)a. \tag{1.50}$$

Multiplying (1.50) by $T(u,a)(t)$ and integrating on $[0,\omega]$, we obtain, in accordance with Remark 1.1.1,

$$-\int_0^\omega u'^2(s)ds + \lambda \int_0^\omega g(T(u,a)(s))T(u,a)(s)ds = \lambda \int_0^\omega h(s, T(u,a)(s))T(u,a)(s)ds + \\ -(1-\lambda)a \int_0^\omega T(u,a)(s)ds.$$

Then

$$\int_0^\omega {u'}^2(s)ds = \lambda \int_0^\omega g(T(u,a)(s))T(u,a)(s)ds - \lambda \int_0^\omega h(s,T(u,a)(s))T(u,a)(s)ds + \\ + (1-\lambda)a \int_0^\omega T(u,a)(s)ds \quad (1.51)$$

is fulfilled.

On the other hand, from (1.42) and (1.43) we achieve the existence of $\varepsilon_0 > 0$ and $r_0 > 0$ such that

$$\frac{g(x)}{x} \leq g^* + \varepsilon_0 < \left(\frac{\pi}{\omega}\right)^2 \qquad \text{for } x \geq r_0 \tag{1.52}$$

and

$$\limsup_{x\to+\infty} \frac{\int_0^\omega \eta(s,x)ds}{x} < \frac{4}{\omega}\left(1-(g^*+\varepsilon_0)\left(\frac{\omega}{\pi}\right)^2\right). \tag{1.53}$$

As follows, (1.41) implies that

$$M_g = \sup\left\{g(x) : x \in (0, r_0]\right\} < +\infty. \tag{1.54}$$

Hence, from (1.52) and (1.54) we obtain

$$g(x) \leq (g^* + \varepsilon_0)x + M_g \qquad \text{for } x > 0. \tag{1.55}$$

Now, (1.55) implies

$$\int_0^\omega g(T(u,a)(s))T(u,a)(s)ds \leq (g^*+\varepsilon_0)\int_0^\omega \big(T(u,a)(s)\big)^2 ds + M_g \int_0^\omega T(u,a)(s)ds. \tag{1.56}$$

Using Lema 1.1.5 in (1.56) we arrive to

$$\int_0^\omega g(T(u,a)(s))T(u,a)(s)ds \le (g^* + \varepsilon_0)\left(\frac{\omega}{\pi}\right)^2 \int_0^\omega {u'}^2(s)ds +$$
$$+ \left((g^* + \varepsilon_0)2e^a + M_g\right)\int_0^\omega T(u,a)(s)ds. \quad (1.57)$$

If we use the inequalities (1.48), (1.57) and the hypothesis $a \le a_0$ in (1.51) we prove

$$\left[1 - (g^* + \varepsilon_0)\left(\frac{\omega}{\pi}\right)^2\right] \int_0^\omega {u'}^2(s)ds + \le \int_0^\omega \xi(s, e^{a_0} + M - m)T(u,a)(s)ds$$
$$+ K_0 \int_0^\omega T(u,a)(s)ds \quad (1.58)$$

where

$$K_0 = (g^* + \varepsilon_0)2e^{a_0} + M_g + a_0.$$

Without lost generality, we can suppose that $M \neq m$, e. g. $M - m > 0$; then, let $\varepsilon = \dfrac{e^{a_0}}{M - m}$. In addition

$$\varepsilon \to 0 \qquad \text{if} \qquad M - m \to +\infty, \quad (1.59)$$

and

$$T(u,a)(t) \le (1+\varepsilon)(M - m) \qquad \text{for } t \in [0, \omega].$$

Then,

$$\left[1 - (g^* + \varepsilon_0)\left(\frac{\omega}{\pi}\right)^2\right] \int_0^\omega {u'}^2(s)ds \le \left(K_0\omega + \int_0^\omega \xi(s, (1+\varepsilon)(M-m))ds\right)(1+\varepsilon)(M-m).$$

Using the inequality (1.18) of Lema 1.1.3, from the last inequality we get

$$\frac{4}{\omega}\left[1 - (g^* + \varepsilon_0)\left(\frac{\omega}{\pi}\right)^2\right] \le \frac{(1+\varepsilon)^2\left(K_0\omega + \int_0^\omega \xi(s,y)ds\right)}{y} \quad (1.60)$$

where $y = (1+\varepsilon)(M-m)$. Finally, (1.49), (1.53), (1.59) and (1.60) imply the existence of a constant K such that (1.44) is verified.

Remark 1.1.2 Note that from the inequality (1.44), in view of (1.14), it also follows that $\|u\|_\infty \leq K$.

Lemma 1.1.8 *Let us assume that*

$$\int_0^1 [f(s)]_+ \, ds < +\infty \tag{1.61}$$

or

$$\int_0^1 [f(s)]_- \, ds < +\infty. \tag{1.62}$$

Furthermore, assume that (1.41) *is verified. Then, for each* $a_0 \geq 0$ *and* $K > 0$ *there exists a constant* $K_1 > 0$ *such that every solution* (u, a) *of* (1.13)–(1.15) *with*

$$\|u\|_\infty \leq K \qquad \text{and} \qquad a \leq a_0 \tag{1.63}$$

verifies the boundary

$$\|u'\|_\infty \leq \lambda K_1 + a_0 \omega. \tag{1.64}$$

Proof 8 Assume that the condition (1.61) is fulfilled. Let (u, a) be a solution of (1.13)–(1.15), then u is a periodic function and in addition there exist $t_0, t_1 \in [0, \omega]$ such that

$$u(t_0) = m, \qquad u(t_1) = M \tag{1.65}$$

where

$$M = \max\{u(t) : t \in [0, \omega]\}, \qquad m = \min\{u(t) : t \in [0, \omega]\}.$$

We integrate in (1.50) on the interval $[t_0, t] \subseteq [t_0, t_0 + \omega]$ obtaining

$$\vartheta(u')(t) + \lambda \int_{t_0}^{t} f(\vartheta(T(u,a))(s))\vartheta(u')(s)ds + \lambda \int_{t_0}^{t} g(\vartheta(T(u,a))(s))ds =$$
$$= \lambda \int_{t_0}^{t} \vartheta_1(h)(s, \vartheta(T(u,a))(s))ds - (1-\lambda)a(t-t_0)$$

where $\vartheta : C\big([0,\omega];\mathbb{R}\big) \to C\big([0,2\omega];\mathbb{R}\big)$ and $\vartheta_1 : \mathrm{Car}\,\big([0,\omega] \times \mathbb{R}_+;\mathbb{R}\big) \to \mathrm{Car}\,\big([0,2\omega] \times \mathbb{R};\mathbb{R}\big)$, respectively, those are the periodic extension operators

$$\vartheta(v)(t) = \begin{cases} v(t) & \text{if } t \in [0,\omega], \\ v(t-\omega) & \text{if } t \in (\omega, 2\omega], \end{cases} \tag{1.66}$$

$$\vartheta_1(h)(t,x) = \begin{cases} h(t,x) & \text{if } t \in [0,\omega], \\ h(t-\omega,x) & \text{if } t \in (\omega, 2\omega]. \end{cases} \tag{1.67}$$

Obviously,

$$-\vartheta(u')(t) = \lambda \int_{t_0}^{t} f(\vartheta(T(u,a))(s))\vartheta(u')(s)ds + \lambda \int_{t_0}^{t} g(\vartheta(T(u,a))(s))ds$$
$$- \lambda \int_{t_0}^{t} \vartheta_1(h)(s, \vartheta(T(u,a))(s))ds + (1-\lambda)a(t-t_0). \tag{1.68}$$

Using (1.63) and (1.65) we get

$$0 < T(u,a)(t_0) \le T(u,a)(t) \le T(u,a)(t_1) \le e^{a_0} + 2K \qquad \text{for } t \in [0,\omega]. \tag{1.69}$$

Then, by (1.41) and the fact that $h \in \mathrm{Car}\,\big([0,\omega] \times \mathbb{R}_+;\mathbb{R}\big)$, the number μ and the function σ defined by

$$\mu = \sup\big\{[g(s)]_+ : s \in (0, e^{a_0}+2K]\big\}, \quad \sigma(s) = \sup\big\{\,|h(s,x)| : x \in [0, e^{a_0}+2K]\big\}, \tag{1.70}$$

satisfies

$$0 \le \mu < +\infty, \qquad \sigma \in L\big([0,\omega];\mathbb{R}_+\big). \tag{1.71}$$

Using (1.63), (1.69)–(1.71) and $t_0 \le t \le t_0+\omega$, in the equation (1.68), we obtain

$$-\vartheta(u')(t) \le \lambda \int_0^{e^{a_0}+2K} [f(s)]_+\, ds + \lambda\omega\mu + \lambda\|\sigma\|_1 + \omega a_0. \tag{1.72}$$

Defining $K_1 = \int_0^{e^{a_0}+2K} [f(s)]_+\, ds + \omega\mu + \|\sigma\|_1$, we have, from (1.72),

$$-\vartheta(u')(t) \le \lambda K_1 + \omega a_0 \qquad \text{for } t \in [t_0, t_0+\omega]. \tag{1.73}$$

We integrate on the interval $[t, t_1+\omega] \subseteq [t_1, t_1+\omega]$ the equation (1.50), obtaining

$$\begin{aligned}\vartheta(u')(t) = \lambda \int_t^{t_1+\omega} f(\vartheta(T(u,a))(s))\vartheta(u')(s)ds + \lambda \int_t^{t_1+\omega} g(\vartheta(T(u,a))(s))ds \\ - \lambda \int_t^{t_1+\omega} \vartheta_1(h)(s, \vartheta(T(u,a))(s))ds + (1-\lambda)a(t_1+\omega-t).\end{aligned} \tag{1.74}$$

Using (1.63), (1.69)–(1.71) and $t_1 \le t \le t_1+\omega$ in the equation (1.74), we have

$$\vartheta(u')(t) \le \lambda K_1 + \omega a_0 \qquad \text{for } t \in [t_1, t_1+\omega]. \tag{1.75}$$

From (1.73) and (1.75) we conclude that (1.64) is verified. Therefore the proof is finished for this case.

Now we suppose that (1.62) is fulfilled. By defining

$$v(t) = u(\omega - t) \qquad \text{for } t \in [0,\omega] \tag{1.76}$$

we obtain that

$$v''(t)-\lambda f(T(v,a)(t))v'(t)+\lambda g(T(v,a)(t)) = \lambda\widetilde{h}(t,T(v,a)(t))-(1-\lambda)a \quad \text{for a. e. } t\in[0,\omega],$$

where

$$\widetilde{h}(t,x) = h(\omega-t,x) \qquad \text{for a. e. } t\in[0,\omega], \quad x\in\mathbb{R}_+.$$

If we do an analogical reasoning, using (1.62) by (1.61), we arrive at

$$\|v'\|_\infty \le \lambda K_1 + a_0\omega \tag{1.77}$$

with

$$K_1 = \int_0^{e^{a_0}+2K} [f(s)]_- ds + \omega\mu + \|\sigma\|_1 .$$

Now, since (1.76) and (1.77) we obtain (1.64).

Remark 1.1.3 If we take $a_0 = 0$ in Lema 1.1.8, we would obtain that

$$\|u'\|_\infty \le \lambda K_1 \tag{1.78}$$

Whenever (u,a) is a solution of (1.13)–(1.15) with $a \le 0$.

Lemma 1.1.9 *We suppose that*

$$\lim_{x\to 0_+} g(x) = -\infty, \qquad \int_0^1 g(s)ds = -\infty, \tag{1.79}$$

and (1.61) *or* (1.62) *is verified. Then, for each $K > 0$ there exists a constant $a_1 > 0$ such that every solution (u,a) of* (1.13)–(1.15) *with*

$$\|u\|_\infty \le K \qquad \text{and} \qquad a \le 0 \tag{1.80}$$

admits the bound

$$-a_1 \leq a. \tag{1.81}$$

Proof 9 We define σ as in (1.70) with $a_0 = 0$. Obviously, because (1.79) and $h \in \mathrm{Car}\big([0,\omega]\times\mathbb{R}_+;\mathbb{R}\big)$, we arrive to $\sigma \in L\big([0,\omega];\mathbb{R}_+\big)$. Since (u,a) is solution of (1.13)–(1.15), from (1.15), by (1.70) and (1.80), we have

$$\begin{aligned}\frac{a(1-\lambda)}{\lambda} &= -\frac{1}{\omega}\left[\int_0^\omega g(T(u,a)(s))ds - \int_0^\omega h(s,T(u,a)(s))ds\right] \\ &\geq -\frac{1}{\omega}\int_0^\omega g(T(u,a)(s))ds - \frac{1}{\omega}\left\|\sigma\right\|_1,\end{aligned}$$

hence

$$-\frac{1}{\omega}\int_0^\omega g(T(u,a)(s))ds \leq \frac{a(1-\lambda)}{\lambda} + \frac{1}{\omega}\left\|\sigma\right\|_1.$$

By (1.80) we obtain

$$-\int_0^\omega g(T(u,a)(s))ds \leq \left\|\sigma\right\|_1. \tag{1.82}$$

On the other hand, (1.79) implies that there exists $s_0 > 0$ such that

$$g(s) < -\frac{\left\|\sigma\right\|_1}{\omega} \leq 0 \qquad \text{for } \ s \in (0,s_0). \tag{1.83}$$

We denote by $t_m \in [0,\omega]$ the point where $u(t_m) = \min\big\{u(t) : t \in [0,\omega]\big\}$. Obviously, either

$$T(u,a)(t_m) = e^a \geq s_0, \tag{1.84}$$

or

$$T(u,a)(t_m) = e^a < s_0. \tag{1.85}$$

Clearly, if we get a bound like (1.81) in the case (1.85), that same bound would be useful also for every solution (u,a) of (1.13)–(1.15) verifying (1.84). Hence, we can suppose that (1.85) is fulfilled without loss of generality.

If $T(u,a)(t) < s_0$ for every $t \in [0,\omega]$, by using (1.82) and (1.83) we would obtain a contradiction. Then there exists points $t_1, t_2 \in (t_m, t_m+\omega)$ such that

$$\vartheta(T(u,a))(t) < s_0 \qquad \text{for } t \in [t_m, t_1), \qquad \vartheta(T(u,a))(t_1) = s_0, \tag{1.86}$$

$$\vartheta(T(u,a))(t) < s_0 \qquad \text{for } t \in (t_2, t_m+\omega], \qquad \vartheta(T(u,a))(t_2) = s_0, \tag{1.87}$$

where ϑ is the operator defined by (1.66). Since $a \leq 0$, then

$$\frac{\lambda}{\omega}\left[\int_0^\omega g(T(u,a)(s))ds - \int_0^\omega h(s, T(u,a)(s))ds\right] \geq 0,$$

so

$$u''(t) + \lambda f(T(u,a)(t))u'(t) + \lambda g(T(u,a)(t)) \geq \lambda h(t, T(u,a)(t)) \qquad \text{for a. e. } t \in [0,\omega].$$

Obviously,

$$[\vartheta(u')(t)]' + \lambda f(\vartheta(T(u,a))(t))\vartheta(u')(t) + \lambda g(\vartheta(T(u,a))(t)) \geq$$
$$\geq \lambda \vartheta_1(h)(t, \vartheta(T(u,a))(t)) \quad \text{for a. e. } t \in [0, 2\omega] \tag{1.88}$$

where ϑ and ϑ_1 are operators defined by (1.66) and (1.67), respectively.

first, let us assume that (1.61) is verified. Integrating on $[t_m, t_1]$ the inequality (1.88) we obtain

$$\vartheta(u')(t_1) + \lambda \int_{t_m}^{t_1} f(\vartheta(T(u,a))(s))\vartheta(u')(s)ds + \lambda \int_{t_m}^{t_1} g(\vartheta(T(u,a))(s))ds \geq$$
$$\geq \lambda \int_{t_m}^{t_1} \vartheta_1(h)(s, \vartheta(T(u,a))(s))ds;$$

By a variable change and using (1.85) and (1.86) we have

$$\vartheta(u')(t_1) + \lambda \int_{e^a}^{s_0} f(s)ds - \lambda \int_{t_m}^{t_1} \vartheta_1(h)(s, \vartheta(T(u,a))(s))ds \geq -\lambda \int_{t_m}^{t_1} g(\vartheta(T(u,a))(s))ds.$$

According to Lema 1.1.8, Remark 1.1.3 and the conditions (1.79) and (1.80) we obtain that there exists a constant $K_1 > 0$ such that (1.78) is fulfilled. Using (1.78), (1.85), the inequality $\lambda > 0$ and the fact of $x \leq [x]_+$ for any $x \in \mathbb{R}$ we obtain

$$-\int_{t_m}^{t_1} g(\vartheta(T(u,a))(s))ds \leq K_2 \tag{1.89}$$

where

$$K_2 = K_1 + \int_0^{s_0} [f(s)]_+ ds + \|\sigma\|_1 .$$

Multiplying by K_1 in the inequality (1.89), then

$$-K_1 \int_{t_m}^{t_1} g(\vartheta(T(u,a))(s))ds \leq K_2 K_1.$$

Using (1.78), (1.83) we obtain

$$-\int_{t_m}^{t_1} g(\vartheta(T(u,a))(s))\vartheta(u')(s)ds \leq K_2 K_1.$$

After a simple change of variable and using (1.85) and (1.86) we arrive to

$$-\int_{e^a}^{s_0} g(s)ds \leq K_2 K_1. \tag{1.90}$$

Using (1.79) we ensure the existence of $a_1 > 0$ such that (1.81) is fulfilled.

As a second part, we suppose that (1.62) is true. Integrating on $[t_2, t_m+\omega]$ the inequality (1.88) and doing similar steps as before, using (1.87) in place of (1.86), we arrive to (1.90) with

$$K_2 = K_1 + \int_0^{s_0} [f(s)]_- ds + \|\sigma\|_1 .$$

Then, the condition (1.79) implies the existence of a constant $a_1 > 0$ such that (1.81) is fulfilled.

1.1.4 Main result and consequences

Now we are at the point to prove our abstract existence result to periodic solutions of (1.1) in the repulsive case and to discuss some consequences.

Theorem 1.1.3 *Let $\eta \in \mathrm{Car}\big([0,\omega] \times \mathbb{R}_+; \mathbb{R}_+\big)$ non–decreasing function with respect to the second variable, $h_0 \in L\big([0,\omega]; \mathbb{R}\big)$, $\rho \in C\big(\mathbb{R}^+; \mathbb{R}^+\big)$ non–decreasing and $r > 0$ such that is fulfilled*

1. $-\eta(t,x) \le h(t,x) \le h_0(t)\rho(x) \qquad$ *for a. e.* $t \in [0,\omega], \quad x \ge r,$
2. $g(x) \ge \overline{h}_0 \rho(x) \qquad$ *for* $x \ge r,$
3. $\lim\limits_{x \to 0_+} g(x) = -\infty, \qquad \int_0^1 g(x)dx = -\infty,$
4. $g^* \overset{def}{=} \limsup\limits_{x \to +\infty} \dfrac{[g(x)]_+}{x} < \left(\dfrac{\pi}{\omega}\right)^2,$
5. $\limsup\limits_{x \to +\infty} \dfrac{\int_0^\omega \eta(t,x)dt}{x} < \dfrac{4}{\omega}\left(1 - g^* \left(\dfrac{\omega}{\pi}\right)^2\right),$
6. $\int_0^1 [f(s)]_+ ds < +\infty \qquad$ *or* $\qquad \int_0^1 [f(s)]_- ds < +\infty.$

Then, there exists at least one solution of problem (1.1), (1.3).

Proof 10 The result immediately follows from Lemma 1.1.2, Lemmas 1.1.6–1.1.9, and Remark 1.1.2.

Many classical papers consider the case where the right-hand side only depends on t and f is continuous at zero, that is, $f \in C\big(\mathbb{R}_+;\mathbb{R}\big)$ and $\delta = 0$. We consider this case in a separated corollary.

Corollary 1.1.2 *Let us consider the problem* (1.2), (1.3) *where* $g \in C\big(\mathbb{R}^+;\mathbb{R}\big)$ *verifies the conditions*

1. $\lim\limits_{x\to 0+} g(x) = -\infty$,
2. $\int_0^1 g(x)dx = -\infty$,
3. $\limsup\limits_{x\to+\infty} \dfrac{g(x)}{x} < \left(\dfrac{\pi}{\omega}\right)^2$,
4. *there exists* $r > 0$ *such that* $g(x) \geq \overline{h}_0$ *for every* $x \geq r$.

Then, there exists a positive solution of problem (1.2), (1.3).

Proof 11 It is enough to apply Theorem 1.1.3 with $h(t,x) = h_0(t)$, $\eta(t,x) = [h_0(t)]_-$ and $\rho \equiv 1$.

Let us observe that the condition 3. is in some sense optimal, since in [8] the authors have constructed an example of $h \in C\big([0,\omega];\mathbb{R}\big)$ such that the equation

$$u'' + \left(\frac{\pi}{\omega}\right)^2 u - \frac{1}{u^3} = h(t)$$

has no periodic solution. This is because of the last equation is resonant at infinity (i.e. it does not fulfil the first hypothesis of Theorem 1.1.1 when $a(t) = (\pi/\omega)^2$). Thus the phenomenon of resonance plays a crucial role to the existence of periodic solution with repulsive singularities. However, our condition 3., roughly speaking, says that (1.1) is a non-resonant equation at infinity. Although we will not consider resonant equations, in literature there is a extensive literature about it, starting by the first paper to a non-singular equation [40] whose technique was based on original application of *Schauder fixed point theorem*. Afterwards appeared some interesting papers as [43, 38, 37, 44, 18] all of them are recommendable to read if one is interesting about the topic, in special see [43, 44] for a survey on the topic. Other recent papers to study the resonance to singular equations are [17, 8, 20].

Corollary 1.1.2 covers the classical model equation of Lazer-Solimini [41] (profoundly studied in the next chapter). It also improves the following result by Mawhin.

Theorem 1.1.4 ([45]) *Let us assume that* $f(x) \equiv c \in \mathbb{R}$. *Fix* $0 < a < \dfrac{1}{2\omega^2 e^{2|c|\omega}}$ *and* $b > 0$ *such that*

1. $g(x) \leq ax + b \qquad$ *for* $x > 0$,
2. $\lim\limits_{x \to 0+} g(x) = -\infty$,
3. $\int_0^1 g(x)dx = -\infty$,
4. $\liminf\limits_{x \to +\infty} g(x) > \overline{h}_0$.

Then there exists at least one positive solution of problem (1.2), (1.3).

Obviously, our Corollary 1.1.2 improves conditions 1. and 4. of Theorem 1.1.4. The rest of conditions are the same and thy are classically assumed.

Another related result was proved by Habets and Sanchez.

Theorem 1.1.5 ([27]) *Let there exist constants* $c > 0$ *and* $0 < r_0 < 1 < r_1 < +\infty$ *such that*

1. $g(x) - h_0(t) \leq c \qquad$ *for* $t \in [0, \omega], \quad x > 0$,

2. $g(x) < \overline{h}_0$ *for all* $x < r_0$*,*

3. $\int_0^1 g(x)dx = -\infty$*,*

4. $g(x) > \overline{h}$ *for all* $x > r_1$*.*

5. $\int_0^\omega h_0^2(s)ds < +\infty$

Then the problem (1.2)*,* (1.3) *has at least one positive solution.*

Our Corollary 1.1.2 improves Theorem 1.1.5, except for the condition 1., which it is not assumed by Habets-Sanchez.

1.2 Attractive singularities

Intuitively speaking, if the singularity is of attractive type, one can expect to use the *lower and upper functions theory* in order to obtain existence of solutions for boundary value problems, in particular for our periodic problem. In this case, usually the expected results are "better" than when the nonlinearity presents a repulsive singularity.

These type of equations will be mainly emphasized in this work. Because it can establish an original sharp condition guaranteeing the existence of periodic solutions. More concretely, it seems that there an inusual relationship between the "order" of the singularity and the smoothness of the coeficientes in the equation. This fact will be studied in depth in the next chapters.

Such as it was mentioned we need to introduce a new tool to study existence of periodic solutions to (1.1): the lower and upper functions method. The main theorem of this part is a general existence result of periodic solutions to the above-mentioned equation based (with attractive singularity) on this method.

1.2.1 The method of lower and upper functions

The method of upper and lower functions is one of the most fruitful techniques in Nonlinear Analysis and the main idea can be traced back at least to Picard. The monograph [11] presents a nice and complete historical review of the subject. In our context, the definition of upper and lower functions is as follows.

Definition 1.2.1 *A function $\alpha \in AC^1\big([0,\omega];\mathbb{R}\big)$ is called a lower–function to the problem (1.1), (1.3) if $\alpha(t) > 0$ for every $t \in [0,\omega]$ and*

$$\alpha''(t) + f(\alpha(t))\alpha'(t) + g(\alpha(t)) \geq h(t,\alpha(t)) \qquad \textit{for a. e. } t \in [0,\omega],$$
$$\alpha(0) = \alpha(\omega), \qquad \alpha'(0) \geq \alpha'(\omega).$$

Definition 1.2.2 *A function $\beta \in AC^1\big([0,\omega];\mathbb{R}\big)$ is called an upper–function to the problem (1.1), (1.3) if $\beta(t) > 0$ for every $t \in [0,\omega]$ and*

$$\beta''(t) + f(\beta(t))\beta'(t) + g(\beta(t)) \leq h(t,\beta(t)) \qquad \textit{for a. e. } t \in [0,\omega],$$
$$\beta(0) = \beta(\omega), \qquad \beta'(0) \leq \beta'(\omega).$$

Next theorem is well-known in the related literature (see, e.g., [11] or more general case in [48, Theorem 8.12]).

Proposition 1.2.1 *Let α and β be lower and upper functions to the problem (1.1), (1.3) such that*

$$\alpha(t) \leq \beta(t) \qquad \textit{for } t \in [0,\omega].$$

Then there exists a positive solution u to the problem (1.1), (1.3) such that

$$\alpha(t) \leq u(t) \leq \beta(t) \qquad \textit{for } t \in [0,\omega].$$

At this point we can prove the intuitive reason why we use this method when the equation has an attractive singular non-linearity and why many times it obtained better result in this case. To see that, we propose to consider the following equation

$$u''(t) \mp ku(t) \pm \frac{1}{u^{\lambda}(t)} = 0 \qquad \text{for } t \in [0, \omega],$$

where $k > 0$ and $\lambda > 0$. Then, if the type of singularity is attractive, one can easily find constant well-ordered lower and upper functions and to apply Proposition 1.2.1. On the contrary, if the type of singularity is repulsive, it can find revers-ordered lower and upper functions, thus, in this case, we can not apply Proposition 1.2.1. Nevertheless there are theorems which ensure existence of periodic solution when we have lower and upper functions in reversed order under an additional condition which is often used to avoid resonances (see [28, 56, 1, 2]).

The objective of this section is to develop a new technique for construction of upper and lower functions to the problem (1.1), (1.3).

1.2.2 Auxiliary results

Given $x_1 \in \mathbb{R}^+$ and $x_0 \in \mathbb{R}_+$ fixed constants, let us define the operator $T : C^1\big([0, \omega]; \mathbb{R}\big) \to C^1\big([0, \omega]; \mathbb{R}\big)$ by

$$T(u)(t) = x_1 + x_0 \left(u(t) - \min\left\{u(s) : s \in [0, \omega]\right\}\right) \qquad \text{for } t \in [0, \omega] \tag{1.91}$$

and consider the auxiliar problem

$$u''(t) + f(T(u)(t))u'(t) = q(t) \qquad \text{for a. e. } t \in [0, \omega], \tag{1.92}$$

$$u(0) = 0, \qquad u(\omega) = 0, \tag{1.93}$$

where $f \in C(\mathbb{R}^+;\mathbb{R})$ and $q \in L([0,\omega];\mathbb{R})$. By a solution to the problem (1.92), (1.93) we understand a function $u \in AC^1([0,\omega];\mathbb{R})$ which satisfies (1.92) almost everywhere on $[0,\omega]$, and verifies (1.93).

Lemma 1.2.1 *For every possible solution u to the problem*

$$u''(t) + \lambda f(T(u)(t))u'(t) = \lambda q(t) \qquad \text{for a. e. } t \in [0,\omega], \tag{1.94}$$

$$u(0) = 0, \qquad u(\omega) = 0 \tag{1.95}$$

with $\lambda \in (0,1]$, the estimate

$$M - m \le \frac{\omega}{4} \max\left\{\int_0^\omega [q(s)]_+ ds, \int_0^\omega [q(s)]_- ds\right\} \tag{1.96}$$

holds (see M and m in Basic Notation).

Proof 12 Multiplying (1.94) by u and integrating on $[0,\omega]$, we get

$$-\int_0^\omega u'^2(s)ds = \lambda \int_0^\omega q(s)u(s)ds.$$

Hence,

$$\int_0^\omega u'^2(s)ds \le \lambda \left(M \int_0^\omega [q(s)]_- ds - m \int_0^\omega [q(s)]_+ ds\right). \tag{1.97}$$

Note that (1.95) implies $M \ge 0$, $m \le 0$. Therefore, from (1.97) we obtain

$$\int_0^\omega u'^2(s)ds \le \max\left\{\int_0^\omega [q(s)]_+ ds, \int_0^\omega [q(s)]_- ds\right\}(M - m). \tag{1.98}$$

Now, (1.96) is a direct consequence of Lemma 1.1.3 and (1.98).

Next lemma is a generalized version of a lemma proved by Mawhin in [45, Lemma 6.2.].

Lemma 1.2.2 *For every $x_1 \in \mathbb{R}^+$, $x_0 \in \mathbb{R}_+$ and $q \in L\big([0,\omega];\mathbb{R}\big)$ there exists a solution u to the problem (1.92), (1.93). Furthermore,*

$$u'(\omega) - u'(0) = \int_0^\omega q(s)ds \tag{1.99}$$

and (1.96) is fulfilled.

Proof 13 Let u be a possible solution to (1.94), (1.95) with $\lambda \in (0,1)$. According to Lemma 1.2.1 we have

$$\|u\|_\infty \leq \frac{\omega}{4}\|q\|_1. \tag{1.100}$$

On the other hand, it is obvious that there exists $t_0 \in [0,\omega]$ such that

$$u'(t_0) = 0. \tag{1.101}$$

The integration of (1.94) from t_0 to t with respect to (1.91), (1.101), (1.96), and the inclusion $\lambda \in (0,1)$, yields

$$|u'(t)| \leq \left|\int_{t_0}^t f(T(u)(s))u'(s)ds - \int_{t_0}^t q(s)ds\right| \leq \int_{x_1}^{x_1+x_0\frac{\omega}{4}\|q\|_1} |f(s)|ds + \|q\|_1 \quad \text{for } t \in [0,\omega],$$

whence we obtain

$$\|u'\|_\infty \leq \left(M_f x_0 \frac{\omega}{4} + 1\right)\|q\|_1, \tag{1.102}$$

where

$$M_f = \max\left\{|f(x)| : x_1 \leq x \leq x_1 + x_0\frac{\omega}{4}\|q\|_1\right\}.$$

Therefore, in view of (1.100) and (1.102), u satisfies $\|u\|_{C^1} \leq r$ with

$$r = \left[(1 + x_0 M_f)\frac{\omega}{4} + 1\right]\|q\|_1.$$

Define $F : C^1\big([0,\omega];\mathbb{R}\big) \to C^1\big([0,\omega];\mathbb{R}\big)$ by

$$F(v)(t) = \frac{1}{\omega}\Bigg[(\omega - t)\int_0^t s\big(f(T(v)(s))v'(s) - q(s)\big)ds \\ + t\int_t^\omega (\omega - s)\big(f(T(v)(s))v'(s) - q(s)\big)ds\Bigg] \qquad \text{for } t \in [0,\omega].$$

Then, every solution to $F(u) = \lambda u$ with $\lambda \in (0,1)$ is a solution to (1.94), (1.95) and thus according to Theorem 1.1.2 the problem (1.92), (1.93) has at least one solution u. Integrating (1.92) from 0 to ω we obtain (1.99). The estimate (1.96) immediately follows from Lemma 1.2.1.

Remark 1.2.1 Theorem 1.1.2 is enounced to the special Banach space $C^1\big([0,\omega];\mathbb{R}\big)\times\mathbb{R}$ with the norm $\|(u,a)\| = \|u\|_{C^1} + |a|$, in this way it can be used directly to prove the results in the previous section. However it can be used in the same way to a general Banach space $(X, \|\cdot\|)$.

Lemma 1.2.3 *Let $h \in L\big([0,\omega];\mathbb{R}\big)$. Then,*

$$\lim_{n\to+\infty}\int_0^\omega [h(s) - n]_+ ds = 0 \tag{1.103}$$

and

$$\lim_{n\to+\infty}\int_0^\omega [h(s) + n]_- ds = 0. \tag{1.104}$$

Proof 14 Let us define

$$h_n(t) = \begin{cases} n & \text{if } h(t) > n, \\ h(t) & \text{if } h(t) \le n, \end{cases} \qquad \text{for a. e. } t \in [0,\omega], \quad n \in \mathbb{N}. \tag{1.105}$$

Then,

$$h(t) = h_n(t) + [h(t) - n]_+ \qquad \text{for a. e. } t \in [0,\omega], \quad n \in \mathbb{N}. \tag{1.106}$$

Integrating (1.106) over a period,

$$\int_0^\omega h(s)ds = \int_0^\omega h_n(s)ds + \int_0^\omega [h(s) - n]_+ ds \qquad \text{for } n \in \mathbb{N}. \tag{1.107}$$

On the other hand, from (1.105) and (1.106) we get

$$-[h(t)]_- \le h_n(t) \le h(t) \qquad \text{for a. e. } t \in [0, \omega], \quad n \in \mathbb{N}$$

and

$$\lim_{n \to +\infty} h_n(t) = h(t) \qquad \text{for a. e. } t \in [0, \omega].$$

Thus, according to *Lebesgue Theorem* we have

$$\lim_{n \to +\infty} \int_0^\omega h_n(s)ds = \int_0^\omega h(s)ds. \tag{1.108}$$

Now, from (1.107) and (1.108) we get (1.103). The identity (1.104) can be proved by similar arguments.

1.2.3 Construction of lower functions

The first result of this section gives sufficient conditions for the construction of a lower function.

Proposition 1.2.2 *Let* $h_0 \in L\big([0, \omega]; \mathbb{R}\big)$, $\rho_0 \in C\big(\mathbb{R}^+; \mathbb{R}^+\big)$ *be a non–decreasing function,* $0 < x_1 \le x_2 < +\infty$, *and* $c \in \mathbb{R}$ *be such that*

$$h(t, x) \le h_0(t)\rho_0(x) \qquad \textit{for a. e. } t \in [0, \omega], \quad x \in [x_1, x_2], \tag{1.109}$$

$$\frac{g(x)}{\rho_0(x)} \ge c \ge \overline{h}_0 \qquad \textit{for } x \in [x_1, x_2], \tag{1.110}$$

and

$$\rho_0(x_2)\frac{\omega}{4}\Phi_+ \leq x_2 - x_1 \leq \rho_0(x_1)\frac{\omega}{4}\Phi_-, \tag{1.111}$$

where

$$\varphi(t) = h_0(t) - c \qquad \text{for a. e. } t \in [0,\omega]. \tag{1.112}$$

Then there exists a lower function α to the problem (1.1), (1.3) such that

$$x_1 \leq \alpha(t) \leq x_2 \qquad \text{for } t \in [0,\omega].$$

Proof 15 By the definition of φ and (1.110), we obtain $\Phi_- \geq \Phi_+ \geq 0$. As a first case we suppose that

$$\Phi_+ > 0.$$

Put

$$x_0 = \frac{4(x_2 - x_1)}{\omega\Phi_-\Phi_+}, \tag{1.113}$$

$$q(t) = \Phi_-[\varphi(t)]_+ - \Phi_+[\varphi(t)]_- \qquad \text{for a. e. } t \in [0,\omega], \tag{1.114}$$

and let $T : C^1\big([0,\omega];\mathbb{R}\big) \to C^1\big([0,\omega];\mathbb{R}\big)$ be the operator defined by (1.91). Note that

$$\int_0^\omega q(s)ds = 0. \tag{1.115}$$

According to Lemma 1.2.2 there exists a solution u to (1.92), (1.93) such that (1.96) and (1.99) hold. By using (1.114) and (1.115), we obtain

$$M - m \leq \frac{\omega}{4}\Phi_+\Phi_-, \tag{1.116}$$

$$u'(0) = u'(\omega). \tag{1.117}$$

Put

$$\alpha(t) = T(u)(t) \qquad \text{for } t \in [0,\omega]. \tag{1.118}$$

Then, according to (1.91)–(1.93), (1.113), (1.114) and (1.116)–(1.118) we arrive at

$$\alpha''(t) + f(\alpha(t))\alpha'(t) = x_0\Phi_-[\varphi(t)]_+ - x_0\Phi_+[\varphi(t)]_- \qquad \text{for a. e. } t \in [0,\omega], \tag{1.119}$$

$$\alpha(0) = \alpha(\omega), \qquad \alpha'(0) = \alpha'(\omega), \tag{1.120}$$

$$x_1 \le \alpha(t) \le x_2 \qquad \text{for } t \in [0,\omega]. \tag{1.121}$$

Using that ρ_0 is a non–decreasing function, from the inequality (1.121) we obtain

$$\rho_0(x_1) \le \rho_0(\alpha(t)) \le \rho_0(x_2) \qquad \text{for } t \in [0,\omega]. \tag{1.122}$$

From the inequality (1.111), by virtue of (1.113), we get

$$x_0\Phi_+ \le \rho_0(x_1), \qquad \rho_0(x_2) \le x_0\Phi_-. \tag{1.123}$$

Now (1.122) and (1.123) imply

$$x_0\Phi_+ \le \rho_0(\alpha(t)) \le x_0\Phi_- \qquad \text{for } t \in [0,\omega]. \tag{1.124}$$

Using (1.124) in (1.119) we get

$$\alpha''(t) + f(\alpha(t))\alpha'(t) \ge \rho_0(\alpha(t))\varphi(t) \qquad \text{for a. e. } t \in [0,\omega]. \tag{1.125}$$

On the other hand, we can prove, using (1.110), (1.112) and (1.121), that

$$\varphi(t) \ge h_0(t) - \frac{g(\alpha(t))}{\rho_0(\alpha(t))} \qquad \text{for a. e. } t \in [0,\omega]. \tag{1.126}$$

From (1.125), on account of (1.109), (1.121) and (1.126), it follows that

$$\alpha''(t) + f(\alpha(t))\alpha'(t) + g(\alpha(t)) \geq h(t, \alpha(t)) \qquad \text{for a. e. } t \in [0, \omega]. \tag{1.127}$$

Consequently, (1.120), (1.121) and (1.127) ensure us that α is a lower function to the problem (1.1), (1.3).

Now, we consider the remaining case

$$\Phi_+ = 0.$$

Of course, in this case

$$\varphi(t) \leq 0 \qquad \text{for a. e. } t \in [0, \omega].$$

Then, defining α by

$$\alpha(t) = x_1 \qquad \text{for } t \in [0, \omega] \tag{1.128}$$

we can prove easily that α is a positive function which fulfils (1.120) and (1.125). Again, from (1.110), (1.112) and (1.128) we obtain (1.126) and using (1.109), (1.126) and (1.128) in (1.125) we arrive at (1.127). Finally, also in this case, (1.120), (1.127) and (1.128) imply that α is a lower function to the problem (1.1), (1.3).

A simplified version of the latter proposition is presented below.

Proposition 1.2.3 *Let* $h_0 \in L\big([0, \omega]; \mathbb{R}\big)$, $\rho_0 \in C\big(\mathbb{R}^+; \mathbb{R}^+\big)$ *be a non–decreasing function,* $x_0 > 0$*, and* $c \in \mathbb{R}$ *be such that*

$$h(t, x) \leq h_0(t)\rho_0(x) \qquad \textit{for a. e. } t \in [0, \omega], \quad 0 < x \leq x_0, \tag{1.129}$$

$$\frac{g(x)}{\rho_0(x)} \geq c \geq \overline{h}_0 \qquad \textit{for } 0 < x \leq x_0, \tag{1.130}$$

and, in addition, let there exist $x_2 \in (0, x_0]$ *such that*

$$x_2 - \rho_0(x_2)\frac{\omega}{4}\Phi_+ > 0, \tag{1.131}$$

$$\rho_0(x_2)\Phi_+ \leq \rho_0\left(x_2 - \rho_0(x_2)\frac{\omega}{4}\Phi_+\right)\Phi_- \tag{1.132}$$

where $\varphi(t) = h_0(t) - c$ *for almost every* $t \in [0, \omega]$*. Then there exists a lower function* α *to the problem* (1.1), (1.3) *with*

$$0 < \alpha(t) \leq x_2 \qquad \textit{for } t \in [0, \omega].$$

Proof 16 In order to apply Proposition 1.2.2, we define

$$x_1 = x_2 - \rho_0(x_2)\frac{\omega}{4}\Phi_+. \tag{1.133}$$

By (1.131), $x_1 > 0$. Then, it is clear that (1.129) and (1.130) imply (1.109) and (1.110). It remains to show that (1.111) holds. Indeed, by the definition of x_1 we have

$$x_2 - x_1 = \frac{\omega}{4}\rho_0(x_2)\Phi_+. \tag{1.134}$$

On the other hand, using (1.133) in (1.132) we get

$$\frac{\omega}{4}\rho_0(x_2)\Phi_+ \leq \frac{\omega}{4}\rho_0(x_1)\Phi_-. \tag{1.135}$$

Therefore, (1.134) and (1.135) imply (1.111).

The following corollaries are direct consequences of Proposition 1.2.3.

Corollary 1.2.1 *Let* $x_0 > \frac{\omega}{8}\|h_0 - \overline{h}_0\|_1$ *be such that*

$$g(x) \geq \overline{h}_0 \qquad \textit{for } 0 < x \leq x_0.$$

Then there exists a lower function α to the problem (1.2), (1.3) with

$$0 < \alpha(t) \leq x_0 \qquad \textit{for } t \in [0, \omega]. \tag{1.136}$$

Proof 17 The assertion immediately follows from Proposition 1.2.3 with $h(t,x) \equiv h_0(t)$ if we put $c = \overline{h}_0$, $\rho_0 \equiv 1$, and $x_2 = x_0$. Note also that in this case $\|h_0 - \overline{h}_0\|_1 = \Phi_+ + \Phi_- = 2\Phi_+$.

Corollary 1.2.2 *Let $h_0 \in L\big([0,\omega];\mathbb{R}\big)$, $\rho_0 \in C\big(\mathbb{R}^+;\mathbb{R}^+\big)$ be a non–decreasing function, $x_0 > 0$, and $c \in \mathbb{R}$ be such that (1.129) and (1.130) are fulfilled. Let, moreover, there exist a sequence $\{y_n\}_{n=1}^{+\infty}$ of positive numbers such that*

$$\lim_{n \to +\infty} y_n = 0, \tag{1.137}$$

$$\lim_{n \to +\infty} \frac{\rho_0(y_n)}{y_n} = 0, \tag{1.138}$$

and let there exist $\varepsilon \in (0,1)$ and $n_0 \in \mathbb{N}$ such that

$$\frac{\rho_0(y_n)}{\rho_0(y_n(1-\varepsilon))}\Phi_+ \leq \Phi_- \qquad \textit{for } n \geq n_0 \tag{1.139}$$

where $\varphi(t) = h_0(t) - c$ for almost every $t \in [0,\omega]$. Then there exists a lower function α to the problem (1.1), (1.3) satisfying (1.136).

Proof 18 According to Proposition 1.2.3, it is sufficient to prove that (1.131) and (1.132) are fulfilled for some $x_2 \in (0, x_0]$. According to (1.137) and (1.138), there exists $n_1 \geq n_0$ such that

$$y_n \leq x_0 \qquad \text{for } n \geq n_1,$$

$$-\frac{\omega}{4}\Phi_+ \rho_0(y_n) \geq -\varepsilon y_n \qquad \text{for } n \geq n_1, \tag{1.140}$$

Adding y_n to both sides of the inequality (1.140) and applying that ρ_0 is a non–decreasing function, we obtain

$$\rho_0\left(y_n - \frac{\omega}{4}\Phi_+\rho_0(y_n)\right) \geq \rho_0\left(y_n(1-\varepsilon)\right) \qquad \text{for } n \geq n_1. \tag{1.141}$$

Now, if we put $x_2 = y_{n_1}$ we obtain, on account of (1.139)–(1.141) that (1.131) and (1.132) are fulfilled.

Corollary 1.2.3 *Let $h_0 \in L\big([0,\omega];\mathbb{R}\big)$, $\rho_0 \in C\big(\mathbb{R}^+;\mathbb{R}^+\big)$ be a non–decreasing function, $x_0 > 0$, and $c \in \mathbb{R}$ be such that (1.129) and (1.130) are fulfilled. If $\dfrac{\rho_0(x)}{x}$ is a non–increasing function and*

$$\frac{\omega}{4}\Phi_+\Phi_-\frac{\rho_0(x_0)}{x_0} \leq \Phi_- - \Phi_+ \tag{1.142}$$

where $\varphi(t) = h_0(t) - c$ for almost every $t \in [0,\omega]$, then there exists a lower function α to the problem (1.1), (1.3) satisfying (1.136).

Proof 19 According to Proposition 1.2.3, it is sufficient to prove that (1.131) and (1.132) are satisfied with $x_2 = x_0$. From (1.142) we easily obtain (1.131). On the other hand, since the function $\frac{\rho_0(x)}{x}$ is non–increasing,

$$\frac{\rho_0\left(x_0 - \rho_0(x_0)\frac{\omega}{4}\Phi_+\right)}{x_0 - \rho_0(x_0)\frac{\omega}{4}\Phi_+} \geq \frac{\rho_0(x_0)}{x_0}.$$

Consequently,

$$\rho_0\left(x_0 - \rho_0(x_0)\frac{\omega}{4}\Phi_+\right) \geq \rho_0(x_0)\left(1 - \frac{\omega}{4}\Phi_+\frac{\rho_0(x_0)}{x_0}\right). \tag{1.143}$$

Multiplying both sides of (1.143) by Φ_- and using the inequality (1.142) we get (1.132).

1.2.4 Construction of upper functions

The following assertions dealing with the existence of an upper function to the problem considered can be proved analogously to the proviously results formulated, therefore, their proofs are omitted.

Proposition 1.2.4 *Let* $h_1 \in L\big([0,\omega];\mathbb{R}\big)$, $\rho_1 \in C\big(\mathbb{R}^+;\mathbb{R}^+\big)$ *be a non–decreasing function,* $0 < x_1 \le x_2 < +\infty$, *and* $c \in \mathbb{R}$ *be such that*

$$h(t,x) \ge h_1(t)\rho_1(x) \qquad \text{for a. e. } t \in [0,\omega], \quad x \in [x_1,x_2], \tag{1.144}$$

$$\frac{g(x)}{\rho_1(x)} \le c \le \overline{h}_1 \qquad \text{for } x \in [x_1,x_2], \tag{1.145}$$

and

$$\rho_1(x_2)\frac{\omega}{4}\Phi_- \le x_2 - x_1 \le \rho_1(x_1)\frac{\omega}{4}\Phi_+ \tag{1.146}$$

where

$$\varphi(t) = h_1(t) - c \qquad \text{for a. e. } t \in [0,\omega]. \tag{1.147}$$

Then there exists an upper function β *to the problem* (1.1), (1.3) *such that*

$$x_1 \le \beta(t) \le x_2 \qquad \text{for } t \in [0,\omega].$$

Proposition 1.2.5 *Let* $h_1 \in L\big([0,\omega];\mathbb{R}\big)$, $\rho_1 \in C\big(\mathbb{R}^+;\mathbb{R}^+\big)$ *be a non–decreasing function,* $x_0 > 0$, *and* $c \in \mathbb{R}$ *be such that*

$$h(t,x) \ge h_1(t)\rho_1(x) \qquad \text{for a. e. } t \in [0,\omega], \quad x \ge x_0, \tag{1.148}$$

$$\frac{g(x)}{\rho_1(x)} \le c \le \overline{h}_1 \qquad \text{for } x \ge x_0, \tag{1.149}$$

and, in addition, let there exist $x_1 \geq x_0$ *such that*

$$\rho_1\left(x_1 + \rho_1(x_1)\frac{\omega}{4}\Phi_+\right)\Phi_- \leq \rho_1(x_1)\Phi_+ \tag{1.150}$$

where $\varphi(t) = h_1(t) - c$ *for almost every* $t \in [0, \omega]$*. Then there exists an upper function* β *to the problem* (1.1), (1.3) *with*

$$x_1 \leq \beta(t) \qquad \text{for } t \in [0, \omega]. \tag{1.151}$$

Corollary 1.2.4 *Let there exists* $x_0 > 0$ *such that*

$$g(x) \leq \overline{h}_0 \qquad \text{for } x \geq x_0.$$

Then there exists an upper function β *to the problem* (1.2), (1.3) *with*

$$\beta(t) \geq x_0 \qquad \text{for } t \in [0, \omega]. \tag{1.152}$$

Corollary 1.2.5 *Let* $h_1 \in L\big([0, \omega]; \mathbb{R}\big)$*,* $\rho_1 \in C\big(\mathbb{R}^+; \mathbb{R}^+\big)$ *be a non–decreasing function,* $x_0 > 0$*, and* $c \in \mathbb{R}$ *be such that* (1.148) *and* (1.149) *hold, and let there exist a sequence* $\{y_n\}_{n=1}^{+\infty}$ *of positive numbers such that*

$$\lim_{n \to +\infty} y_n = +\infty, \tag{1.153}$$

$$\lim_{n \to +\infty} \frac{\rho_1(y_n)}{y_n} = 0. \tag{1.154}$$

Furthermore, let there exist $\varepsilon > 0$ *and* $n_0 \in \mathbb{N}$ *such that*

$$\frac{\rho_1(y_n(1+\varepsilon))}{\rho_1(y_n)}\Phi_- \leq \Phi_+ \qquad \text{for } n \geq n_0 \tag{1.155}$$

where $\varphi(t) = h_1(t) - c$ *for almost every* $t \in [0, \omega]$*. Then there exists an upper function*

β to the problem (1.1), (1.3) *satisfying* (1.152).

Corollary 1.2.6 *Let $h_1 \in L\big([0,\omega];\mathbb{R}\big)$, $\rho_1 \in C\big(\mathbb{R}^+;\mathbb{R}^+\big)$ be a non–decreasing function, $x_0 > 0$, and $c \in \mathbb{R}$ be such that* (1.148) *and* (1.149) *are fulfilled. If $\frac{\rho_1(x)}{x}$ is a non–increasing function such that*

$$\frac{\omega}{4}\Phi_+\Phi_-\frac{\rho_1(x_0)}{x_0} \le \Phi_+ - \Phi_- \tag{1.156}$$

where $\varphi(t) = h_1(t) - c$ for almost every $t \in [0,\omega]$, then there exists an upper function β to the problem (1.1), (1.3) *satisfying* (1.152).

1.2.5 Main results and consequences

Now we can prove our abstract theorem on the existence of periodic solution to (1.1) with attractive type of singularity and to discuss some consequences.

Theorem 1.2.1 *Let $\rho_0, \rho_1 \in C\big(\mathbb{R}^+;\mathbb{R}^+\big)$ be non–decreasing functions, $h_0, h_1 \in L\big([0,\omega];\mathbb{R}\big)$, and $0 < r_0 \le r_1 < +\infty$ be such that*

$$h(t,x) \le h_0(t)\rho_0(x) \qquad \text{for a. e. } t \in [0,\omega], \quad 0 < x \le r_0, \tag{1.157}$$

$$h(t,x) \ge h_1(t)\rho_1(x) \qquad \text{for a. e. } t \in [0,\omega], \quad x \ge r_1, \tag{1.158}$$

and let there exist $c_0, c_1 \in \mathbb{R}$ such that

$$\frac{g(x)}{\rho_0(x)} \ge c_0 \ge \overline{h}_0 \qquad \text{for } 0 < x \le r_0, \tag{1.159}$$

$$\frac{g(x)}{\rho_1(x)} \le c_1 \le \overline{h}_1 \qquad \text{for } x \ge r_1. \tag{1.160}$$

Furthermore, let us suppose that ρ_0 fulfils at least one of the following conditions:

a) there exists a sequence $\{x_n\}_{n=1}^{+\infty}$ *of positive numbers such that*

$$\lim_{n\to+\infty} x_n = 0, \qquad \lim_{n\to+\infty} \frac{\rho_0(x_n)}{x_n} = 0,$$

and there exist $\varepsilon_0 \in (0,1)$ *and* $n_0 \in \mathbb{N}$ *such that*

$$\frac{\rho_0(x_n)}{\rho_0(x_n(1-\varepsilon_0))}\Phi_+ \leq \Phi_- \qquad \text{for } n \geq n_0,$$

where $\varphi(t) = h_0(t) - c_0$ *for almost every* $t \in [0,\omega]$*;*

b) the function $\frac{\rho_0(x)}{x}$ *is non–increasing and*

$$\frac{\omega}{4}\Phi_+\Phi_-\frac{\rho_0(r_0)}{r_0} \leq \Phi_- - \Phi_+,$$

where $\varphi(t) = h_0(t) - c_0$ *for almost every* $t \in [0,\omega]$*.*

Besides, let us suppose that ρ_1 *fulfils at least one of the following conditions:*

c) there exists a sequence $\{y_n\}_{n=1}^{+\infty}$ *of positives numbers such that*

$$\lim_{n\to+\infty} y_n = +\infty, \qquad \lim_{n\to+\infty} \frac{\rho_1(y_n)}{y_n} = 0,$$

and there exist $\varepsilon_1 > 0$ *and* $n_1 \in \mathbb{N}$ *such that*

$$\frac{\rho_1(y_n(1+\varepsilon_1))}{\rho_1(y_n)}\Psi_- \leq \Psi_+ \qquad \text{for } n \geq n_1,$$

where $\psi(t) = h_1(t) - c_1$ *for almost every* $t \in [0,\omega]$*;*

d) the function $\frac{\rho_1(x)}{x}$ *is non–increasing and*

$$\frac{\omega}{4}\Psi_+\Psi_-\frac{\rho_1(r_1)}{r_1} \leq \Psi_+ - \Psi_-,$$

where $\psi(t) = h_1(t) - c_1$ *for almost every* $t \in [0, \omega]$*.*

Then there exists at least one positive solution to the problem (1.1)*,* (1.3)*.*

Proof 20 According to Corollaries 1.2.2, 1.2.3, 1.2.5, and 1.2.6, the conditions of the theorem guarantee a well–ordered couple of lower and upper functions, therefore the result is a direct consequence of Proposition 1.2.1.

Remark 1.2.2 Note that (1.159) (resp. (1.160)) implies $\Phi_- \geq \Phi_+$ (resp. $\Psi_+ \geq \Psi_-$). In addition, the conditions $a)$ and $c)$ are verified if, for instance, $\Phi_- \neq \Phi_+$ and $\Psi_- \neq \Psi_+$, $\rho_i(x) = x^{\mu_i}$ $(i = 0, 1)$ with $\mu_0 > 1 > \mu_1 \geq 0$. On the other hand, conditions $b)$ and $d)$ are fulfilled if, for instance, $\rho_i(x) = x$ $(i = 0, 1)$ and

$$\frac{\omega}{4}\Phi_+\Phi_- \leq \Phi_- - \Phi_+, \qquad \frac{\omega}{4}\Psi_+\Psi_- \leq \Psi_+ - \Psi_-. \tag{1.161}$$

Next, we formulate some corollaries which can be obtained immediately from Theorem 1.2.1 and Remark 1.2.2.

Corollary 1.2.7 *Let* $h_0, h_1 \in L\big([0, \omega]; \mathbb{R}\big)$*,* $\mu_0 > 1 > \mu_1 \geq 0$ *and* $0 < r_0 \leq r_1 < +\infty$ *be such that*

$$h(t, x) \leq h_0(t)x^{\mu_0} \qquad \textit{for a. e. } t \in [0, \omega], \quad 0 < x \leq r_0,$$

$$h(t, x) \geq h_1(t)x^{\mu_1} \qquad \textit{for a. e. } t \in [0, \omega], \quad x \geq r_1,$$

$$\liminf_{x \to 0_+} \frac{g(x)}{x^{\mu_0}} > \overline{h}_0, \qquad \limsup_{x \to +\infty} \frac{g(x)}{x^{\mu_1}} < \overline{h}_1.$$

Then there exists at least one positive solution to the problem (1.1)*,* (1.3)*.*

Proof 21 According to Remark 1.2.2, one can apply Theorem 1.2.1 with $\rho_0(x) = x^{\mu_0}$, $\rho_1(x) = x^{\mu_1}$.

Corollary 1.2.8 *Let $h_0, h_1 \in L\big([0,\omega];\mathbb{R}\big)$ and $0 < r_0 \le r_1 < +\infty$ be such that*

$$h(t,x) \le h_0(t)x \qquad \text{for a. e. } t \in [0,\omega], \quad 0 < x \le r_0,$$

$$h(t,x) \ge h_1(t)x \qquad \text{for a. e. } t \in [0,\omega], \quad x \ge r_1,$$

$$\overline{h}_0 < \liminf_{x\to 0_+} \frac{g(x)}{x} < +\infty, \qquad \overline{h}_1 > \limsup_{x\to+\infty} \frac{g(x)}{x} > -\infty. \tag{1.162}$$

In addition, we suppose that

$$\frac{\omega}{4} H_0^+ H_0^- < H_0^- - H_0^+, \qquad \frac{\omega}{4} H_1^+ H_1^- < H_1^+ - H_1^-, \tag{1.163}$$

where

$$H_0^+ = \int_0^\omega [h_0(t) - g_*]_+ dt, \qquad H_0^- = \int_0^\omega [h_0(t) - g_*]_- dt,$$
$$H_1^+ = \int_0^\omega [h_1(t) - g^*]_+ dt, \qquad H_1^- = \int_0^\omega [h_1(t) - g^*]_- dt,$$

and

$$g_* = \liminf_{x\to 0_+} \frac{g(x)}{x}, \qquad g^* = \limsup_{x\to+\infty} \frac{g(x)}{x}.$$

Then there exists at least one positive solution to the problem (1.1), (1.3).

Proof 22 From (1.162) and (1.163) we obtain that there exists $\varepsilon > 0$ small enough such that $\varepsilon < \min\big\{g_* - \overline{h}_0, \overline{h}_1 - g^*\big\}$ and (1.161) is verified, where

$$\varphi(t) = h_0(t) - g_* + \varepsilon \quad \text{for a. e. } t \in [0,\omega], \qquad \psi(t) = h_1(t) - g^* - \varepsilon \quad \text{for a. e. } t \in [0,\omega].$$

Hence, setting $c_0 = g_* - \varepsilon$, $c_1 = g^* + \varepsilon$ and $\rho_i(x) = x$ $(i = 0, 1)$, the corollary follows from Theorem 1.2.1.

Corollary 1.2.9 *Let* $h_0, h_1 \in L\big([0,\omega];\mathbb{R}\big)$ *and* $0 < r_0 \le r_1 < +\infty$ *be such that*

$$h(t,x) \le h_0(t)x \qquad \textit{for a. e. } t \in [0,\omega], \quad 0 < x \le r_0,$$

$$h(t,x) \ge h_1(t)x \qquad \textit{for a. e. } t \in [0,\omega], \quad x \ge r_1,$$

$$\lim_{x\to 0+} \frac{g(x)}{x} = +\infty, \qquad \lim_{x\to +\infty} \frac{g(x)}{x} = -\infty. \tag{1.164}$$

Then there exists at least one positive solution to the problem (1.1)*,* (1.3)*.*

Proof 23 Using (1.164) and Lemma 1.2.3, we can find $c_0 > \overline{h}_0$ and $c_1 < \overline{h}_1$ such that (1.161) is fulfilled where $\varphi(t) = h_0(t) - c_0$, $\psi(t) = h_1(t) - c_1$. Moreover, $g(x) \ge c_0 x$ nearby zero and $g(x) \le c_1 x$ nearby $+\infty$. Hence, taking $\rho_i(x) = x$ $(i = 0, 1)$, the corollary follows from Theorem 1.2.1.

In conclusion, the conditions nearby zero guarantee the existence of a positive lower function, whereas the conditions nearby infinite guarantee the existence of an upper function. Both ideas can be combined in order to get a wide variety of results.

We finish the section with two results dealing with the classical singular Liénard equation (1.2).

Theorem 1.2.2 *Let* $\frac{\omega}{8}\|h_0 - \overline{h}_0\|_1 < r_0 \le r_1 < +\infty$ *be such that*

$$g(x) \ge \overline{h}_0 \qquad \textit{for } 0 < x \le r_0,$$

$$g(x) \le \overline{h}_0 \qquad \textit{for } x \ge r_1.$$

Then there exists at least one positive solution to the problem (1.2)*,* (1.3)*.*

Proof 24 It is a direct consequence of Corollaries 1.2.1 and 1.2.4.

Theorem 1.2.3 *Let*

$$\limsup_{x\to 0_+} g(x) = +\infty$$

and let $r_1 > 0$ *be such that*

$$g(x) \leq \overline{h}_0 \qquad \textit{for } x \geq r_1.$$

If

$$\text{ess sup}\left\{h_0(t) : t \in [0,\omega]\right\} < +\infty,$$

then there exists at least one positive solution to the problem (1.2), (1.3).

Proof 25 The existence of a lower function follows from Proposition 1.2.2 with $h(t,x) = h_0(t)$, $\rho_0 \equiv 1$,

$$c = \text{ess sup}\left\{h_0(t) : t \in [0,\omega]\right\},$$

and $x_1 = x_2 > 0$ sufficiently small such that $g(x_1) \geq c$.

The existence of an upper function follows from Proposition 1.2.5 with $h(t,x) = h_0(t)$, $h_1 \equiv h_0$, $\rho_1 \equiv 1$, $c = \overline{h}_0$, and $x_0 = x_1 = r_1$.

Consequently, the assertion follows from Proposition 1.2.1.

It is interesting to compare our results with the existing ones in the related literature. For instance, it is easy to verify that Theorem 1.2.3 generalises in some sense the result of Lazer and Solimini [41, Theorem 2.1] for the equation

$$u'' + g(u) = h(t) \tag{1.165}$$

with attractive singularity and without friction.

On the other hand, if [48, Lemma 8.19] is applied to (1.165), the following result is obtained.

Theorem 1.2.4 *Assume that there exist* $0 < r_0 < r_1 < +\infty$ *such that*

1. $r_0 > 2\omega\|h - \overline{h}\|_1$,
2. $g(x) \geq \overline{h}$ *if* $0 < x \leq r_0$,
3. $g(x) \leq \overline{h}$ *if* $x \geq r_1$.

Then the problem (1.165), (1.3) *has at least one positive solution.*

Note that Theorem 1.2.2 is more general. First, it works for the equation with friction. Besides, since $\frac{\omega}{8}\left\|h - \bar{h}\right\|_1 < 2\omega\left\|h - \bar{h}\right\|_1$, it is evident that the assumption of Theorem 1.2.2 is better.

A related interesting result can be found in [49].

Theorem 1.2.5 (see [49, Corollary 3.3]) *Assume that*

1. $\limsup_{x \to +\infty} g(x) < \overline{h}_0$,
2. *there exists* $r > 0$ *such that* $h_0(t) \leq g(r)$ *for a.e.* $t \in [0, \omega]$.

Then the problem (1.165), (1.3) *has at least one positive solution.*

Theorem 1.2.3 shows that Theorem 1.2.5 is still valid also in the case when the term $f(x)x'$ is incorporated to the equation, and even in the case when f has a singularity at zero.

Chapter 2

The Lazer and Solimini equation

In this chapter we consider a class of second order scalar differential equations with periodic forcing, zero damping, and a restoring force which become infinite at a finite displacement, which we take to be zero. As a model we have in mind the equation

$$u''(t) \pm \frac{1}{u^{\lambda}(t)} = h(t) \qquad \text{for a. e. } t \in [0,\omega], \tag{2.1}$$

where $h \in L\big([0,\omega];\mathbb{R}\big)$. This simple equation can describe the dynamic of a particle with unitary mass moving on the right side of another fixed particle subjected to an external restoring force, when the positive sign is considered. However, (2.1) with negative sign can model a charged electrical particle under the influence of an electrical field generated by a fixed particle at origen with contrary electrical charge. Logically the type of external force h will determinate the dynamic of the equation. In our case, we will always assume that h is periodic (i.e. it is extended periodically on whole $\mathbb{R}$).

From mathematical point of view equation (2.1) is considered as the start point to develop the actual theory of *singular equations*, however, previously, there were some authors who studied very particular singular equations, see [36, 22]. More concretely the topic of singular equation was instigated by the pioneer paper of Lazer and Solimini [41]. In spite of the passed time of the publication of the paper, there has been no contributions to this equation unless that a more general equation is considered.

In this chapter will be to present the results of Lazer and Solimini [41] with the respective used tools. In this way we put to the reader in a suitable status to understand the main contribution of this Thesis: *to find a pioneer relation to attractive singular equations between the regularity of the external restoring force h and the order of the singularity, λ.*

The presented results in [41] concerns to the equation

$$u''(t) + g(u(t)) = h(t) \qquad \text{for a. e. } t \in [0, \omega], \tag{2.2}$$

with $g \in C(\mathbb{R}^+; \mathbb{R}^+)$. This equation has essentially the same properties that (2.1) because both functions $1/x^\lambda$ and $g(x)$ has the some asymptotic properties. Thus the arguments to (2.1) can be easily extended to (2.2).

2.1 Repulsive singularity

In this part we consider a class of problems which includes (2.1) taking the minus sign, i.e.

$$u''(t) - \frac{1}{u^\lambda(t)} = h(t) \qquad \text{for a. e. } t \in [0, \omega] \tag{2.3}$$

with $h \in L([0, \omega]; \mathbb{R})$. As always, we understand by periodic solution a positive function which is solution of the problem (2.3), (1.3).

Concerning to equation (2.2) they proved under the assumptions $\lim_{x \to +\infty} g(x) = 0$ and

$$\lim_{x \to 0+} g(x) = -\infty, \qquad \int_0^1 g(x)dx = -\infty \tag{2.4}$$

the following theorem.

Theorem 2.1.1 (see [41, Theorem 3.12]) *If $g > 0$ then (2.2), (1.3) has at least one* *positive solution if and only if* $\overline{h} > 0$.

They did a truncation to the function g in order to define it on whole $\mathbb{R}$. Then, using such a function, it defines a nonsingular problem which is possible to apply the existence theorem proved in [39]. Finally using the strong condition (2.4) they proved that the solution to the modified problem coincides with the solution to (2.2), (1.3). However, we can prove it by Corollary 1.1.2.

The previous result applied to equation (2.3) means:

Corollary 2.1.1 *If* $\lambda \geq 1$ *then* (2.3), (1.3) *has at least one positive solution if and only if* $\overline{h} < 0$.

Naturally it comes up the question on the "optimality" of the result. There they proved a necessary and sufficient condition under a strong condition, (2.4). The question is *does Theorem 2.1.1 remain still valid if this condition is violated?* The answer was publish in the some paper proving the following result.

Theorem 2.1.2 (see [41, Theorem 4.1]) *For every* $\omega > 0$ *there exists* M_0 *such that for any* $M > M_0$ *there is, a* $T-$*periodic continuous and nonnegative function, such that* (2.2), (1.3) *has no solution and* $\overline{h} = M$.

In addition, since h is taken as a continuous function, this proves that we cannot aspire to find a relationship between the smoothness of h and the order of the singularity λ which allows to guarantee existence of periodic solutions to (2.3) even when the singularity is weak ($\lambda < 1$). This is the reason why there are less papers dealing with weak singularities, being it a rather unexplored field.

2.2 Attractive singularity

Having in mind the equation (2.1) taking the sign $+$, i.e.

$$u''(t) + \frac{1}{u^\lambda(t)} = h(t) \qquad \text{for a. e. } t \in [0, \omega] \tag{2.5}$$

with $h \in L\big([0,\omega];\mathbb{R}\big)$, Lazer and Solimini proved that if h is continuous and g satisfies

$$\lim_{x\to 0+} g(x) = +\infty, \qquad \lim_{x\to +\infty} g(x) = 0, \tag{2.6}$$

then there exists a positive solution to (2.1), (1.3) in the space $C^2\big([0,\omega];\mathbb{R}\big)$ if and only if $\overline{h} > 0$. The way of the proof of Lazer and Solimini allows us to formulate their result as follows:

Theorem 2.2.1 (see [41, Theorem 2.1]) *Let $g \in C\big(\mathbb{R}^+;\mathbb{R}^+\big)$ satisfy (2.6) and let*

$$h \in L^\infty\big([0,\omega];\mathbb{R}\big). \tag{2.7}$$

Then there exists a positive solution $u \in AC^1\big([0,\omega];\mathbb{R}\big)$ to (2.1), (1.3) if and only if $\overline{h} > 0$.

The proof was done using a simple and elegant application of the method of lower and upper functions. Also we can obtain it as immediate consequence of our Theorem 1.2.3. Nevertheless both arguments are essentially the same, only we can do in a more general framework.

At this point, in view of the results in the previous subsection, it comes up the natural question *why did they not consider $h \in L\big([0,\omega];\mathbb{R}\big)$?*. There are other papers in the literature where the problem of the type (2.1), (1.3) is studied in the framework of the

Carathéodory theory, i.e., if $h \in L([0,\omega];\mathbb{R})$ and a positive function $u \in AC^1([0,\omega];\mathbb{R})$ is understood as a solution to (2.1), (1.3) (see, e.g., [27, 49, 48, 29, 35] and references therein). However, also in the works [27, 49], the boundedness of the function h is needed. In [48, 29, 35], the condition (2.7) is replaced by another condition dealing with the oscillation of the primitive of h.

On the other hand, the major part of the results dealing with the continuous input functions can be formulated also in the framework of the Carathéodory theory without any essential changes. This fact may encourage one's expectation that also Theorem 2.2.1 can be extended to the case when $h \in L([0,\omega];\mathbb{R})$ without any other additional conditions. Such a question was formally posted in [29, Open Problem 4.1]. Despite of all expectations, the answer is negative. The condition (2.7) is essential and, as shown we will prove (see Example 2.2.1), Theorem 2.2.1 is not valid anymore if the condition (2.7) is withdrawn unless the additional assumptions are involved—such an additional condition is, e.g., the relation (2.27) in Theorem 2.2.2 formulated below. Moreover, we prove that the condition (2.27) is optimal in a certain sense. More precisely, if we consider the equation (2.5) from our results, for every $p \in [1,+\infty)$, we obtain

- If $\lambda \geq 1/(2p-1)$ and $h \in L^p([0,\omega];\mathbb{R})$ then (2.5), (1.3) has a positive solution if and only if $\overline{h} > 0$, and such a solution is unique.
- If $0 < \lambda < 1/(2p-1)$ then there exists a function $h \in L^p([0,\omega];\mathbb{R})$ with $\overline{h} > 0$ such that (2.5), (1.3) has no positive solution.

At this point we would like to emphasize the following: *for $h \in L^p([0,\omega];\mathbb{R})$, there exists a relation between p and the order of singularity, λ. In other words, there exists a critical value depending on p such that if the power of the singularity λ is greater than or equal to this value then there exists a positive periodic solution. Moreover, if $h \in L^\infty([0,\omega];\mathbb{R})$, then also $h \in L^p([0,\omega];\mathbb{R})$ for every $p \in [1,+\infty)$, and so applying* *our results for p sufficiently large we obtain that* (2.5), (1.3) *has a positive solution for every* $\lambda > 0$ *(provided* $h \in L^\infty\big([0,\omega];\mathbb{R}\big)$*).* Thus Theorem 2.2.1 can be understood as a limit case of Theorem 2.2.2 formulated below.

Theorem 2.2.2 deals also with the uniqueness of a solution in the case when g is strictly decreasing function. This fact is worth mentioning here because in the original paper of Lazer and Solimini, the question of the uniqueness was not discussed.

2.2.1 Auxiliary results

The following results are important to prove our main results

Lemma 2.2.1 *Let* $h \in L^p\big([0,\omega];\mathbb{R}\big)$ *and let* $u \in AC^1\big([0,\omega];\mathbb{R}\big)$ *satisfy*

$$u''(t) \leq h(t) \qquad \text{for a. e. } t \in [0,\omega]. \tag{2.8}$$

Then

$$|u'(t)| \leq \left(\frac{(2p-1)}{p}\|h\|_p\right)^{p/(2p-1)} \big(u(t)\big)^{(p-1)/(2p-1)} \qquad \text{for } t \in [0,\omega]. \tag{2.9}$$

Proof 26 Let $t_0 \in \mathbb{R}$ be arbitrary but such that $u'(t_0) \neq 0$, and let $\sigma = \operatorname{sgn} u'(t_0)$. Then there exists $s_0 \in I(t_0, t_0 - \sigma\omega)$ such that

$$\sigma u'(t) > 0 \qquad \text{for } t \in I(s_0, t_0), \qquad u'(s_0) = 0.$$

Multiplying both sides of (2.8) by $|u'(t)|^{(p-1)/p}$ and integrating over $I(s_0, t_0)$ we get

$$\frac{p}{2p-1}|u'(t_0)|^{(2p-1)/p} = \sigma \int_{s_0}^{t_0} u''(t)|u'(t)|^{(p-1)/p}dt \leq \sigma \int_{s_0}^{t_0} h(t)|u'(t)|^{(p-1)/p}dt \leq$$
$$\|h\|_p \left(\sigma \int_{s_0}^{t_0} |u'(t)|dt\right)^{(p-1)/p} = \|h\|_p\big(u(t_0) - u(s_0)\big)^{(p-1)/p} \leq \|h\|_p\big(u(t_0)\big)^{(p-1)/p}. \tag{2.10}$$

Thus (2.9) follows from (2.10).

Lemma 2.2.2 *Let* $u \in AC^1\big([0,\omega];\mathbb{R}\big)$, $g \in C\big(\mathbb{R}^+;\mathbb{R}^+\big)$, *and let*

$$g_*(x) \overset{def}{=} \inf\big\{g(s) : s \in (0,x]\big\}. \tag{2.11}$$

Let, moreover, $s_0, t_0 \in \mathbb{R}$ *be such that* $s_0 < t_0$ *and*

$$u'(s_0) = 0, \qquad u'(t_0) = 0. \tag{2.12}$$

Then

$$\int_{s_0}^{t_0} u''(t) g_*^{p-1}(u(t))dt \geq 0. \tag{2.13}$$

Proof 27 Let $g_n \in C\big(\mathbb{R}^+;\mathbb{R}^+\big)$ be a sequence of non-increasing functions which are continuous together with their derivatives and such that

$$\lim_{n\to+\infty} \|g_n^{p-1} \circ u - g_*^{p-1} \circ u\|_\infty = 0. \tag{2.14}$$

Obviously, since g_n are non-increasing, in view of (2.12) we have

$$\int_{s_0}^{t_0} u''(t) g_n^{p-1}(u(t))dt = -(p-1)\int_{s_0}^{t_0} {u'}^2(t) g_n^{p-2}(u(t)) g_n'(u(t))dt \geq 0. \tag{2.15}$$

Now, (2.13) follows from (2.14) and (2.15), because

$$\left|\int_{s_0}^{t_0} u''(t) g_n^{p-1}(u(t))dt - \int_{s_0}^{t_0} u''(t) g_*^{p-1}(u(t))dt\right| \leq \|g_n^{p-1} \circ u - g_*^{p-1} \circ u\|_\infty \int_{s_0}^{t_0} |u''(t)|dt.$$

Lemma 2.2.3 *Let* $h \in L^p\big([0,\omega];\mathbb{R}\big)$, $g \in C\big(\mathbb{R}^+;\mathbb{R}^+\big)$, *and let* $u \in AC^1\big([0,\omega];\mathbb{R}\big)$ *be a periodic function (i.e. (1.3) holds) such that*

$$u''(t) + g(u(t)) \leq h(t) \qquad \text{for a. e. } t \in [0,\omega]. \tag{2.16}$$

Let, moreover, $s_0, t_0 \in \mathbb{R}$ be such that $s_0 < t_0$ and (2.12) is fulfilled. Then

$$\int_{s_0}^{t_0} g(u(t))g_*^{p-1}(u(t))dt \leq \|h\|_p^p \tag{2.17}$$

where g_* is given by (2.11).

Proof 28 Multiplying both sides of (2.16) by $g_*^{p-1}(u(t))$ and integrating from s_0 to t_0, in view of (2.12) and according to Lemma 2.2.2, we arrive at

$$\int_{s_0}^{t_0} g(u(t))g_*^{p-1}(u(t))dt \leq \int_{s_0}^{t_0} h(t)g_*^{p-1}(u(t))dt \leq \|h\|_p \left(\int_{s_0}^{t_0} g_*^p(u(t))dt\right)^{(p-1)/p},$$

whence in view of (2.11) we get (2.17).

Lemma 2.2.4 *Let $u_i \in AC^1\big([0,\omega];\mathbb{R}_+\big)$ $(i = 1, 2)$ be fulfilling (1.3) such that*

$$\operatorname{meas}\big\{t \in [0,\omega] : u_i(t) = 0\big\} = 0 \qquad (i = 1, 2), \tag{2.18}$$

$$u_i''(t) + \widetilde{g}(u_i(t)) = h(t) \qquad \text{for a. e. } t \in [0,\omega] \quad (i = 1, 2), \tag{2.19}$$

where

$$\widetilde{g}(x) = \begin{cases} g(x) & \text{for } x > 0, \\ 0 & \text{for } x = 0, \end{cases} \tag{2.20}$$

$g \in C\big(\mathbb{R}^+;\mathbb{R}^+\big)$ is a decreasing function, and $h \in L\big([0,\omega];\mathbb{R}\big)$. Then $u_1 \equiv u_2$.

Proof 29 First assume that

$$u_1(t) \geq u_2(t) \qquad \text{for } t \in [0,\omega]. \tag{2.21}$$

Put

$$z(t) = u_1(t) - u_2(t) \qquad \text{for } t \in [0,\omega]. \tag{2.22}$$

Then, in view of (2.18)–(2.21) we have

$$z''(t) \geq 0 \qquad \text{for a. e. } t \in [0, \omega], \qquad z(0) - z(\omega) = 0 = z'(0) - z'(\omega). \tag{2.23}$$

However, (2.23) implies that z is a constant function, i.e., with respect to (2.18), (2.19), (2.20), and (2.22), we have

$$0 = u_1''(t) - u_2''(t) = -g(u_1(t)) + g(u_2(t)) \qquad \text{for a. e. } t \in [0, \omega]. \tag{2.24}$$

Now from (2.24), according to the fact that g is assumed to be decreasing, it follows that $u_1 \equiv u_2$.

Further suppose that (2.21) is not valid. Then there exist $t_0, t_1 \in \mathbb{R}$ such that $t_0 < t_1$ and

$$u_1(t) > u_2(t) \qquad \text{for } t \in (t_0, t_1), \qquad u_1(t_0) = u_2(t_0), \qquad u_1(t_1) = u_2(t_1). \tag{2.25}$$

Define z by (2.22). Then, in view of (2.18)–(2.20), and (2.25) we have

$$z''(t) \geq 0 \qquad \text{for a. e. } t \in [t_0, t_1], \qquad z(t_0) = 0, \qquad z(t_1) = 0. \tag{2.26}$$

However, (2.26) implies $z(t) \leq 0$ for $t \in [t_0, t_1]$, which, on account of (2.22) contradicts (2.25).

2.2.2 Main result

This subsection is devoted to prove the following theorem

Theorem 2.2.2 *Let* $h \in L^p\big([0, \omega]; \mathbb{R}\big)$ *and let* $g \in C\big(\mathbb{R}^+; \mathbb{R}^+\big)$ *verify* (2.6). *Let, more-*

over,

$$\lim_{x\to 0+}\int_x^1 g\left(w(s)\right) g_*^{p-1}\big(w(s)\big)ds = +\infty, \tag{2.27}$$

where

$$w(s) \stackrel{def}{=} s^{(2p-1)/p} \qquad \textit{for } s\in\mathbb{R}^+. \tag{2.28}$$

Then the problem (2.2), (1.3) has a positive solution if and only if $\overline{h}>0$*. If, in addition,* g *is a decreasing function, then such a solution is unique.*

Proof 30 If u is a positive solution to (2.2), (1.3) then the integration of (2.2) from 0 to ω results in

$$\int_0^\omega g(u(s))ds = \int_0^\omega h(s)ds,$$

whence it follows that $\overline{h}>0$ as the function g is positive.

Now suppose that $\overline{h}>0$. Together with (2.2), for every $k\in\mathbb{N}$, consider the auxiliary equation

$$u''(t)+g(u(t)) = h_k(t) \qquad \text{for a. e. } t\in[0,\omega], \tag{2.29}$$

where

$$h_k(t) = \begin{cases} k & \text{if } h(t)>k, \\ h(t) & \text{if } h(t)\le k, \end{cases} \qquad \text{for a. e. } t\in[0,\omega], \quad k\in\mathbb{N}. \tag{2.30}$$

Obviously,

$$h_k(t)\le h_m(t)\le h(t) \qquad \text{for a. e. } t\in[0,\omega], \quad k\le m, \tag{2.31}$$

$$|h_k(t)|\le|h(t)| \qquad \text{for a. e. } t\in[0,\omega], \quad k\in\mathbb{N}, \tag{2.32}$$

$$\lim_{k\to+\infty}\overline{h}_k = \overline{h}. \tag{2.33}$$

According to (2.6), (2.33), and $\overline{h} > 0$, there exist $x_0 > 0$ and $k_0 \in \mathbb{N}$ such that

$$g(x) \leq \overline{h}_{k_0} \qquad \text{for } \; x \geq x_0. \tag{2.34}$$

Let z be a solution to the Dirichlet problem

$$z''(t) = h_{k_0}(t) - \overline{h}_{k_0}, \qquad z(0) = 0, \qquad z(\omega) = 0 \tag{2.35}$$

and put

$$\beta(t) = \widetilde{z}(t) + r \qquad \text{for } \; t \in [0, \omega], \tag{2.36}$$

where $\widetilde{z}$ is an ω-periodic prolongation of z to the real axis and $r > 0$ is large enough such that

$$x_0 \leq \beta(t) \qquad \text{for } \; t \in [0, \omega]. \tag{2.37}$$

Obviously, $\beta \in AC^1\big([0,\omega];\mathbb{R}\big)$ and in view of (2.31) and (2.34)–(2.37),

$$\beta''(t) + g(\beta(t)) \leq h_k(t) \qquad \text{for a. e. } \; t \in [0, \omega], \quad k \geq k_0. \tag{2.38}$$

On the other hand, on account of (2.6) and (2.30), for every $k \geq k_0$ there exists $x_k \in (0, x_0)$ such that

$$g(x_k) \geq h_k(t) \qquad \text{for a. e. } \; t \in [0, \omega]. \tag{2.39}$$

If we put $\alpha_k(t) = x_k$ for $t \in [0, \omega]$ then, in view of (2.37) and (2.39), we have

$$\alpha_k''(t) + g(\alpha_k(t)) \geq h_k(t) \qquad \text{for a. e. } \; t \in [0, \omega], \quad k \geq k_0, \tag{2.40}$$

$$\alpha_k(t) \leq \beta(t) \qquad \text{for } \; t \in [0, \omega], \quad k \geq k_0. \tag{2.41}$$

Thus, for every $k \geq k_0$, there exists a pair of well-ordered upper and lower functions to the problem (2.2), (1.3), which is a particular case of the problem (1.1), (1.3). According to Proposition 1.2.1, there exists a sequence of solutions $\{u_k\}_{k=k_0}^{+\infty}$ to (2.29), (1.3) such that

$$0 < \alpha_k(t) \le u_k(t) \le \beta(t) \qquad \text{for } \ t \in [0,\omega]. \tag{2.42}$$

From Lemma 2.2.1 and (2.42) it follows that

$$\|u_k'\|_\infty \le \left(\frac{(2p-1)}{p} \|h\|_p \right)^{p/(2p-1)} \|\beta\|_\infty^{(p-1)/(2p-1)} \qquad \text{for } \ k \ge k_0. \tag{2.43}$$

Further, we show that the set of functions $\{u_k\}_{k=k_0}^{+\infty}$ is bounded from below. The integration of (2.29) from 0 to ω, in view of (2.31), yields

$$\int_0^\omega g(u_k(s))ds \le \omega \overline{h}. \tag{2.44}$$

On the other hand, (2.6) implies the existence of $y_0 > 0$ (which does not depend on k) such that

$$g(x) > \overline{h} \qquad \text{for } \ x \in (0, y_0). \tag{2.45}$$

From (2.44) and (2.45) it follows that for every $k \ge k_0$ we have

$$\|u_k\|_\infty \ge y_0. \tag{2.46}$$

Let $r_k \in [0,\omega]$ and $\xi_k \in [r_k - \omega, r_k)$ be such that

$$u_k(r_k) = \max\big\{u_k(t) : t \in [0,\omega]\big\}, \qquad u_k(\xi_k) = \min\big\{u_k(t) : t \in [0,\omega]\big\}. \tag{2.47}$$

Obviously, $u_k'(\xi_k) = 0$, $u_k'(r_k) = 0$, and in view of (2.46), (2.47), and Lemmas 2.2.1 and

2.2.3 we have

$$\frac{2p-1}{p}\int_{(u_k(\xi_k))^{p/(2p-1)}}^{y_0^{p/(2p-1)}} g\left(w(s)\right) g_*^{p-1}\big(w(s)\big)ds \le \int_{\xi_k}^{r_k} \frac{u_k'(t)g\left(u_k(t)\right) g_*^{p-1}\big(u_k(t)\big)}{\big(u_k(t)\big)^{(p-1)/(2p-1)}}dt \le$$
$$\left(\frac{(2p-1)}{p}\|h\|_p\right)^{p/(2p-1)} \int_{\xi_k}^{r_k} g\left(u_k(t)\right) g_*^{p-1}\big(u_k(t)\big)dt \le \left(\frac{2p-1}{p}\right)^{p/(2p-1)} \|h\|_p^{2p^2/(2p-1)},$$

where w is given by (2.28).

Thus the assumption (2.27) implies the existence of an $\varepsilon > 0$ such that

$$\varepsilon \le u_k(t) \qquad \text{for } t \in [0,\omega], \quad k \ge k_0. \tag{2.48}$$

Finally, using (2.32), (2.42), and (2.48), from (2.29) we obtain

$$|u_k''(t)| \le g^* + |h(t)| \qquad \text{for a. e. } t \in [0,\omega], \quad k \ge k_0$$

where

$$g^* = \max\big\{g(x) : x \in [\varepsilon, \|\beta\|_\infty]\big\}.$$

Thus, the sequences $\{u_k\}_{k=k_0}^{+\infty}$ and $\{u_k'\}_{k=k_0}^{+\infty}$ are uniformly bounded and equicontinuous. Therefore, according to *Arzelà–Ascoli Theorem*, we can assume without loss of generality that there exist $u_0, v_0 \in C\big([0,\omega];\mathbb{R}\big)$ such that

$$\lim_{k\to+\infty} \|u_k - u_0\|_\infty = 0, \qquad \lim_{k\to+\infty} \|u_k' - v_0\|_\infty = 0. \tag{2.49}$$

Moreover, since u_k are solutions to (2.29), (1.3), in view of (2.30), (2.48), and (2.49) we have $u_0 \in AC^1\big([0,\omega];\mathbb{R}\big)$, $u_0' \equiv v_0$, and u_0 is a positive solution to (2.2), (1.3).

The uniqueness of a solution in the case when g is a decreasing function follows from Lemma 3.3.1.

Remark 2.2.1 Always we can assume without lost of generality that a function $y : [0, \omega] \to \mathbb{R}$ is defined by periodicity on whole $\mathbb{R}$ using the introduced prolongation operators in (1.66) and (1.67). However, with the aim of keeping the exposition at a rather simple level we omite to write such an operators.

Remark 2.2.2 Note that assumption $h \in L^p\big([0, \omega]; \mathbb{R}\big)$ in Theorem 2.2.2 can be weakened to $h \in L\big([0, \omega]; \mathbb{R}\big)$, $[h]_+ \in L^p\big([0, \omega]; \mathbb{R}\big)$, where $[h]_+$ is a non-negative part of the function h, i.e.,

$$[h]_+(t) \stackrel{def}{=} \frac{|h(t)| + h(t)}{2} \qquad \text{for a. e. } t \in [0, \omega].$$

2.2.3 Optimality and Counter-Example

A particular case of the equation (2.2) is the equation (2.5) where $\lambda > 0$. For this equation, Theorems 2.2.1 and 2.2.2 yield the following assertions:

Corollary 2.2.1 *Let*

$$h \in L^\infty\big([0, \omega]; \mathbb{R}\big), \qquad \lambda > 0. \tag{2.50}$$

Then the problem (2.5), (1.3) has a positive solution if and only if $\overline{h} > 0$, and such a solution is unique.

Corollary 2.2.2 *Let $p \in [1, +\infty)$ and*

$$h \in L^p\big([0, \omega]; \mathbb{R}\big), \qquad \lambda \geq 1/(2p - 1). \tag{2.51}$$

Then the problem (2.5), (1.3) has a positive solution if and only if $\overline{h} > 0$, and such a solution is unique.

Remark 2.2.3 In spite of the fact that nor in Theorem 2.2.1 nor in the original result of Lazer and Solimini, the uniqueness of a positive solution to (2.2), (1.3) is discussed, Corollary 2.2.1 is valid—the uniqueness of a solution follows from Lemma 3.3.1.

Before we formulate other theorem, we introduce an example:

Counter–example 2.2.1 Let $p \geq 1$ and $\lambda \in \big(0, \frac{1}{2p-1}\big)$. Choose $\mu \in \big(2 - \frac{1}{p\lambda}, \frac{1}{p}\big)$, $\varepsilon \in \big(0, \frac{\omega}{4}\big)$, and put

$$\varphi(t) = \begin{cases} -t^{-\mu} & \text{for } t \in (0, \varepsilon], \\ 0 & \text{for } t \in \big(\varepsilon, \frac{\omega}{2} - \varepsilon\big), \\ \big(\frac{\omega}{2} - t\big)^{-\mu} & \text{for } t \in \big[\frac{\omega}{2} - \varepsilon, \frac{\omega}{2}\big), \end{cases} \qquad \varphi(t) = \varphi(\omega - t) \quad \text{for } t \in \left(\frac{\omega}{2}, \omega\right),$$

$$v(t) = \int_t^{\frac{\omega}{2}} \int_s^{\frac{\omega}{2}} \varphi(\xi) d\xi ds \qquad \text{for } t \in [0, \omega).$$

If we periodically extend the functions φ and v to the whole real axis, we obviously obtain

$$\varphi \in L^p\big([0, \omega]; \mathbb{R}\big), \quad v \in AC^1\big([0, \omega]; \mathbb{R}_+\big), \quad v''(t) = \varphi(t) \quad \text{for a. e. } t \in [0, \omega], \tag{2.52}$$

and by a direct calculation, the following relations can be verified:

$$v(t) > 0 \qquad \text{for } t \in [0, \omega/2) \cup (\omega/2, \omega], \qquad v(\omega/2) = 0, \tag{2.53}$$

$$v(t) = \frac{|\omega/2 - t|^{2-\mu}}{(2-\mu)(1-\mu)} \qquad \text{for } t \in \left(\frac{\omega}{2} - \varepsilon, \frac{\omega}{2} + \varepsilon\right).$$

Now it can be easily seen that

$$\frac{1}{v^\lambda} \in L^p\big([0, \omega]; \mathbb{R}\big). \tag{2.54}$$

Put

$$h(t) \stackrel{def}{=} \varphi(t) + \frac{1}{v^{\lambda}(t)} \qquad \text{for a. e. } t \in [0, \omega]. \tag{2.55}$$

Obviously, in view of (2.52)–(2.54) we have $h \in L^p\big([0,\omega];\mathbb{R}\big)$ and $\overline{h} > 0$. Consider the problem (2.5), (1.3) and suppose that there exists a positive solution u to (2.5), (1.3). According to (2.52), (2.53), (2.55), and Lemma 3.3.1, it follows that $u(t) = v(t)$ for $t \in [0, \omega]$. However, that is impossible, because $v(\omega/2) = 0$. Thus, (2.5), (1.3) has no positive solution with h defined by (2.55).

Example 2.2.1 proves the following assertion:

Theorem 2.2.3 *Let $p \in [1, +\infty)$, $0 < \lambda < 1/(2p-1)$. Then there exists $h \in L^p\big([0,\omega];\mathbb{R}\big)$ with $\overline{h} > 0$ such that the problem (2.5), (1.3) has no positive solution.*

According to Theorem 2.2.3, it can be seen that the condition (2.27) in Theorem 2.2.2 is essential and cannot be omitted. Moreover, Theorem 2.2.3 shows that the condition (2.51) in Corollary 2.2.2 is unimprovable.

Chapter 3

The Rayleigh-Plesset equation

In this section we will use our mathematical result in Chapter 1 to study the Rayleigh-Plesset equation, which models the oscillations of a spherical bubble in a liquid subjected to a periodic acoustic field. The Rayleigh-Plesset equation plays a prominent role in Dynamics of Fluids. It can be derived by taking spherical coordinates in Euler equations and assuming some physically admissible simplifications, as shown in many reviews and monographs (see for instance [6, 12, 23, 47, 58]). A variety of physical, biological and medical models rely on this equation (see bibliographies of the cited references), in connection with the physical phenomena of cavitation and sonoluminescence.

Following [23], the evolution in time of the radius $R(t)$ of the bubble is ruled by

$$\rho\left[R\ddot{R}+\frac{3}{2}\dot{R}^2\right]=[P_v-P_\infty(t)]+P_{g_0}\left(\frac{R_0}{R}\right)^{3k}-\frac{2S}{R}-\frac{4\mu\dot{R}}{R}. \tag{3.1}$$

Here, at the left-hand side $\dot{R}$ and $\ddot{R}$ are the first and second derivatives of the bubble radius with respect to time and ρ is the density of the liquid. At the right-hand side we have four different terms. The first one is $P_v - P_\infty(t)$, which measures the difference between the vapour pressure P_v inside the bubble and the applied pressure, which is time-periodic. The second term is related with the non-condensability of the gas. More exactly, P_{g_0} and R_0 correspond, respectively, to the gas pressure and initial radius of the bubble, while k is the polytropic coefficient, which contents information about thermic transmission behaviour of the system liquid–gas. If the behaviour is isothermal then the coefficient k is equal to one. The most usual case considered in the cited references is when polytropic coefficient is greater than or equal to one, but possibly it is any real number. In this paper, we consider the adiabatic case (when $k \geq 1$). The third terms corresponds to surface tension, i.e., the energy which is needed to increase the surface of a liquid by area unit. Finally, the last term corresponds to the viscosity of liquid.

When the surface tension and viscosity effects are neglected (a physically admissible simplification for bubbles of big radius), we may obtain the classical Rayleigh equation

$$\rho \left[R\ddot{R} + \frac{3}{2}\dot{R}^2 \right] = P_v - P_\infty(t),$$

which was proposed in 1907 by Rayleigh. Furthermore, we observe that when the applied pressure is constant, the Rayleigh equation has a first integral

$$\dot{R}^2 = \frac{2}{3}\frac{P_v - P_\infty}{\rho}\left[1 - \left(\frac{R_0}{R} \right)^3 \right].$$

Nevertheless, when the applied pressure $P_\infty(t)$ is time-varying, most of the present knowledge about the dynamics of this models is based on numerical computations.

If the change of variables $R = u^{\frac{2}{5}}$ is introduced in the Rayleigh-Plesset equation, we obtain

$$\ddot{u} = \frac{5\left[P_v - P_\infty(t) \right]}{2\rho} u^{\frac{1}{5}} + \left(\frac{5P_{g0}R_0^{3k}}{2\rho} \right) \frac{1}{u^{\frac{6k-1}{5}}} - \frac{5S}{u^{\frac{1}{5}}} - 4\mu\frac{\dot{u}}{u^{\frac{4}{5}}},$$

Consequently, the class of equations

$$u''(t) + \frac{cu'(t)}{u^\mu(t)} + \frac{g_1}{u^\nu(t)} - \frac{g_2}{u^\gamma(t)} = h_0(t)u^\delta(t) \qquad \text{for a. e. } t \in [0, \omega] \qquad (3.2)$$

with non-negative constants g_1, g_2, δ, $\nu > 0$ and real numbers c, μ, γ, and $h_0 \in$

$L\big([0,\omega];\mathbb{R}\big)$, plays an important role in fluid mechanics.

The aim of this chapter is to find periodic solutions to equations (3.1) and (3.2). For that it will be convenient to consider the general equation (1.1) and to use the proved results of Chapter 1.

3.1 The model equation

This part is devoted to study equation (3.2). For that we will consider three cases depending on the considered type of nonlinearity: repulsive case ($\gamma > \nu > 0$), attractive case ($0 < \gamma < \nu$) and the case $\gamma \leq 0$.

3.1.1 The repulsive case

In this part we consider the equation (3.2) with repulsive singularity, concretely when $\nu < \gamma$. From Theorem 1.1.3 it finds a direct application to (3.2) in the sublinear case $\delta < 1$.

Theorem 3.1.1 *Let us assume* $0 \leq \delta < 1$, $\gamma > \nu$, $\gamma \geq 1$, $g_2 > 0$. *If* $\overline{h}_0 \leq 0$ *and* $g_1 + |\overline{h}_0| > 0$, *then there exists at least one positive solution to the problem* (3.2), (1.3).

Proof 31 It can be proved by applying Theorem 1.1.3 with $\eta(t,x) = [h_0(t)]_- x^\delta$, $\rho(x) = x^\delta$ and $h(t,x) = h_0(t)x^\delta$. Indeed, hypotheses 1, 3, 4, 5, and 6 of Theorem 1.1.3 are straightforward. Finally, hypothesis 2 one can easily prove by using the inequality $g_1 + \big|\overline{h}_0\big| > 0$.

The linear case $\delta = 1$ is also covered by Theorem 1.1.3 as follows.

Theorem 3.1.2 *Let us assume* $\delta = 1$, $\gamma > \nu$, $\gamma \geq 1$, $g_2 > 0$. *If* $\overline{h}_0 \leq 0$, $g_1 + |\overline{h}_0| > 0$ *and*

$$\int_0^\omega [h_0(s)]_- ds < \frac{4}{\omega}, \tag{3.3}$$

then there exists at least one positive solution to the problem (3.2), (1.3).

Proof 32 It can be proved by applying Theorem 1.1.3 with $\eta(t,x) = [h_0(t)]_- x$, $\rho(x) = x$, $h(t,x) = h_0(t)x$ and reasoning as we did in Theorem 3.1.1.

Such as Condition *1.* in Theorem 1.1.1, the above-mentioned condition (3.3) is used to avoid the resonant phenomenon at infinity. In spite of both conditions are used to the some work, they are independents and the best in some sense.

We previously mentioned the importance to take into account the strong singular condition ($\gamma \geq 1$) when (3.2) is considered. However, when h_0 verifies

$$h_0(t) \leq \sup\left\{\frac{g_1}{x^{\nu+\delta}} - \frac{g_2}{x^{\gamma+\delta}} : x \in \mathbb{R}^+\right\} \qquad \text{for a. e. } t \in [0,\omega], \tag{3.4}$$

one can find a constant lower function, taking $\alpha \equiv r_0 \in \mathbb{R}^+$ such that

$$\frac{g_1}{r_0^{\nu+\delta}} - \frac{g_2}{r_0^{\gamma+\delta}} = \sup\left\{\frac{g_1}{x^{\nu+\delta}} - \frac{g_2}{x^{\gamma+\delta}} : x \in \mathbb{R}^+\right\}.$$

Combining this with the obtained results to construct lower and upper functions, we can find an ordered pair of lower and upper functions under some conditions.

Theorem 3.1.3 *Let* $0 \leq \delta < 1$, $\gamma > \nu$, $g_1 > 0$ *and* $g_2 > 0$. *If* $\overline{h}_0 > 0$ *and* (3.4), *then there exists at least one positive solution to the problem* (3.2), (1.3).

Proof 33 In view of the previous discussion, we only need to prove the existence of an upper function.

To prove such a existence, we apply Corollary 1.2.5 taking $\{y_n\}_{n=1}^{+\infty}$ an arbitrary sequence of positive numbers satisfying (1.153), $c \in (0, \overline{h}_0)$, $\rho_1(x) = x^\delta$, $h_1 \equiv h_0$, $x_0 > r_0$ large enough, and $\varepsilon > 0$ small enough such that

$$(1+\varepsilon)^\delta \Phi_- \leq \Phi_+. \tag{3.5}$$

Consequently, the assertion follows from Proposition 1.2.1.

Theorem 3.1.4 *Let $\delta = 1$, $\gamma > \nu$, $g_1 > 0$ and $g_2 > 0$. Assume that $\overline{h}_0 > 0$,*

$$h_0(t) \leq \sup\left\{\frac{g_1}{x^{\nu+1}} - \frac{g_2}{x^{\gamma+1}} : x \in \mathbb{R}^+\right\} \qquad \text{for a. e. } t \in [0,\omega], \tag{3.6}$$

and

$$\frac{\omega}{4}\int_0^\omega [h_0(s)]_+ ds \int_0^\omega [h_0(s)]_- ds < \int_0^\omega [h_0(s)]_+ ds - \int_0^\omega [h_0(s)]_- ds. \tag{3.7}$$

Then there exists at least one positive solution to the problem (3.2), (1.3).

Proof 34 As in the proof of Theorem 3.1.3, we can check that there exists a constant $r_0 \in \mathbb{R}^+$ such that $\alpha(t) = r_0$ for $t \in [0,\omega]$ is a lower function to the problem (3.2), (1.3). On the other hand, from (3.7) it follows that there exists a sufficiently small constant $c > 0$ such that $c < \overline{h}_0$ and

$$\frac{\omega}{4}\Phi_+\Phi_- \leq \Phi_+ - \Phi_-,$$

where

$$\varphi(t) = h_0(t) - c \quad \text{for a. e. } t \in [0,\omega].$$

Therefore, if we put $\rho_1(x) = x$ and $h_1 \equiv h_0$, taking into account that

$$\lim_{x\to+\infty} \frac{g_1}{x^{1+\nu}} - \frac{g_2}{x^{1+\gamma}} = 0,$$

the existence of an upper function large enough follows from Corollary 1.2.6.

Consequently, the assertion follows from Proposition 1.2.1.

Theorem 3.1.5 *Let $\delta > 1$, $\gamma > \nu$, $g_1 > 0$ and $g_2 > 0$. If*

$$0 \le h_0(t) \le \sup \left\{ \frac{g_1}{x^{\nu+\delta}} - \frac{g_2}{x^{\gamma+\delta}} : x \in \mathbb{R}^+ \right\} \qquad \text{for a. e. } t \in [0,\omega] \tag{3.8}$$

and $\overline{h}_0 > 0$, then there exists at least one positive solution to the problem (3.2), (1.3).

Proof 35 Analogously to the previous proofs, there exists a constant $r_0 \in \mathbb{R}^+$ such that the function $\alpha(t) = r_0$ for $t \in [0,\omega]$ is a lower function to the problem (3.2), (1.3).

On the other hand, from the first inequality of (3.8) it follows that $h_0(t)x^\delta \ge h_0(t)$ for almost every $t \in [0,\omega]$ and $x \ge 1$. Thus, the existence of an upper function to the problem (3.2), (1.3) follows from Corollary 1.2.5 by taking $h_1 \equiv h_0$, $\rho_1 \equiv 1$, an arbitrary sequence $\{y_n\}_{n=1}^{+\infty}$ of positive numbers such that (1.153) holds, $c \in (0, \overline{h}_0]$, $\varepsilon > 0$ arbitrary and $x_0 > r_0$ large enough.

Consequently, the assertion follows from Proposition 1.2.1.

3.1.2 The attractive case

We will consider (3.2) with attractive singularity (i.e. $\nu > \gamma$).

Theorem 3.1.6 *Let $0 \le \delta < 1$, $\gamma < \nu$ and $g_1 > 0$. If $\overline{h}_0 > 0$ and*

$$\text{ess sup} \left\{ h_0(t) : t \in [0,\omega] \right\} < +\infty, \tag{3.9}$$

then there exists at least one positive solution to the problem (3.2), (1.3).

Proof 36 According to (3.9), we can choose $K > 0$ such that

$$K \geq h_0(t) \qquad \text{for a. e. } t \in [0, \omega].$$

As $\lim\limits_{x \to 0+} \frac{g_1}{x^\nu} - \frac{g_2}{x^\gamma} = +\infty$, there exits $x_1 > 0$ such that

$$\frac{g_1}{x_1^{\nu+\delta}} - \frac{g_2}{x_1^{\gamma+\delta}} \geq K.$$

Obviously, $\alpha \equiv x_1$ is a constant lower function to the problem (3.2), (1.3).

To prove the existence of an upper function we apply Corollary 1.2.5 taking $\{y_n\}_{n=1}^{+\infty}$ an arbitrary sequence of positive numbers satisfying (1.153), $c \in (0, \overline{h}_0)$, $\rho_1(x) = x^\delta$, $h_1 \equiv h_0$, $x_0 > K$ large enough, and $\varepsilon > 0$ small enough such that (3.5) holds.

Consequently, the assertion follows from Proposition 1.2.1.

Theorem 3.1.7 *Let $0 \leq \delta < 1$, $\gamma < \nu$, $g_1 > 0$ and $g_2 > 0$. If $\overline{h}_0 \leq 0$, (3.9) is fulfilled and*

$$h_0(t) \geq \inf \left\{ \frac{g_1}{x^{\nu+\delta}} - \frac{g_2}{x^{\gamma+\delta}} : x \in \mathbb{R}^+ \right\} \qquad \textit{for a. e. } t \in [0, \omega], \tag{3.10}$$

then there exists at least one positive solution to the problem (3.2), (1.3).

Proof 37 In this case,

$$\inf \left\{ \frac{g_1}{x^{\nu+\delta}} - \frac{g_2}{x^{\gamma+\delta}} : x \in \mathbb{R}^+ \right\} > -\infty$$

and there exists $x_0 > 0$ such that

$$\frac{g_1}{x_0^{\nu+\delta}} - \frac{g_2}{x_0^{\gamma+\delta}} = \inf \left\{ \frac{g_1}{x^{\nu+\delta}} - \frac{g_2}{x^{\gamma+\delta}} : x \in \mathbb{R}^+ \right\}.$$

According to (3.10), it can be easily verified that the function $\beta(t) = x_0$ for $t \in [0, \omega]$ is an upper function to the problem (3.2), (1.3).

To obtain a lower function, we proceed as in the proof of Theorem 3.1.6 choosing x_1 small enough.

Consequently, the assertion follows from Proposition 1.2.1.

Remark 3.1.1 Note that the conditions guaranteeing solvability of the problem (3.2), (1.3) in the case where $0 \le \delta < 1$, $\gamma = \nu$, and $g_1 > g_2$ can be derived from Theorem 3.1.6. However, in that case $\gamma = \nu$ and $g_1 < g_2$, only the conditions sufficient for the existence of non–ordered lower and upper functions are known to the authors. Thus our analysis is incomplete and the case $\gamma = \nu < 1$, $g_1 < g_2$ remains as an open problem.

Theorem 3.1.8 *Let $\delta = 1$, $g_1 > 0$ and $g_2 = 0$. If $\overline{h}_0 > 0$ and (3.7) is fulfilled, then there exists at least one positive solution to the problem (3.2), (1.3).*

Proof 38 Put $\rho_0(x) = x$. According to Lemma 1.2.3 we can choose $c_0 > \overline{h}_0$ large enough such that the condition b) of Theorem 1.2.1 is fulfilled. Obviously, also $r_0 > 0$ can be chosen such that (1.159) is satisfied.

On the other hand, put $h_1 \equiv h_0$ and $\rho_1(x) = x$. Then, in view of (3.7), there exists a constant $c_1 > 0$ such that $c_1 \le \overline{h}_1$ and the condition d) of Theorem 1.2.1 and (1.160) are fulfilled with a suitable $r_1 > r_0$.

Consequently, the assertion follows from Theorem 1.2.1.

Theorem 3.1.9 *Let $\delta = 1$, $\gamma < \nu$, $g_1 > 0$ and $g_2 > 0$. If $\overline{h}_0 \ge 0$ and*

$$\frac{\omega}{4} \int_0^\omega [h_0(s)]_+ ds \int_0^\omega [h_0(s)]_- ds \le \int_0^\omega [h_0(s)]_+ ds - \int_0^\omega [h_0(s)]_- ds, \tag{3.11}$$

then there exists at least one positive solution to the problem (3.2), (1.3).

Proof 39 The proof is similar to that of Theorem 3.1.8. The only difference is that the inequality $g_1 x^{-(\nu+1)} - g_2 x^{-(\gamma+1)} < 0$ for x sufficiently large allows one to choose a constant c_1 equal to zero.

Theorem 3.1.10 *Let $\delta = 1$, $\gamma < \nu$, $g_1 > 0$ and $g_2 > 0$. If $\overline{h}_0 \leq 0$ and*

$$h_0(t) \geq \inf\left\{\frac{g_1}{x^{\nu+1}} - \frac{g_2}{x^{\gamma+1}} : x \in \mathbb{R}^+\right\} \qquad \text{for a. e. } t \in [0,\omega], \tag{3.12}$$

then there exists at least one positive solution to the problem (3.2), (1.3).

Proof 40 As in the proof of Theorem 3.1.7, we can verify that there exists a constant $x_1 > 0$ such that $\beta(t) = x_1$ for $t \in [0,\omega]$ is an upper function to the problem (3.2), (1.3).

On the other hand, put $\rho_0(x) = x$, and choose $c \geq \overline{h}_0$ and $x_0 > 0$ such that $x_0 < x_1$ and the conditions of Corollary 1.2.3 are fulfilled. Note that the existence of $c \geq \overline{h}_0$ large enough such that (1.142) holds follows from Lemma 1.2.3. Therefore, there exists a lower function.

Consequently, the assertion follows from Proposition 1.2.1.

Theorem 3.1.11 *Let $1 < \delta$, $\gamma < \nu$ and $g_1 > 0$, $g_2 > 0$. If* (3.10) *is fulfilled, then there exists at least one positive solution to the problem* (3.2), (1.3).

Proof 41 The upper function is constructed as in the proof of Theorem 3.1.7.

On the other hand, put $\rho_0(x) = x^\delta$, and choose $\{y_n\}_{n=1}^{+\infty}$ a sequence of posivite numbers satisfying (1.137), $c > \overline{h}_0$, and $\varepsilon \in (0,1)$ such that

$$(1-\varepsilon)^{-\delta}\Phi_+ \leq \Phi_-.$$

Then there exists $x_0 > 0$ sufficiently small such that all the conditions of Corollary 1.2.2 are fulfilled. Consequently, there exists a lower function $\alpha(t) \le x_0$.

Now the assertion follows from Proposition 1.2.1.

3.1.3 The case $\gamma \le 0$

Finally two results dealing with the problem (3.2), (1.3) in the case when the parameter γ is non–positive. This case is also interesting from the physical point of view.

Theorem 3.1.12 *Let $0 \le \delta < 1$, $-\gamma > \delta$, $g_1 > 0$ and $g_2 > 0$. If (3.9) is fulfilled, then there exists at least one positive solution to the problem (3.2), (1.3).*

Proof 42 The assertion immediately follows from Theorem 1.2.1 b) and c) with $h(t,x) = h_0(t)x^\delta$, $h_1 \equiv h_0$, $\rho_i(x) = x^\delta$ $(i = 0,1)$, $c_0 = \text{ess sup}\left\{h_0(t) : t \in [0,\omega]\right\}$, $c_1 = \overline{h}_0 - 1$, and $g(x) = g_1 x^{-\nu} - g_2 x^{-\gamma}$.

Theorem 3.1.13 *Let $0 \le \delta < 1$, $\gamma \le 0$, $|\gamma| \le \delta$, $g_1 > 0$ and $g_2 > 0$, and $\overline{h}_0 > 0$. If (3.9) is fulfilled, then there exists at least one positive solution to the problem (3.2), (1.3).*

Proof 43 The assertion immediately follows from Theorem 1.2.1 b) and c) with $h(t,x) = h_0(t)x^\delta$, $h_1 \equiv h_0$, $\rho_i(x) = x^\delta$ $(i = 0,1)$, $c_0 = \text{ess sup}\left\{h_0(t) : t \in [0,\omega]\right\}$, $c_1 = 0$, and $g(x) = g_1 x^{-\nu} - g_2 x^{-\gamma}$.

3.2 The physical model

A direct application of Theorem 3.1.1 gives the following result.

Theorem 3.2.1 *Let us assume $k \geq 1$ and $P_v \leq \overline{P}_\infty$. Then there exists at least one positive periodic solution to the equation* (3.1).

As far as we know, this is the first analytical proof of a well-known numerical evidence exposed in many related works, see for instance [23].

Also, Theorem 3.1.3 implies

Theorem 3.2.2 *Let $k > \frac{1}{3}$, $P_v > \overline{P}_\infty$ and*

$$\frac{5(P_v - P_\infty(t))}{2\rho} \leq \left(\frac{6k-2}{5}\right)\left[\frac{(\frac{2}{5})^{\frac{2}{5}}(5S)^{\frac{6k}{5}}}{(\frac{6k}{5})^{\frac{6k}{5}}\left(\frac{5P_{g0}R_0^{3k}}{2\rho}\right)^{\frac{2}{5}}}\right]^{\frac{5}{6k-2}} \qquad \textit{for } t \in [0,\omega].$$

Then there exists at least one positive periodic solution to the equation (3.1).

Theorems 3.1.6 and 3.1.7, respectively, lead to

Theorem 3.2.3 *Let $\frac{1}{6} < k < \frac{1}{3}$, $P_v > \overline{P}_\infty$, and*

$$\text{ess inf}\left\{P_\infty(t) : t \in [0,\omega]\right\} > -\infty. \tag{3.13}$$

Then there exists at least one positive periodic solution to the equation (3.1).

Theorem 3.2.4 *Let $\frac{1}{6} < k < \frac{1}{3}$, $P_v \leq \overline{P}_\infty$,* (3.13) *holds, and*

$$\frac{5(P_v - P_\infty(t))}{2\rho} \geq -\left(\frac{2-6k}{5}\right)\left[\frac{(\frac{6k}{5})^{\frac{6k}{5}}\left(\frac{5P_{g0}R_0^{3k}}{2\rho}\right)^{\frac{2}{5}}}{(\frac{2}{5})^{\frac{2}{5}}(5S)^{\frac{6k}{5}}}\right]^{\frac{5}{2-6k}} \qquad \textit{for } t \in [0,\omega].$$

Then there exists at least one positive periodic solution to the equation (3.1).

Applying Theorem 3.1.6 with $g_1 = 5S - \frac{5P_{g_0}R_0^{3k}}{2\rho}$, $g_2 = 0$, and $\nu = 1/5$ we get

Theorem 3.2.5 *Let $k = \frac{1}{3}$, $P_v > \overline{P}_\infty$, $2\rho S > P_{g_0}R_0^{3k}$, and let (3.13) holds. Then there exists at least one positive periodic solution to equation (3.1).*

Remark 3.2.1 The open problem posed in Remark 3.1.1 corresponds to this last result when $2\rho S < P_{g_0}R_0^{3k}$.

Applying Theorems 3.1.12 and 3.1.13, we get, respectively,

Theorem 3.2.6 *Let $k < 0$ and let (3.13) holds. Then there exists at least one positive periodic solution to equation (3.1).*

Theorem 3.2.7 *Let $0 \leq k \leq \frac{1}{6}$, $P_v > \overline{P}_\infty$, and let (3.13) holds. Then there exists at least one positive periodic solution to equation (3.1).*

Let us finish by pointing out the presented results have a direct physical reading. For example, we can conclude that as a general rule a high density coefficient ρ of the liquid should benefit the presence of oscillating bubbles, an effect that seems physically plausible.

3.3 Playing with singularities

Taking into account that (2.1) is a particular equation of (3.2), we will use the studied theory in the previous chapter in order to motivate some questions. When $\nu > \gamma$, $g_1 > 0$, having in mind the equation (2.5), one can expect that either $\nu \geq 1$ or (3.9) is needed. However we should think that (3.2) has also singularity at the friction type term $c\frac{u'}{u^\mu}$

and a new sub-linear term when $\delta \in [0,1)$. Playing with these elements we can weaken the attractive singularity even when $h_0 \in L\big([0,\omega];\mathbb{R}\big)$.

To find a novel relation between the nonlinearities of (3.2) will be our objective, showing the following general property: *to consider singularity at friction type term helps to periodic solution exits.*

We consider the general equation (1.1) considered at the beginning of this work. By using some previous results proved in Chapter 1 (Section 1.2) and other ones news, we will obtain a general existence result to (1.1), (1.3) which we will set out in an independent section.

3.3.1 Main result and consequences

The next result is based on a new method of lower and upper functions which will be proven below.

Theorem 3.3.1 *Let $\rho_0 \in C^1\big(\mathbb{R}^+;\mathbb{R}^+\big)$ and $\rho_1 \in C\big(\mathbb{R}^+;\mathbb{R}^+\big)$ be non-decreasing functions, $h_0, h_1 \in L\big([0,\omega];\mathbb{R}\big)$, and $x_0 > 0$ be such that*

$$h_1(t)\rho_1(x) \leq h(t,x) \leq h_0(t)\rho_0(x) \qquad \text{for a. e. } t \in [0,\omega], \quad x \geq x_0, \tag{3.14}$$

and let there exist $c_0, c_1 \in \mathbb{R}$ such that

$$\frac{g(x)}{\rho_0(x)} \leq c_0 < \overline{h}_0 \qquad \text{for } x \geq x_0, \tag{3.15}$$

$$\frac{g(x)}{\rho_1(x)} \leq c_1 \leq \overline{h}_1 \qquad \text{for } x \geq x_0. \tag{3.16}$$

Let, moreover, there exist $\lambda \in [0,1]$ *such that*

$$\int_0^1 \frac{ds}{\rho_0^\lambda(s)} < +\infty, \tag{3.17}$$

$$\lim_{x \to 0_+} \frac{g(x)}{\rho_0^\lambda(x)} = +\infty, \tag{3.18}$$

and let either

$$\int_0^1 \left(\frac{[f(s)]_+}{\rho_0^\lambda(s)} + \frac{[g(s)]_+}{\rho_0^{2\lambda}(s)} \right) ds = +\infty, \qquad \int_0^1 \frac{[f(s)]_-}{\rho_0^\lambda(s)} ds < +\infty \tag{3.19}$$

or

$$\int_0^1 \left(\frac{[f(s)]_-}{\rho_0^\lambda(s)} + \frac{[g(s)]_+}{\rho_0^{2\lambda}(s)} \right) ds = +\infty, \qquad \int_0^1 \frac{[f(s)]_+}{\rho_0^\lambda(s)} ds < +\infty. \tag{3.20}$$

Furthermore, let us suppose that ρ_0 *fulfills at least one of the following conditions:*

(a) there exists a sequence $\{y_n\}_{n=1}^{+\infty}$ *of positive numbers such that*

$$\lim_{n \to +\infty} y_n = +\infty, \qquad \lim_{n \to +\infty} \frac{\rho_0^{1-\lambda}(y_n)}{\sigma(y_n)} = 0, \tag{3.21}$$

and there exist $\varepsilon_0 > 0$, $\varepsilon_1 \in (0, \varepsilon_0]$, *and* $n_0 \in \mathbb{N}$ *such that*

$$\frac{\rho_0^{1-\lambda}((1+\varepsilon_0)y_n)}{\rho_0^{1-\lambda}(y_n)} \Phi_- \leq \Phi_+ - \varepsilon_0 \qquad \text{for } n \geq n_0, \tag{3.22}$$

$$(1+\varepsilon_1)\sigma(y_n) \leq \sigma((1+\varepsilon_0)y_n) \qquad \text{for } n \geq n_0, \tag{3.23}$$

where $\varphi(t) = h_0(t) - c_0$ *for almost every* $t \in [0, \omega]$ *and*

$$\sigma(x) = \int_0^x \frac{ds}{\rho_0^\lambda(s)}; \tag{3.24}$$

(b) the function $\frac{\rho_0^{1-\lambda}(x)}{\sigma(x)}$ *is non–increasing and*

$$\frac{\omega}{4}\Phi_+\Phi_-\frac{\rho_0^{1-\lambda}(x_0)}{\sigma(x_0)} < \Phi_+ - \Phi_-, \tag{3.25}$$

where $\varphi(t) = h_0(t) - c_0$ *for almost every* $t \in [0, \omega]$ *and* σ *is given by* (3.24).

Besides, let us suppose that ρ_1 *fulfills at least one of the following conditions:*

(c) there exists a sequence $\{z_n\}_{n=1}^{+\infty}$ *of positive numbers such that*

$$\lim_{n\to+\infty} z_n = +\infty, \qquad \lim_{n\to+\infty} \frac{\rho_1(z_n)}{z_n} = 0,$$

and there exist $\varepsilon_2 > 0$ *and* $n_1 \in \mathbb{N}$ *such that*

$$\frac{\rho_1(z_n(1+\varepsilon_2))}{\rho_1(z_n)}\Psi_- \leq \Psi_+ \qquad \text{for } n \geq n_1,$$

where $\psi(t) = h_1(t) - c_1$ *for almost every* $t \in [0, \omega]$*;*

(d) the function $\frac{\rho_1(x)}{x}$ *is non–increasing and*

$$\frac{\omega}{4}\Psi_+\Psi_-\frac{\rho_1(x_0)}{x_0} \leq \Psi_+ - \Psi_-,$$

where $\psi(t) = h_1(t) - c_1$ *for almost every* $t \in [0, \omega]$.

Then there exists at least one solution to the problem (1.1), (1.3).

Remark 3.3.1 Under the assumptions of Theorem 2.2.2, note that there exists a suitable ε_1 such that (3.23) holds, e.g., if

$$\limsup_{x\to+\infty} \frac{\rho_0^{\lambda}((1+\varepsilon_0)x)}{\rho_0^{\lambda}(x)} < 1 + \varepsilon_0.$$

Indeed, if (3.21) holds then

$$\int_0^{+\infty} \frac{ds}{\rho_0^\lambda(s)} = +\infty,$$

so that using the previous part (3.23) is proven by a classical arguments.

For the equation (3.2), from Theorem 3.3.1 we get the following assertion.

Corollary 3.3.1 *Let* $g_1 > 0$, $g_2 \geq 0$, $0 \leq \delta < 1$, $\nu > \gamma$ *and*

$$\textit{either} \qquad (\mu + \delta)\,\mathrm{sgn}\,|c| \geq 1 \qquad \textit{or} \qquad \nu + 2\delta \geq 1.$$

If

$$\overline{h}_0 > -\lim_{x \to +\infty} \frac{g_2}{x^{\gamma+\delta}},$$

then (3.2), (1.3) *has at least one solution.*

Remark 3.3.2 In the previous chapter, there is proven, among others, that the equation

$$u'' + \frac{1}{u^\nu} = h_0(t) \tag{3.26}$$

with $h_0 \in L\big([0,\omega];\mathbb{R}\big)$ and $\overline{h}_0 > 0$, has a positive ω-periodic solution if $\nu \geq 1$. Moreover, there is also introduced an example showing that for any $\nu \in (0,1)$, there exists $h_0 \in L\big([0,\omega];\mathbb{R}\big)$ with $\overline{h}_0 > 0$ such that (3.26), (1.3) has no positive solution.

Corollary 3.3.1 says that if a friction-like term or sub-linear term are added to (3.26), the condition $\nu \geq 1$ can be weakened. For example

$$u'' + \frac{u'}{u^\mu} + \frac{1}{u^\nu} = h_0(t)$$

has a positive solution satisfying (1.3) for any $\nu > 0$ if $\mu \geq 1$, provided $\overline{h}_0 > 0$. Also the equation

$$u'' + \frac{1}{u^\nu} = h_0(t)u^\delta$$

subjected to the boundary conditions (1.3) is solvable for any $\nu > 0$ if $\delta \in [1/2, 1)$, provided $\overline{h}_0 > 0$.

Corollary 3.3.2 *Let $g_1 > 0$, $g_2 \geq 0$, $\nu > \gamma$. Let, moreover, either $g^* = -\infty$ or*

$$\overline{h}_0 > g^* > -\infty, \qquad \frac{\omega}{4}\Phi_+\Phi_- < \Phi_+ - \Phi_-, \tag{3.27}$$

where $\varphi(t) = h_0(t) - g^*$ for almost every $t \in [0, \omega]$, and

$$g^* = -\lim_{x\to+\infty} \frac{g_2}{x^{\gamma+1}}.$$

Then the problem (3.2), (1.3) with $\delta = 1$ has at least one solution.

Remark 3.3.3 According to [30] and Theorem 2.2.1, it can be easily verified that the problem

$$u'' + \frac{g_1}{u^\nu} = h_0(t)u; \qquad u(0) = u(\omega), \quad u'(0) = u'(\omega) \tag{3.28}$$

with $g_1 > 0$ and $\nu > 0$, has a positive solution if and only if the inclusion $\mathcal{L}[0, -h_0] \in V^-$ holds (see notation in [30]).

Indeed, according to [30, Definition 1.1], the inclusion $\mathcal{L}[0, -h_0] \in V^-$ implies the existence of a positive solution v to the problem

$$v'' = h_0(t)v - g_1; \qquad u(0) = u(\omega), \quad u'(0) = u'(\omega).$$

Therefore there exist $x > 0$ and $y > 0$ such that $x^{1+\nu} \leq v(t) \leq y^{1+\nu}$ for $t \in [0, \omega]$. By setting

$$\alpha(t) \stackrel{def}{=} \frac{v(t)}{y^\nu}, \qquad \beta(t) \stackrel{def}{=} \frac{v(t)}{x^\nu} \qquad \text{for } t \in [0,\omega],$$

one can easily realized that α and β are lower and upper functions to (3.28), respectively, satisfying (2.29). Now the existence of a positive solution to (3.28) follows from Theorem 2.2.1.

On the other hand, the existence of a positive solution to (3.28) implies the inclusion $\mathcal{L}[0,-h_0] \in V^-$ (see [30, Theorem 2.1]).

However, one of the optimal effective conditions guaranteeing such an inclusion is $h_0 \not\equiv 0$ and

$$\frac{\omega}{4}\int_0^\omega [h_0(s)]_+ ds \int_0^\omega [h_0(s)]_- ds \leq \int_0^\omega [h_0(s)]_+ ds - \int_0^\omega [h_0(s)]_- ds.$$

(see [30, Corollary 2.5]). Therefore, the condition (3.27) is natural, in a certain sense.

When the right-hand side of the equation (1.3) does not depend on u, i.e., when $h(t,x) \equiv h_0(t)$, then (1.3) has the form (1.2). From Theorem 3.3.1, for the equation (1.2) we get the following assertion.

Corollary 3.3.3 *Let there exist $x_0 > 0$ and $c_0 \in \mathbb{R}$ such that*

$$g(x) \leq c_0 < \overline{h}_0 \qquad \textit{for } x \geq x_0$$

and let

$$\lim_{x\to 0_+} g(x) = +\infty.$$

Let, moreover, either

$$\int_0^1 ([f(s)]_+ + [g(s)]_+)\, ds = +\infty, \qquad \int_0^1 [f(s)]_- ds < +\infty$$

or

$$\int_0^1 \left([f(s)]_- + [g(s)]_+\right) ds = +\infty, \qquad \int_0^1 [f(s)]_+ ds < +\infty.$$

Then there exists at least one solution to the problem (1.2), (1.3).

In the following result, the assumptions do not depend on the friction-like term. On the other hand, a certain smallness of oscillation of the primitive to h_0 is supposed. Clearly, Theorems 3.3.1 and 3.3.2 are independent.

Theorem 3.3.2 *Let $\rho_0 \in C^1\big(\mathbb{R}^+;\mathbb{R}^+\big)$ and $\rho_1 \in C\big(\mathbb{R}^+;\mathbb{R}^+\big)$ be non–decreasing functions, $h_0, h_1 \in L\big([0,\omega];\mathbb{R}\big)$, and $0 < x_0 \le x_1 < +\infty$ be such that*

$$h(t,x) \le h_0(t)\rho_0(x) \qquad \text{for a. e. } t \in [0,\omega], \quad 0 < x \le x_0, \tag{3.29}$$

$$h(t,x) \ge h_1(t)\rho_1(x) \qquad \text{for a. e. } t \in [0,\omega], \quad x \ge x_1. \tag{3.30}$$

Let, moreover,

$$\frac{\omega}{8}\|h_0 - \overline{h}_0\|_1 < \int_0^{x_0} \frac{ds}{\rho_0(s)} < +\infty \tag{3.31}$$

$$\frac{g(x)}{\rho_0(x)} \ge \overline{h}_0 \qquad \text{for } 0 < x \le x_0, \tag{3.32}$$

and let there exist $c_1 \in \mathbb{R}$ such that

$$\frac{g(x)}{\rho_1(x)} \le c_1 \le \overline{h}_1 \qquad \text{for } x \ge x_1. \tag{3.33}$$

Besides, let us suppose that ρ_1 fulfills at least one of the conditions (c) or (d) of Theorem 3.3.1. Then there exists at least one solution to the problem (1.1), (1.3).

In particular case when the equation (1.1) has the form (3.2), the following assertion immediately follows from Theorem 3.3.2.

Corollary 3.3.4 *Let $0 \le \delta < 1$, and let $0 < x_0 \le x_1 < +\infty$ be such that*

$$(1-\delta)\frac{\omega}{8}\|h_0 - \overline{h}_0\|_1 < x_0^{1-\delta}.$$

Let, moreover,

$$\frac{g_1}{x^{\nu+\delta}} - \frac{g_2}{x^{\gamma+\delta}} \ge \overline{h}_0 \qquad \text{if} \qquad 0 < x \le x_0,$$
$$\frac{g_1}{x^{\nu+\delta}} - \frac{g_2}{x^{\gamma+\delta}} \le \overline{h}_0 \qquad \text{if} \qquad x \ge x_1.$$

Then the problem (3.2), (1.3) has at least one solution.

Remark 3.3.4 The consequence of Theorem 3.3.2 for the problem (1.2), (1.3) coincides with the obtained result Theorem 1.2.2.

3.3.2 Auxiliary propositions

In what follows we will show the existence of a solution to the equation

$$u''(t) + f(u(t))u'(t) + g(u(t)) = h_0(t)\rho_0(u(t)) \qquad \text{for a. e. } t \in [0,\omega], \tag{3.34}$$

satisfying the boundary conditions (1.3). Here, $\rho_0 \in C(\mathbb{R}^+;\mathbb{R}^+)$ is a non-decreasing function, $h_0 \in L([0,\omega];\mathbb{R})$, and $f, g \in C(\mathbb{R}^+;\mathbb{R})$. Together with (3.34), for every $k \in \mathbb{N}$, consider the auxiliary equation

$$u''(t) + f(u(t))u'(t) + g(u(t)) = h_{0k}(t)\rho_0(u(t)) \qquad \text{for a. e. } t \in [0,\omega], \tag{3.35}$$

where

$$h_{0k}(t) = \begin{cases} k & \text{if } h_0(t) > k \\ h_0(t) & \text{if } h_0(t) \le k \end{cases} \qquad \text{for a. e. } t \in [0, \omega], \quad k \in \mathbb{N}. \tag{3.36}$$

Obviously,

$$h_{0k}(t) \le h_{0m}(t) \le h_0(t) \qquad \text{for a. e. } t \in [0, \omega], \quad k \le m, \tag{3.37}$$

and

$$\lim_{k \to +\infty} \overline{h}_{0k} = \overline{h}_0. \tag{3.38}$$

The following three results respectively correspond to Corollaries 1.2.5, 1.2.6 and 1.2.1. However a suitable formulation should be done into in our actual framework.

Lemma 3.3.1 *Let $x_0 > 0$ and $c \in \mathbb{R}$ be such that*

$$\frac{g(x)}{\rho_0(x)} \le c \le \overline{h}_0 \qquad \text{for } x \ge x_0. \tag{3.39}$$

Let, moreover, there exist a sequence $\{y_n\}_{n=1}^{+\infty}$ of positive numbers such that

$$\lim_{n \to +\infty} y_n = +\infty, \qquad \lim_{n \to +\infty} \frac{\rho_0(y_n)}{y_n} = 0, \tag{3.40}$$

and let there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that

$$\frac{\rho_0((1+\varepsilon)y_n)}{\rho_0(y_n)} \Phi_- \le \Phi_+ \qquad \text{for } n \ge n_0,$$

where $\varphi(t) = h_0(t) - c$ for almost every $t \in [0, \omega]$. Then there exists an upper function β to the problem (3.34), (1.3) satisfying

$$\beta(t) \ge x_0 \qquad \text{for } t \in [0, \omega]. \tag{3.41}$$

Lemma 3.3.2 *Let $x_0 > 0$ and $c \in \mathbb{R}$ be such that (3.39) holds. If $\frac{\rho_0(x)}{x}$ is a non–increasing function such that*

$$\frac{\omega}{4}\Phi_+\Phi_-\frac{\rho_0(x_0)}{x_0} \le \Phi_+ - \Phi_-$$

where $\varphi(t) = h_0(t) - c$ for almost every $t \in [0,\omega]$, then there exists an upper function β to the problem (3.34), (1.3) satisfying (3.41).

Lemma 3.3.3 *Let $x_0 > \frac{\omega}{8}\|h_0 - \overline{h}_0\|_1$ be such that*

$$g(x) \ge \overline{h}_0 \qquad \text{for } 0 < x \le x_0.$$

Then there exists a lower function α to the problem (1.36), (1.3) with

$$0 < \alpha(t) \le x_0 \qquad \text{for } t \in [0,\omega]. \tag{3.42}$$

Lemma 3.3.4 *Let $x_0 > 0$ and $c_0 \in \mathbb{R}$ be such that (3.15) holds. Let, moreover, there exist a sequence $\{y_n\}_{n=1}^{+\infty}$ of positive numbers such that (3.40) is fulfilled, and let there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that*

$$\frac{\rho_0((1+\varepsilon)y_n)}{\rho_0(y_n)}\Phi_- \le \Phi_+ - \varepsilon \qquad \text{for } n \ge n_0, \tag{3.43}$$

where $\varphi(t) = h_0(t) - c_0$ for almost every $t \in [0,\omega]$. Then there exist $k_0 \in \mathbb{N}$ and an upper function β to the problems (3.35), (1.3) for $k \ge k_0$ satisfying (3.41).

Proof 44 Put

$$\varphi_k(t) = h_{0k}(t) - c_0 \qquad \text{for a. e. } t \in [0,\omega], \tag{3.44}$$

$$\Phi_{k+} = \int_0^\omega [\varphi_k(s)]_+ ds, \qquad \Phi_{k-} = \int_0^\omega [\varphi_k(s)]_- ds. \tag{3.45}$$

Then, obviously, in view of (3.36), we have

$$\lim_{k\to+\infty} \Phi_{k+} = \Phi_+, \qquad \lim_{k\to+\infty} \Phi_{k-} = \Phi_- \tag{3.46}$$

and, consequently, on account of (3.15), (3.38), (3.43), and (3.46), there exists $k_0 \in \mathbb{N}$ such that

$$\frac{g(x)}{\rho_0(x)} \le c_0 \le \overline{h}_{0k_0} \le \overline{h}_0 \qquad \text{for } x \ge x_0, \tag{3.47}$$

$$\frac{\rho_0((1+\varepsilon)y_n)}{\rho_0(y_n)}\Phi_{k_0-} \le \Phi_{k_0+} \qquad \text{for } n \ge n_0. \tag{3.48}$$

Therefore, according to Lemma 3.3.1, there exists an upper function β to (3.35), (1.3) with $k = k_0$ satisfying (3.41). Obviously, in view of (3.37) and the non-negativity of ρ_0 it follows that β is also an upper function to (3.35), (1.3) for $k \ge k_0$.

Lemma 3.3.5 *Let $x_0 > 0$ and $c_0 \in \mathbb{R}$ be such that (3.15) holds. If $\frac{\rho_0(x)}{x}$ is a non–increasing function such that*

$$\frac{\omega}{4}\Phi_+\Phi_-\frac{\rho_0(x_0)}{x_0} < \Phi_+ - \Phi_- \tag{3.49}$$

where $\varphi(t) = h_0(t) - c_0$ for almost every $t \in [0,\omega]$, then there exist $k_0 \in \mathbb{N}$ and an upper function β to the problems (3.35), (1.3) for $k \ge k_0$ satisfying (3.41).

Proof 45 Define φ_k, Φ_{k+}, and Φ_{k-} by (3.44) and (3.45). Then, obviously, in view of (3.36), we have that (3.46) holds, and, consequently, on account of (3.15), (3.38), (3.46), and (3.49), there exists $k_0 \in \mathbb{N}$ such that (3.47) is valid and

$$\frac{\omega}{4}\Phi_{k_0+}\Phi_{k_0-}\frac{\rho_0(x_0)}{x_0} \le \Phi_{k_0+} - \Phi_{k_0-} \tag{3.50}$$

Therefore, according to Lemma 3.3.2, there exists an upper function β to (3.35), (1.3) with $k = k_0$ satisfying (3.41). Obviously, in view of (3.37) and the non-negativity of ρ_0 it follows that β is also an upper function to (3.35), (1.3) for $k \geq k_0$.

Lemma 3.3.6 *Let*

$$\liminf_{x \to 0+} g(x) > -\infty, \tag{3.51}$$

and let either

$$\int_0^1 [f(s)]_+ ds < +\infty \tag{3.52}$$

or

$$\int_0^1 [f(s)]_- ds < +\infty. \tag{3.53}$$

Then, for every $K > 0$, there exists a constant $K_1 > 0$ such that for any $k \in \mathbb{N}$ and any positive solution u of (3.35), (1.3) with

$$\|u\|_\infty \leq K \tag{3.54}$$

we have the estimate

$$\|u'\|_\infty \leq K_1. \tag{3.55}$$

Proof 46 Assume that (3.53) is fulfilled. Let u be a positive solution to (3.35), (1.3) satisfying (3.54). Then there exist $t_0, t_1 \in [0, \omega]$ such that

$$u(t_0) = \min\{u(t) : t \in [0,\omega]\}, \qquad u(t_1) = \max\{u(t) : t \in [0,\omega]\}. \tag{3.56}$$

Define the operator ϑ of ω-periodic prolongation by

$$\vartheta(v)(t) = \begin{cases} v(t) & \text{if } t \in [0,\omega] \\ v(t-\omega) & \text{if } t \in (\omega, 2\omega] \end{cases}. \tag{3.57}$$

Then, obviously, from (3.35) and (1.3) it follows that

$$\vartheta(u)''(t) + f(\vartheta(u)(t))\vartheta(u)'(t) + g(\vartheta(u)(t)) = \vartheta(h_{0k})(t)\rho_0(\vartheta(u)(t)) \quad \text{for a. e. } t \in [0, 2\omega]. \tag{3.58}$$

The integration of (3.58) from t_0 to t, on account of (3.56), yields

$$\vartheta(u)'(t) = -\int_{t_0}^{t} f(\vartheta(u)(s))\vartheta(u)'(s)ds - \int_{t_0}^{t} g(\vartheta(u)(s))ds + \int_{t_0}^{t} \vartheta(h_{0k})(s)\rho_0(\vartheta(u)(s))ds \qquad \text{for } t \in [t_0, t_0 + \omega]. \tag{3.59}$$

From (3.54), (3.56), and (3.57) it follows that

$$0 < \vartheta(u)(t_0) \le \vartheta(u)(t) \le K \qquad \text{for } t \in [t_0, t_0 + \omega]. \tag{3.60}$$

Put

$$\mu = \sup\big\{[g(s)]_- : s \in (0, K]\big\}. \tag{3.61}$$

According to (3.51) we have

$$0 \le \mu < +\infty. \tag{3.62}$$

Thus, using (3.36), (3.53), (3.54), and (3.60)–(3.62) in (3.59) we arrive at

$$\vartheta(u)'(t) \le \int_0^K [f(s)]_- ds + \omega\mu + \|h_0\|_1\rho_0(K) \qquad \text{for } t \in [t_0, t_0 + \omega]. \tag{3.63}$$

Put

$$K_1 = \int_0^K [f(s)]_- ds + \omega\mu + \|h_0\|_1\rho_0(K).$$

Then, on account of (3.57) and (3.63) we have

$$u'(t) \leq K_1 \qquad \text{for } \ t \in [0, \omega]. \tag{3.64}$$

On the other hand, the integration of (3.58) from t to $t_1 + \omega$, with respect to (3.56), results in

$$\vartheta(u)'(t) = \int_t^{t_1+\omega} f(\vartheta(u)(s))\vartheta(u)'(s)ds + \int_t^{t_1+\omega} g(\vartheta(u)(s))ds$$
$$- \int_t^{t_1+\omega} \vartheta(h_{0k})(s)\rho_0(\vartheta(u)(s))ds \qquad \text{for } \ t \in [t_1, t_1 + \omega]. \tag{3.65}$$

Now, using (3.36), (3.53), (3.54), and (3.60)–(3.62) in (3.65) we obtain

$$- \vartheta(u)'(t) \leq K_1 \qquad \text{for } \ t \in [t_1, t_1 + \omega]. \tag{3.66}$$

Therefore, in view of (3.57), from (3.66) we get

$$- u'(t) \leq K_1 \qquad \text{for } \ t \in [0, \omega] \tag{3.67}$$

Consequently, (3.64) and (3.67) results in (3.55).

Now suppose that (3.52) is fulfilled. Put

$$v(t) = u(\omega - t) \qquad \text{for } \ t \in [0, \omega]. \tag{3.68}$$

Then, according to (3.35) we have

$$v''(t) - f(v(t))v'(t) + g(v(t)) = \widetilde{h}_{0k}(t)\rho_0(v(t)) \qquad \text{for a. e. } \ t \in [0, \omega], \tag{3.69}$$

where

$$\widetilde{h}_{0k}(t) = h_{0k}(\omega - t) \qquad \text{for a. e. } t \in [0, \omega].$$

Analogously to the above-proven, using (3.52) instead of (3.53), we obtain

$$\|v'\|_\infty \le K_1 \tag{3.70}$$

with

$$K_1 = \int_0^K [f(s)]_+ ds + \omega\mu + \|h_0\|_1 \rho_0(K).$$

Thus, (3.68) and (3.70) yields (3.55).

Lemma 3.3.7 *Let*

$$\lim_{x \to 0_+} g(x) = +\infty \tag{3.71}$$

and let either

$$\int_0^1 \left([f(s)]_+ + [g(s)]_+\right) ds = +\infty, \qquad \int_0^1 [f(s)]_- ds < +\infty \tag{3.72}$$

or

$$\int_0^1 \left([f(s)]_- + [g(s)]_+\right) ds = +\infty, \qquad \int_0^1 [f(s)]_+ ds < +\infty \tag{3.73}$$

Then, for every $K > 0$ there exists a constant $a > 0$ such that for any $k \in \mathbb{N}$ and any positive solution u of (3.35), (1.3) satisfying (3.54) we have the estimate

$$a \le u(t) \qquad \textit{for } t \in [0, \omega]. \tag{3.74}$$

Proof 47 Let u be a positive solution to (3.35), (1.3) satisfying (3.54). Thus, the integration of (3.35) from 0 to ω, in view of (1.3) and (3.37), yields

$$\int_0^\omega g(u(s)) ds \le \|h_0\|_1 \rho_0(K). \tag{3.75}$$

On the other hand, (3.71) implies the existence of $x_0 \in \mathbb{R}^+$ such that

$$g(x) > \frac{\|h_0\|_1 \rho_0(K)}{\omega} \geq 0 \qquad \text{for } x \in (0, x_0). \tag{3.76}$$

Let $t_m \in [0, \omega]$ be such that

$$u(t_m) = \min \big\{ u(t) : t \in [0, \omega] \big\}. \tag{3.77}$$

Obviously, either

$$u(t_m) \geq x_0$$

or

$$u(t_m) < x_0. \tag{3.78}$$

Obviously, it is sufficient to show the estimate (3.74) is valid just in the case when (3.78) is fulfilled. Let, therefore, (3.78) hold.

If $u(t) < x_0$ for $t \in [0, \omega]$, then applying (3.76) in (3.75) we obtain a contradiction. Thus, there exist points $t_1, t_2 \in (t_m, t_m + \omega)$ such that

$$\vartheta(u)(t) < x_0 \qquad \text{for } t \in [t_m, t_1), \qquad \vartheta(u)(t_1) = x_0, \tag{3.79}$$

$$\vartheta(u)(t) < x_0 \qquad \text{for } t \in (t_2, t_m + \omega], \qquad \vartheta(u)(t_2) = x_0, \tag{3.80}$$

where ϑ is the operator defined by (3.57). Obviously, (3.58) holds.

Assume that (3.72) holds. Then, according to Lemma 3.3.6, there exists $K_1 > 0$ such that (3.55) holds. The integration of (3.58) from t_m to t_1, in view of (3.37), (3.54), (3.55), (3.72), (3.76), (3.77), and (3.79) results in

$$\vartheta(u)'(t_1) + \int_{\vartheta(u)(t_m)}^{x_0} [f(s)]_+ ds + \frac{1}{K_1} \int_{\vartheta(u)(t_m)}^{x_0} [g(s)]_+ ds \leq \int_0^{x_0} [f(s)]_- ds + \|h_0\|_1 \rho_0(K). \quad (3.81)$$

Note that in view of (3.79) we have $\vartheta(u)'(t_1) \geq 0$. Consequently, from (3.81) we obtain

$$\int_{\vartheta(u)(t_m)}^{x_0} ([f(s)]_+ + [g(s)]_+) \, ds \leq K_2 \quad (3.82)$$

where

$$K_2 = (K_1 + 1) \left(\int_0^{x_0} [f(s)]_- ds + \|h_0\|_1 \rho_0(K) \right).$$

Note that K_2 does not depend on k. Therefore, if we apply (3.72) in (3.82), it can be easily seen, with respect to (3.77), that there exists a constant $a > 0$ such that (3.74) holds.

If (3.73) holds, we integrate (3.58) from t_2 to $t_m + \omega$ and apply the similar steps as above, just using (3.80) instead of (3.79), finally we arrive at

$$\int_{\vartheta(u)(t_m+\omega)}^{x_0} ([f(s)]_- + [g(s)]_+) \, ds \leq K_2$$

with

$$K_2 = (K_1 + 1) \left(\int_0^{x_0} [f(s)]_+ ds + \|h_0\|_1 \rho_0(K) \right).$$

Therefore, also in this case there exists a constant $a > 0$ such that (3.74) holds.

Lemma 3.3.8 *Let $x_0 > 0$ and $c_0 \in \mathbb{R}$ be such that (3.15) holds. Let, moreover, (3.71) be fulfilled, and let either (3.72) or (3.73) be valid. Let, in addition, there exist a sequence $\{y_n\}_{n=1}^{+\infty}$ of positive numbers such that (3.40) holds, and let there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$*

such that (3.43) *is fulfilled, where* $\varphi(t) = h_0(t) - c_0$ *for almost every* $t \in [0, \omega]$. *Then there exists a positive solution* u *to* (3.34), (1.3).

Proof 48 According to Lemma 3.3.4, there exists $k_0 \in \mathbb{N}$ and an upper function β to the problems (3.35), (1.3) for $k \geq k_0$ satisfying (3.41). On the other hand, in view of (3.37) and (3.71) there exist $x_k \in (0, x_0]$ for $k \geq k_0$ such that

$$g(x_k) \geq h_{0k}(t)\rho_0(x_k) \qquad \text{for a. e. } t \in [0, \omega].$$

Thus, if we put $\alpha_k(t) = x_k$ for $t \in [0, \omega]$, according to Theorem 2.2.1, there exists a solution u_k to (3.35), (1.3) for $k \geq k_0$ satisfying

$$0 < \alpha_k(t) \leq u_k(t) \leq \beta(t) \qquad \text{for } t \in [0, \omega]. \tag{3.83}$$

Moreover, according to Lemmas 3.3.6 and 3.3.7, in view of (3.83), there exist constants $K > 0$, $K_1 > 0$, and $a > 0$, not depending on k, such that

$$\|u_k\|_\infty \leq K, \qquad \|u_k'\|_\infty \leq K_1, \qquad \text{for } k \geq k_0, \tag{3.84}$$

$$a \leq u_k(t) \qquad \text{for } t \in [0, \omega], \quad k \geq k_0, \tag{3.85}$$

$$|u_k''(t)| \leq f_0 K_1 + g_0 + |h_0(t)|\rho_0(K) \qquad \text{for a. e. } t \in [0, \omega], \quad k \geq k_0, \tag{3.86}$$

where

$$f_0 = \max\big\{|f(x)| : x \in [a, K]\big\}, \qquad g_0 = \max\big\{|g(x)| : x \in [a, K]\big\}.$$

Therefore, according to Arzelà–Ascoli Theorem, there exists $u_0 \in C\big([0, \omega]; \mathbb{R}\big)$ and $v_0 \in C\big([0, \omega]; \mathbb{R}\big)$ such that

$$\lim_{k \to +\infty} \|u_k - u_0\|_\infty = 0, \qquad \lim_{k \to +\infty} \|u_k' - v_0\|_\infty = 0. \tag{3.87}$$

Moreover, since u_k are solutions to (3.35), (1.3), in view of (3.36), (3.85), and (3.87), we have $u_0 \in AC^1\big([0,\omega];\mathbb{R}\big)$, $u_0' \equiv v_0$, and u_0 is a positive solution to (3.34), (1.3).

The following assertion can be proven analogously to Lemma 3.3.8, just Lemma 3.3.5 is used instead of Lemma 3.3.4.

Lemma 3.3.9 *Let $x_0 > 0$ and $c_0 \in \mathbb{R}$ be such that (3.15) holds. Let, moreover, (3.71) be fulfilled, and let either (3.72) or (3.73) be valid. Let, in addition, $\frac{\rho_0(x)}{x}$ be a non–increasing function and let (3.49) be fulfilled, where $\varphi(t) = h_0(t) - c_0$ for almost every $t \in [0,\omega]$. Then there exists a positive solution u to (3.34), (1.3).*

Lemma 3.3.10 *Let $\rho_0 \in C^1\big(\mathbb{R}^+;\mathbb{R}^+\big)$ be non-decreasing, $x_0 > 0$, and $c_0 \in \mathbb{R}$ be such that (3.15) holds. Let, moreover, there exist $\lambda \in [0,1]$ such that (3.17) and (3.18) are valid, and let either (3.19) or (3.20) be fulfilled. Let, in addition, there exist a sequence $\{y_n\}_{n=1}^{+\infty}$ of positive numbers such that (3.21) holds and let there exist $\varepsilon_0 > 0$, $\varepsilon_1 \in (0,\varepsilon_0]$, and $n_0 \in \mathbb{N}$ such that (3.22) and (3.23) are fulfilled, where $\varphi(t) = h_0(t) - c_0$ for almost every $t \in [0,\omega]$ and σ is given by (3.24). Then there exists a lower function α to the problem (3.34), (1.3).*

Proof 49 Because ρ_0 is a positive function, from (3.17) and (3.24) we obtain that σ is a positive increasing function. Therefore, there exists an inverse function σ^{-1} to σ which is also increasing. Moreover, in view of (3.17) and (3.24), it follows that

$$\lim_{x\to 0_+} \sigma(x) = 0, \quad \lim_{x\to 0_+} \sigma^{-1}(x) = 0, \quad \lim_{x\to +\infty} \sigma(x) = +\infty, \quad \lim_{x\to +\infty} \sigma^{-1}(x) = +\infty. \tag{3.88}$$

Consider the auxiliary equation

$$u''(t) + f(\sigma^{-1}(u(t)))u'(t) + \frac{g(\sigma^{-1}(u(t)))}{\rho_0^{\lambda}(\sigma^{-1}(u(t)))} = h_0(t)\rho_0^{1-\lambda}(\sigma^{-1}(u(t)))$$

$$\text{for a. e. } t \in [0,\omega]. \quad (3.89)$$

Put $z = \sigma(x)$, $z_0 = \sigma(x_0)$. Then from (3.15) we get

$$\frac{g(\sigma^{-1}(z))}{\rho_0(\sigma^{-1}(z))} \le c_0 < \overline{h}_0 \qquad \text{for } z \ge z_0 \quad (3.90)$$

and, in view of (3.88), from (3.18) we have

$$\lim_{z\to 0_+} \frac{g(\sigma^{-1}(z))}{\rho_0^{\lambda}(\sigma^{-1}(z))} = +\infty. \quad (3.91)$$

Furthermore, the substitution $r = \sigma(s)$ in (3.19), resp (3.20), with respect to (3.24), yields

$$\int_0^1 \left([f(\sigma^{-1}(r))]_+ + \frac{[g(\sigma^{-1}(r))]_+}{\rho_0^{\lambda}(\sigma^{-1}(r))}\right) dr = +\infty, \qquad \int_0^1 [f(\sigma^{-1}(r))]_- dr < +\infty, \quad (3.92)$$

resp.

$$\int_0^1 \left([f(\sigma^{-1}(r))]_- + \frac{[g(\sigma^{-1}(r))]_+}{\rho_0^{\lambda}(\sigma^{-1}(r))}\right) dr = +\infty, \qquad \int_0^1 [f(\sigma^{-1}(r))]_+ dr < +\infty \quad (3.93)$$

Moreover, put $z_n = \sigma(y_n)$ for $n \in \mathbb{N}$. Then from (3.21), in view of (3.88), we get

$$\lim_{n\to+\infty} z_n = +\infty, \qquad \lim_{n\to+\infty} \frac{\rho_0^{1-\lambda}(\sigma^{-1}(z_n))}{z_n} = 0. \quad (3.94)$$

Finally, (3.23) results in

$$\sigma^{-1}((1+\varepsilon_1)z_n) \le (1+\varepsilon_0)y_n \qquad \text{for } \ n \ge n_0,$$

and so, since ρ_0 is a non-decreasing function, from (3.22) we obtain

$$\frac{\rho_0^{1-\lambda}(\sigma^{-1}((1+\varepsilon_1)z_n))}{\rho_0^{1-\lambda}(\sigma^{-1}(z_n))}\Phi_- \le \Phi_+ - \varepsilon_1 \qquad \text{for } \ n \ge n_0. \tag{3.95}$$

Therefore, applying Lemma 3.3.8, according to (3.90)–(3.95), there exists a positive solution u to the problem (3.89), (1.3).

Now we put $\alpha(t) = \sigma^{-1}(u(t))$ for $t \in [0,\omega]$, i.e., in view of (3.24),

$$u(t) = \int_0^{\alpha(t)} \frac{ds}{\rho_0^{\lambda}(s)} \qquad \text{for } \ t \in [0,\omega].$$

Obviously, $\alpha \in AC^1\big([0,\omega];\mathbb{R}\big)$ is a positive function and

$$u'(t) = \frac{\alpha'(t)}{\rho_0^{\lambda}(\alpha(t))} \qquad \text{for } \ t \in [0,\omega],$$

$$u''(t) = \frac{\alpha''(t)}{\rho_0^{\lambda}(\alpha(t))} - \frac{\lambda\alpha'^2(t)\rho_0'(\alpha(t))}{\rho_0^{1+\lambda}(\alpha(t))} \le \frac{\alpha''(t)}{\rho_0^{\lambda}(\alpha(t))} \qquad \text{for a. e. } \ t \in [0,\omega].$$

Thus, it can be easily seen that α is a lower function to the problem (3.34), (1.3).

Analogously to the proof of Lemma 3.3.10, one can prove the following assertion applying Lemma 3.3.9 instead of Lemma 3.3.8.

Lemma 3.3.11 *Let $\rho_0 \in C^1\big(\mathbb{R}^+;\mathbb{R}^+\big)$ be non-decreasing, $x_0 > 0$, and $c_0 \in \mathbb{R}$ be such that (3.15) holds. Let, moreover, there exist $\lambda \in [0,1]$ such that (3.17) and (3.18) are valid, and let either (3.19) or (3.20) be fulfilled. Let, in addition, $\frac{\rho_0^{1-\lambda}(x)}{\sigma(x)}$ be a non–increasing function and let (3.25) be fulfilled, where $\varphi(t) = h_0(t) - c_0$ for almost every*

$t \in [0, \omega]$ and σ is given by (3.24). Then there exists a lower function α to the problem (3.34), (1.3).

3.3.3 Proof of the main results

In this section we will write the proofs of our main results using the previous proved lemmas.

Proof 50 (Proof of Theorem 3.3.1) According to Lemmas 3.3.1, 3.3.2, 3.3.10, and 3.3.11, the conditions of theorem guarantee a well-ordered couple of lower and upper functions, therefore the result is a direct consequence of Theorem 2.2.1.

Proof 51 (Proof of Corollary 3.3.1) It follows from Theorem 3.3.1 with $h_1 \equiv h_0$, $\rho_0(x) = \rho_1(x) = x^\delta$, $\lambda = 1$, and $c_0 = c_1$ such that

$$\overline{h}_0 > c_0 > -\lim_{x \to +\infty} \frac{g_2}{x^{\gamma+\delta}}\,.$$

Then items $a)$ and $c)$ of Theorem 3.3.1 are fulfilled.

Proof 52 (Proof of Corollary 3.3.2) It follows from Theorem 3.3.1 with $h_1 \equiv h_0$, $\rho_0(x) = \rho_1(x) = x$, and $\lambda < 1$ such that $\nu + 2\lambda \geq 1$. Then items $b)$ and $d)$ of Theorem 3.3.1 are fulfilled.

Proof 53 (Proof of Corollary 3.3.3) It immediately follows from Theorem 3.3.1 with $h_1 \equiv h_0$, $\rho_i(x) \equiv 1$ $(i = 0, 1)$.

Proof 54 (Proof of Theorem 3.3.2) Put

$$\sigma(x) = \int_0^x \frac{ds}{\rho_0(s)} \qquad \text{for } x \geq 0. \tag{3.96}$$

Because ρ_0 is a positive function, from (3.31) and (3.96) we obtain that σ is an increasing function. Therefore, there exists an inverse function σ^{-1} to σ which is also increasing.

Consider the auxiliary equation

$$u''(t) + f(\sigma^{-1}(u(t)))u'(t) + \frac{g(\sigma^{-1}(u(t)))}{\rho_0(\sigma^{-1}(u(t)))} = h_0(t) \qquad \text{for a. e. } t \in [0,\omega]. \tag{3.97}$$

Put $z = \sigma(x)$, $z_0 = \sigma(x_0)$. Then from (3.31) and (3.32), in view of (3.96), we get

$$\frac{\omega}{8}\|h_0 - \overline{h}_0\|_1 < z_0$$
$$\frac{g(\sigma^{-1}(z))}{\rho_0(\sigma^{-1}(z))} \geq \overline{h}_0 \qquad \text{for } 0 < z \leq z_0,$$

Therefore, according to Lemma 3.3.3 there exists a lower function w to the problem (3.97), (1.3) satisfying

$$0 < w(t) \leq \sigma(x_0) \qquad \text{for } t \in [0,\omega]. \tag{3.98}$$

Now we put $\alpha = \sigma^{-1}(w(t))$ for $t \in [0,\omega]$, i.e., in view of (3.96),

$$w(t) = \int_0^{\alpha(t)} \frac{ds}{\rho_0(s)} \qquad \text{for } t \in [0,\omega].$$

Obviously, with respect to (3.98), $\alpha \in AC^1\big([0,\omega];\mathbb{R}\big)$ is a positive function satisfying (3.42), and

$$w'(t) = \frac{\alpha'(t)}{\rho_0(\alpha(t))} \qquad \text{for } t \in [0,\omega],$$
$$w''(t) = \frac{\alpha''(t)}{\rho_0(\alpha(t))} - \frac{\alpha'^2(t)\rho_0'(\alpha(t))}{\rho_0^2(\alpha(t))} \leq \frac{\alpha''(t)}{\rho_0(\alpha(t))} \qquad \text{for a. e. } t \in [0,\omega].$$

Thus, on account of (3.29), (3.42), and (3.97), it can be easily seen that α is a lower function to the problem (1.1), (1.3).

The existence of an upper function β to (1.1), (1.3) satisfying

$$\beta(t) \geq x_1 \qquad \text{for } t \in [0, \omega] \tag{3.99}$$

follows from (3.30) and Lemma 3.3.1, resp. 3.3.2.

Obviously, in view of (3.42) and (3.99), we have that (2.29) holds. Thus the theorem follows from Theorem 2.2.1.

Proof 55 (Proof of Corollary 3.3.4) It follows from Theorem 3.3.2 with $h_1 \equiv h_0$, $\rho_0(x) = \rho_1(x) = x^\delta$, and $c_1 = \overline{h}_0$.

Chapter 4

The Brillouin beam equation

The last type of equation which will be studied in this work is the classical Brillouin beam equation

$$u'' + b(1+\cos t)u = \frac{1}{u}, \tag{4.1}$$

depending on a positive parameter b. This equation is a repulsive singular perturbation of a Mathieu equation, as we will explain below. The classical problem to find $2\pi-$periodic solutions of (4.1) arose at the beginning in the sixties in the context of Electronics motivated by some numerical experiments realized in [5], where it was conjectured that *equation* (4.1) *should have a* $2\pi-$*periodic solution whenever* $b \in (0, 1/4)$.

By an acquired intuition in the previous chapters, one expects that under some non-resonant condition should be possible to prove the proposed conjecture. However we will explain below that this problem can be really delicate, and arising doubts on the validity of the result conjectured.

Essentially the unique research line to try to prove this result concerns to use the function $K : [1, +\infty] \to \mathbb{R}$ defined by

$$K(\alpha) = \begin{cases} \dfrac{1}{\pi^2} & \alpha = 1 \\ \dfrac{(\alpha-1)^{1+\frac{1}{\alpha}}}{8\pi^{1-\frac{1}{2\alpha}}\alpha^{1-\frac{1}{\alpha}}(2\alpha-1)^{\frac{1}{\alpha}}} \left(\dfrac{\Gamma\left(\frac{1}{2}-\frac{1}{2\alpha}\right)}{\Gamma\left(1-\frac{1}{2\alpha}\right)}\right)^2 \left(\dfrac{\Gamma(\alpha)}{\Gamma\left(\frac{1}{2}+\alpha\right)}\right)^{\frac{1}{\alpha}} & \alpha \in (1, +\infty) \\ \dfrac{1}{8} & \alpha = +\infty; \end{cases}$$

with this definition, a sufficient condition in order for the Dirichlet problem ($u(0) = 0 = u(2\pi)$) associated with

$$u'' + b(1 + \cos t)u = 0. \tag{4.2}$$

to have a unique solution is that

$$b < \max_{\alpha \in [0,+\infty]} K(\alpha) \approx 0.16448. \tag{4.3}$$

Thus under this non-resonance condition i.e., if $b \in (0, 0.16448)$, M. Zhang proved that (4.1) has at least one 2π-periodic solution in [66]. This last result has been extended to equations where the singularity may be of weak type (see [53]).

In view of [53], one can understand that condition (4.3) implies that (4.2) has an associated Green function which has a positive sign. This makes possible to control the operator

$$T : \Omega \to C([0, 2\pi]; \mathbb{R}), \qquad T[u](t) = \int_0^{2\pi} \frac{G(t,s)}{u(s)} ds,$$

where Ω is a suitable open subset of $C([0, 2\pi]; \mathbb{R})$ in order to apply Kranoselskii fixed point theorem, and $G : [0, 2\pi] \times [0, 2\pi] \to \mathbb{R}$ is the associated Green function to (4.2).

Despite of the expectations, nobody has improved the result of M. Zhang, i.e., the best range of b actually known for the $2\pi-$periodic solvability of (4.2) is the proven in [66], and outside from there the thing is completely unknown. Nevertheless, we can point out that the result cannot be obtained as a consequence of a general equation with strong condition singularity such as it was done in [50] (this is one of the main troubles in the related literature). Indeed, according to [67, Theorem 2.1], it was established an unanimous relation between the stability intervals for the Mathieu equation (4.3) and the existence of periodic solutions for the Yermakov-Pinney equation

$$u'' + b(1 + \cos t)u - \frac{1}{u^3} = 0. \tag{4.4}$$

Notice that the stability intervals of the Mathieu equation

$$(\lambda_0, \lambda_1'), \quad (\lambda_2', \lambda_1), \quad (\lambda_2, \lambda_3'), \quad \ldots,$$

where λ_i, $i = 0, 1, \ldots$ and λ_i', $i = 1, 2, \ldots$, respectively, are the values of the parameter b for which equation (4.2) has, respectively, a genuine π-periodic solution and a genuine 2π-periodic solution, are defined approximately by $\lambda = 0$, $\lambda_1' \approx 1/6$; $\lambda_2' \approx 0.4$, $\lambda_1 \approx 0.95, \ldots$ (see [42, Theorem 2.1] and [10, Figure 1]). Thus the conjectured result in [5] cannot be extended to (4.4).

We will divide in two sections the present chapter. In the first one we use our general results proved in Chapter 1 in order to see what happen with (4.1). In the second one we show that (4.1) may have $2\pi-$periodic solutions also when b belongs to intervals other than $(0, 1/4)$. However the proposed conjecture still remaind as an open problem (see [25] to know the details of the different contributions on this topic).

4.1 A corollary for the Brillouin beam equation

One can easily observe that (4.1) can be considered as a particular equation of (1.1), taking either $g(x) = bx - 1/x$ and $h(t, x) = -bx \cos t$ or $g(x) = -1/x$ and $h(t, x) = -b(1 + \cos t)x$. Thus applying Theorem 1.1.3 follows the following corollaries.

Corollary 4.1.1 *If $b \in (0, 1/(4 + \pi))$, then (4.1) has at least one $2\pi-$periodic solution.*

Proof 56 We take $g(x) = bx - 1/x$ and $h(t,x) = h_0(t)\rho(x)$ where

$$h_0(t) = -b\cos t \qquad \text{for } \ t \in [0, 2\pi], \qquad \rho(x) = x.$$

Defining

$$\eta : [0, 2\pi] \times \mathbb{R}^+ \to \mathbb{R}, \qquad \eta(t,x) = \begin{cases} 0 & t \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right], \\ bx\cos t & t \notin \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]; \end{cases}$$

it deduces that $h(t,x) \geq -\eta(t,x)$ on $[0, 2\pi] \times \mathbb{R}^+$ and, therefore that

$$\frac{1}{x}\int_0^{2\pi} \eta(t,x)dt = 2b.$$

Since η and ρ are non-decreasing functions, the proof follows easily from Theorem 1.1.3.

On the contrary, if we take $g(x) = -1/x$ and $h(t,x) = -b(1 + \cos t)x$, we can only obtain in a natural way the assertion.

Corollary 4.1.2 *If $b \in (0, 1/\pi^2)$, then (4.1) has at least one $2\pi-$periodic solution.*

As always the parameter b should be into the interval $(0, 1/4)$. This is because to prove Theorem 1.1.3 we have implicitly used that $u'' + b(1 + \cos t)u = 0$ has a positive associated Green function.

4.2 A new result for the Brillouin beam equation

In this section we will obtain the first range of parameter b outside of the interval $(0, 1/4)$ thanks to suitable non-resonance assumptions which can be traced back to the work [14] by Fabry. The main abstract tool to obtain such a statement is embodied by the *Poincaré-Bohl fixed point theorem.*

4.2.1 A non-resonance theorem for repulsive singular equations

The proof of our Theorem is based on a non-resonance result which involves nonlinearities with "atypical" linear growing type, and could have interest by itself.

Theorem 4.2.1 *Let us assume that there exist positive constants A_+, B_+ such that*

$$\frac{1}{2\pi}\int_0^{2\pi} \min\left\{\frac{b(1+\cos t)}{B_+},1\right\}dt > \frac{n}{2\sqrt{B_+}}, \tag{4.5}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \max\left\{\frac{b(1+\cos t)}{A_+},1\right\}dt < \frac{n+1}{2\sqrt{A_+}}, \tag{4.6}$$

for some natural number n. Then (4.1) has at least one 2π-periodic solution.

Before introducing the main tools to prove the theorem, a couple of remarks are in order.

Remark 4.2.1 With the aim of keeping the exposition at a rather simple level, and taking into account that our main goal will be to study the existence of 2π-periodic solutions of (4.1), we will always consider equation (4.1) as a starting point. However, the result can be extended, with the same approach and similar computations, to more general equations like

$$u'' + q(t)u - g(t,u) = 0, \tag{4.7}$$

where q is continuous and 2π-periodic, and $g : [0,2\pi] \times (0,+\infty) \to \mathbb{R}$ has a similar behavior as $1/x^\gamma$, with $\gamma \geq 1$, being allowed to grow at most sublinearly at infinity. For instance, as in [21], one can assume that there exist $\sigma > 0$ and a continuous function $f : (0,\sigma] \to \mathbb{R}$ such that $g(t,x) \leq f(x)$, whenever $x \in (0,\sigma]$, and

$$\lim_{r\to 0^+} f(r) = -\infty, \qquad \int_0^\sigma f(r)\,dr = -\infty.$$

Of course, in this case $q(t)$ will replace $b(1+\cos t)$.

Remark 4.2.2 Conditions (4.5) and (4.6) were introduced by Fabry in [14] for the equation

$$u'' + g(t,u) = 0,$$

with

$$p(t) \leq \liminf_{|x|\to+\infty} \frac{g(t,x)}{x} \leq \limsup_{|x|\to+\infty} \frac{g(t,x)}{x} \leq q(t),$$

asking that

$$\sqrt{\lambda_j} < \sup_{\xi>0} \frac{\frac{1}{2\pi}\int_0^{2\pi} \min\{p(t),\xi\}\, dt}{\sqrt{\xi}}, \qquad \inf_{\xi>0} \frac{\frac{1}{2\pi}\int_0^{2\pi} \max\{q(t),\xi\}\, dt}{\sqrt{\xi}} < \sqrt{\lambda_{j+1}},$$

where λ_j is the j-th eigenvalue of the considered 2π-periodic problem. Such conditions are usually coupled with the sign assumption

$$\liminf_{|x|\to+\infty} \operatorname{sgn} x\, f(t,x) > 0$$

(see for instance [15]), which, however, in the model case $g(t,x) = b(1+\cos t)x + f(t,x)$, with $\lim_{|x|\to+\infty} f(t,x) = 0$, is not satisfied. This is one of the main difficulties of the problem considered in this chapter.

As it is easy to see, (4.5) and (4.6) are the counterpart of such conditions for the Dirichlet spectrum (which is the natural one to consider when dealing with problems with a singularity, see [66]).

Remark 4.2.3 As a consequence of Theorem 4.2.1, we can obtain the main results in [13, 21]. Indeed, assume that there exist positive constants A_+, B_+ such that

$$B_+ \leq q(t) \leq A_+ \qquad \text{for every } t \in [0,2\pi]. \tag{4.8}$$

Then, according to [13, 21], there exists at least one 2π-periodic solution of (4.7) under the nonresonance assumption

$$\left(\frac{n}{2}\right)^2 < B_+ \leq A_+ < \left(\frac{n+1}{2}\right)^2,$$

where $n \in \mathbb{N}$. It is easy to obtain this result from Theorem 4.2.1, since from (4.8) we deduce that

$$\frac{q(t)}{A_+} \leq 1 \leq \frac{q(t)}{B_+} \qquad \text{for every } t \in [0, 2\pi].$$

Under (4.8), from the point of view of resonance, the results in [13, 21] are optimal, in view of the counterexample produced in [8]. Thus, Theorem 4.2.1 seems to be optimal whenever we are able to control $q(t)$ with an estimate like (4.8), essentially requiring, in this case, a nonresonance assumption. On the other hand, the mean conditions (4.5) and (4.6) do not ask that $q(t)$ is controlled like in (4.8), allowing it to possibly cross some eigenvalues (cf. [9])) as in our case, being $0 \leq q(t) \leq 2b$.

We are now going to prove Theorem 4.2.1. As it was mentioned previously, we will have to overcome the difficulty to work with nonlinearities with atypical linear growth, since, in our concrete case, the nonlinearity grows linearly towards the function $b(1 + \cos t)x$, which vanishes at some times. For this reason, classic arguments in literature (like the ones in [15, 17, 21]) do not extend as they are to (4.1), because it is not possible to construct an admissible spiral which allows to control the dynamic of the solutions.

We will prove Theorem 4.2.1 by means of some preliminary lemmas. To this aim, it will be convenient to introduce the "norm" application defined as

$$\mathcal{N} : \Lambda \to \mathbb{R}, \qquad \mathcal{N}(x, y) := bx^2 + y^2 - 2\ln x,$$

being Λ the half plane with positive abscissa, i.e. $\Lambda = \{(x, y) \in \mathbb{R}^2 : x > 0\}$. Fixed a value c of the function $\mathcal{N}(x, y)$, we will denote the corresponding level curve by γ_c, i.e.,

$$\gamma_c = \{(x, y) \in \Lambda : \mathcal{N}(x, y) = c\}.$$

It is worth observing that the function $\mathcal{N}(x, y)$ reaches its minimum in the point $P_0 = (1/\sqrt{b}, 0)$, where it takes the value $1 - 2\ln(1/\sqrt{b})$ (possibly negative for some values of the parameter b). For values of the energy greater than $1 - 2\ln(1/\sqrt{b})$, the level curves of $\mathcal{N}$ turn around P_0, being the union of two symmetric arcs joining on the x-axis.
We will look at the solutions of (4.1) in the phase plane, taking thus into account the couple (u, u'). As already mentioned, we are interested in positive solutions, so that we will take into account the dynamics of the solutions in the right half-plane.

The first lemma ensures the global continuability of the solutions, i.e., shows that the maximal domain of every solution of (4.1) is $\mathbb{R}_+$.

Lemma 4.2.1 *Let (u, u') be a solution of (4.1) (not necessarily periodic). Then*

$$\mathcal{N}(u(t), u'(t)) < +\infty \qquad \textit{for every } t \geq 0.$$

Proof 57 Since

$$\lim_{x \to +\infty} \frac{bx^2}{bx^2 - 2\ln x} = 1,$$

taking $C > \max\{1, b\}$ there exists $K_0 > 1$ such that

$$\frac{b}{2}(x^2 + y^2) \leq \frac{C}{2}(bx^2 + y^2 - 2\ln x) \qquad \text{for every } x \geq K_0, \quad y \in \mathbb{R}. \tag{4.9}$$

For every solution $u(t)$ of (4.1), we define the function

$$U : I \to \mathbb{R}, \qquad U(t) = \frac{\mathcal{N}(u(t), u'(t))}{2} = \frac{1}{2}(bu'^2(t) + u'^2(t) - 2\ln u(t)),$$

where I is the maximal domain of $u(t)$. We are going to prove that $I = \mathbb{R}_+$.

Since

$$U'(t) = -bu(t)u'(t)\cos t,$$

for $t \in I$ we have that

$$U'(t) \leq \frac{b(u^2(t) + {u'}^2(t))}{2},$$

from which it can be deduced that

$$U'(t) \leq CU(t) + C\ln K_0 \qquad \text{for every } t \in I. \tag{4.10}$$

Indeed, if $t \geq 0$ is such that $u(t) \geq K_0$, then (4.9) implies that $U'(t) \leq CU(t)$. On the contrary, if $u(t) \leq K_0$, we deduce that either $u(t) \leq 1$, and then $U'(t) \leq CU(t)$, or $1 \leq u(t) \leq K_0$, and thus (4.10) holds. Now, according to the *Gronwall-Bellman Lemma*, the result is proven.

As it was mentioned in the previous discussion, equations like (4.1) do not admit the existence of an admissible spiral controlling the solutions. However, the following result ensures that (4.1) has the "property of elasticity", at least locally. Roughly speaking, this means that if there is a time when the norm of the solution is large enough, then, for every preceding time instant, the solution had to be large (in norm). Precisely, we have the following.

Lemma 4.2.2 *Let $\rho_0 > 0$ be sufficiently large. Then, there exists $R_1 > \rho_0$ such that, for every solution (u, u') of* (4.1) *satisfying*

$$\mathcal{N}(u(t_1), u'(t_1)) \geq R_1$$

for some $t_1 > 0$, *it holds*

$$\mathcal{N}(u(t), u'(t)) \geq \rho_0 \qquad \textit{for every } t \in [0, t_1].$$

Proof 58 We first observe that there exists a constant $M > 0$ such that

$$\frac{b(x^2 + y^2)}{\mathcal{N}(x, y)} < M, \tag{4.11}$$

for every $(x, y) \in \Lambda$. Now, choosing $\rho_0 > 0$ sufficiently large, there exist $u_0^- < 1 < u_0^+$ such that

$$\gamma_{\rho_0} = \mathrm{Graph}(F_0) \cup \mathrm{Graph}(-F_0),$$

where $F_0 : \mathbb{R}^+ \to \mathbb{R}$ is a function such that $F_0(u_0^-) = F_0(u_0^+) = 0$, having constant sign on (u_0^-, u_0^+).

Let us fix L_1 satisfying

$$2L_1 \geq \max_{x \in [u_0^-, u_0^+]} bx^2 - 2 \ln x + 2\rho_0,$$

and consider the set of the couples $(x, y) \in \gamma_{2L_1}$: explicitly,

$$\gamma_{2L_1} = \left\{ (x, y) \in \Lambda : y = \pm\sqrt{2L_1 - (bx^2 - 2 \ln x)} \right\}.$$

Thus, there exist $u_1^- < u_0^- < u_0^+ < u_1^+$ such that, similarly as before,

$$\gamma_{2L_1} = \mathrm{Graph}(F_1) \cup \mathrm{Graph}(-F_1),$$

where $F_1 : \mathbb{R}^+ \to \mathbb{R}$ is defined by $F_1(x) = \sqrt{2L_1 - (bx^2 - 2 \ln x)}$ (and consequently vanishes in u_1^-, u_1^+). On the other hand, we take $L_2 > \mathrm{e}^{2\pi M} L_1$, and consider the level curve γ_{2L_2}, which is explicitly given by

$$\gamma_{2L_2} = \left\{(x,y) \in \Lambda : y = \pm\sqrt{2L_2 - (bx^2 - 2\ln x)}\right\}.$$

Finally, we fix $R_1 > 2L_2$, so that

$$\gamma_{2L_2} \subset \{(x,y) \in \ \Lambda : \mathcal{N}(x,y) \leq R_1\}.$$

Assume that there exists $u(t)$ solving (4.1) such that $\mathcal{N}(u(t_1), u'(t_1)) \geq R_1$, but there is $t_* \in [0, t_1)$ such that $\mathcal{N}(u(t_*), u'(t_*)) \leq \rho_0$. By continuity, we can assume that there exist $t_* < t^*$ such that $(u(t_*), u'(t_*)) \in \gamma_{2L_1}$ and $(u(t^*), u'(t^*)) \in \gamma_{2L_2}$; setting, as in Lemma 4.2.1, $U(t) = \mathcal{N}(u(t), u'(t))/2$, this explicitly means that

$$L_1 < U(t) < L_2 \qquad \text{for every } t \in (t_*, t^*), \qquad U(t_*) = L_1, \quad U(t^*) = L_2. \tag{4.12}$$

According to (4.11) and (4.12), from the definition of $U(t)$ we deduce that

$$U'(t) \leq MU(t), \qquad \text{for every } t \in [t_*, t^*],$$

which implies, thanks to the Gronwall-Bellman Lemma, that

$$U(t) \leq \mathrm{e}^{2\pi M} L_1 \qquad \text{for every } t \in [t_*, t^*].$$

This, however, contradicts (4.12) in view of the definition of L_2.

Now, intuitively speaking, we will prove that either the solutions of (4.1) have the global elasticity property, or their norm in the instant $t = 2\pi$ is lower than in the initial one. This property is useful, and it is similar to the one introduced in [17].

Lemma 4.2.3 *Let $\rho_0 > 0$ be sufficiently large. Then, there exists $R_2 > \rho_0$ such that, for every solution $u(t)$ of (4.1) fulfilling*

$$\max_{t\in[0,2\pi]} \mathcal{N}(u(t), u'(t)) \geq R_2, \tag{4.13}$$

it is either

$$\mathcal{N}(u(t), u'(t)) \geq \rho_0 \quad \textit{for every } t \in [0, 2\pi], \tag{4.14}$$

or

$$\mathcal{N}(u(0), u'(0)) > \mathcal{N}(u(2\pi), u'(2\pi)). \tag{4.15}$$

Proof 59 Let us take R_1 as in the statement of Lemma 4.2.2, for the fixed ρ_0. In the same way, we apply again Lemma 4.2.2, this time with R_1 playing the role of ρ_0, finding the corresponding R_2 for which the statement holds.

Assume now that there exists a solution $u(t)$ of (4.1) satisfying (4.13), for which it is

$$\mathcal{N}(u(0), u'(0)) \leq \mathcal{N}(u(2\pi), u'(2\pi)). \tag{4.16}$$

Since there exists $t_2 \in [0, 2\pi]$ such that $\mathcal{N}(u(t_2), u'(t_2)) \geq R_2$, Lemma 4.2.2 implies that $\mathcal{N}(u(0), u'(0)) \geq R_1$, so that, in view of (4.16), $\mathcal{N}(u(2\pi), u'(2\pi)) \geq R_1$. Consequently, using again Lemma 4.2.2, we obtain that $\mathcal{N}(u(t), u'(t)) \geq \rho_0$ for $t \in [0, 2\pi]$.

We are now able to show that an adaptation of the arguments in [15, 21] to our equation allows to prove that the global elasticity property cannot be fulfilled for solutions of (4.1) with large norm which perform an integer number of revolutions when t goes from 0 to 2π.

Lemma 4.2.4 *Under the hypotheses of Theorem 4.2.1, there exists $R_2 > 0$ such that,* *if $u(t)$ is a solution of (4.1) which satisfies*

$$\max_{t\in[0,2\pi]} \mathcal{N}(u(t),u'(t)) \geq R_2$$

and $(u(t),u'(t))$ performs an integer number of turns around $(1,0)$ in the time interval $[0,2\pi]$, then (4.15) holds.

Proof 60 In view of (4.5), (4.6), there exists a positive number δ such that

$$\frac{1}{2\pi}\int_0^{2\pi} \min\left\{\frac{b(1+\cos t)-\delta}{B_+},1\right\} dt > \frac{n}{2\sqrt{B_+}},$$
$$\frac{1}{2\pi}\int_0^{2\pi} \max\left\{\frac{b(1+\cos t)+\delta}{A_+},1\right\} dt < \frac{n+1}{2\sqrt{A_+}}.$$

In correspondence of δ, we can find $K_\delta > 0$ such that

$$[b(1+\cos t)-\delta)](x-1)^2 - K_\delta < \left[b(1+\cos t)x - \frac{1}{x}\right](x-1)$$
$$< [b(1+\cos t)+\delta)](x-1)^2 + K_\delta \qquad \text{for every } x\in[1,+\infty), \quad t\geq 0. \quad (4.17)$$

Moreover, we choose ρ_1 and B'_+ large, in such a way that the following relations hold:

$$\left(\frac{1}{\sqrt{B_+}}+\frac{1}{\sqrt{B'_+}}\right)^{-1}\left[\frac{1}{2\pi}\int_0^{2\pi}\min\left\{\frac{b(1+\cos t)-\delta}{B_+},1\right\}dt - \frac{K_\delta}{\rho_1}\right] > \frac{n}{2}, \qquad (4.18)$$
$$\sqrt{A_+}\left[\frac{1}{2\pi}\int_0^{2\pi}\max\left\{\frac{b(1+\cos t)+\delta}{A_+},1\right\}dt + \frac{K_\delta}{\rho_1}\right] < \frac{n+1}{2}. \qquad (4.19)$$

In order to perform the estimates leading to the result, we first fix $\rho_0 > 0$ sufficiently large and apply Lemma 1.15 in order to find $R_2 > \rho_0$ such that the statement therein holds. Then, we fix a solution $u(t)$ of (4.1) satisfying (4.13) and such that, in the phase plane, the couple $(u(t),u'(t))$ performs an integer number of revolutions around $(1,0)$ -

say $k \in \mathbb{N}$ - during the time interval $[0, 2\pi]$.

Thus, assume by contradiction that (4.15) is not satisfied; then, in view of Lemma 1.15, $u(t)$ fulfills (4.14). We are now going to estimate the time needed by $(u(t), u'(t))$ to rotate k times around the point $(1, 0)$, by dividing the half-plane Λ in vertical strips and analyzing the behavior of the solution in each strip, following the procedure used in [21].

As a first step, we perform our estimates in the strip $\{x > 1\}$. Passing to modified polar coordinates around $(1, 0)$ by writing

$$-\mu u = -\mu + \rho \sin \vartheta, \qquad u' = \rho \cos \vartheta, \tag{4.20}$$

where $\mu > 0$, we obtain

$$-\dot{\vartheta}(t) = \mu \frac{{u'}^2(t) - u'(t)(u(t) - 1)}{\mu^2 (u(t) - 1)^2 + {u'}^2(t)} \quad \text{for every } t \in [0, 2\pi]. \tag{4.21}$$

Setting

$$J_+ = \{t \in [0, 2\pi] : u(t) \geq 1\}, \qquad J_- = \{t \in [0, 2\pi] : u(t) < 1\},$$

in view of the properties of the modified rotation numbers (see for instance [14]) we have that

$$2\pi \cdot \frac{k}{2} = -\int_{J_+} \dot{\theta}(t)\, dt.$$

Consequently, in view of (4.17),

$$\begin{aligned} \frac{k}{2} \;&\geq\; \frac{\mu}{2\pi} \int_{J_+} \frac{{u'}^2 + [b(1 + \cos t) - \delta](u - 1)^2}{\mu^2 (u - 1)^2 + {u'}^2} dt - \frac{\mu}{2\pi} \int_{J_+} \frac{K_\delta}{\mu^2 (u - 1)^2 + {u'}^2} dt \\ &\geq\; \frac{\mu}{2\pi} \int_{J_+} \frac{\min\left\{\frac{b(1+\cos t) - \delta}{\mu^2}, 1\right\} (u - 1)^2 + (u'/\mu)^2}{(u - 1)^2 + (u'/\mu)^2} dt - \frac{\mu}{2\pi} \int_{J_+} \frac{K_\delta}{\mu^2 (u - 1)^2 + {u'}^2} dt. \end{aligned}$$

Taking into account that the function

$$\Psi : [0, +\infty) \to \mathbb{R}, \qquad \Psi(y) = \frac{\alpha + y}{\beta + y} \tag{4.22}$$

is non-decreasing whenever $\alpha \leq \beta$, choosing $\mu = \sqrt{B_+}$, $\alpha = \min\left\{\frac{b(1+\cos t)-\delta}{\mu^2}, 1\right\}(u-1)^2$, $\beta = (u-1)^2$ and $y = (u'/\mu)^2$ we have

$$\frac{k}{2} \geq \frac{\sqrt{B_+}}{2\pi} \int_{J_+} \min\left\{\frac{b(1+\cos t)-\delta}{B_+}, 1\right\} dt - \frac{\sqrt{B_+}}{2\pi} \int_{J_+} \frac{K_\delta}{B_+(u-1)^2 + {u'}^2}\, dt. \tag{4.23}$$

Without loss of generality, we can assume (up to enlarging ρ_0) that R_2 is sufficiently large, so that

$$B_+(u-1)^2 + {u'}^2 \geq \rho_1, \qquad \text{for every } t \in J_+.$$

Therefore, (4.23) implies

$$\frac{k}{2\sqrt{B_+}} \geq \frac{1}{2\pi} \int_{J_+} \min\left\{\frac{b(1+\cos t)-\delta}{B_+}, 1\right\} dt - \frac{K_\delta}{\rho_1}. \tag{4.24}$$

We now pass to compute the time spent by $(u(t), u'(t))$ to perform $k/2$ revolutions on the "left" half phase plane, i.e. when $u \in (0, 1]$. Preliminarily, we fix

$$\widetilde{\eta} < \frac{2\pi}{\sqrt{B_+'}}, \qquad K = \left(\frac{2\pi}{\widetilde{\eta}}\right)^2 \tag{4.25}$$

and observe that, since

$$\lim_{x \to 0^+} b(1+\cos t)x - \frac{1}{x} = -\infty,$$

there exists $0 < d < 1$ such that

$$b(1+\cos t)x - \frac{1}{x} < -K \quad \text{for every } x \in (0, d]. \tag{4.26}$$

In this way it is possible to define both the sets

$$J_d^- = \{t \in J_- \mid u(t) \le d\}\,, \qquad J_d^+ = \{t \in J_- \mid d < u(t) < 1\}$$

and, correspondingly, the time instants t_1, t_2, t_3 and t_4 (as in [25, Figure 1]) such that, in the time $t_4 - t_1$, the couple $(u(t), u'(t))$ performs half a turn in the "left" half phase plane ($u \in (0, 1]$), and

$$u(t_1) = 1 = u(t_4), \quad u(t_2) = d = u(t_3), \quad (t_1, t_2) \cup (t_3, t_4) \subseteq J_d^+, \qquad [t_2, t_3] \subseteq J_d^-.$$

$$u = 1 + \rho \cos \vartheta, \qquad u' = \rho \sin \vartheta,$$

we arrive at

$$-\,\dot{\vartheta}(t) = \frac{{u'}^2(t) - u'(t)(u(t) - 1)}{(u(t) - 1)^2 + {u'}^2(t)}. \tag{4.27}$$

In view of (4.26), we deduce that

$$-\dot{\vartheta}(t) > K \cos^2 \vartheta(t) + \sin^2 \vartheta(t), \quad t \in [t_2, t_3],$$

so that

$$\begin{aligned} t_3 - t_2 &= \int_{\vartheta(t_3)}^{\vartheta(t_2)} \frac{ds}{K \cos^2 s + \sin^2 s} ds \\ &= \frac{1}{\sqrt{K}} \left[\arctan \left(\frac{\tan \vartheta(t_2)}{K} \right) - \arctan \left(\frac{\tan \vartheta(t_3)}{K} \right) \right] \\ &\le \frac{\pi}{2\sqrt{K}}. \end{aligned}$$

According to (4.25), it follows that $t_3 - t_2 < \widetilde{\eta}/2$; repeating the argument for every revolution made by (u,u') around $(1,0)$ yields

$$\text{meas}\left(J_d^-\right) < \frac{k}{4}\widetilde{\eta}. \tag{4.28}$$

In order to compute $t_2 - t_1$, we observe that, thanks to (4.27), it holds

$$-\dot{\vartheta}(t) \geq \frac{{u'}^2(t) - \widetilde{C}|1-d|}{(1-d)^2 + {u'}^2(t)} \quad \text{for every } t \in [t_1, t_2],$$

where $\widetilde{C} = \max_{x\in[d,1]} 2bx + 1/x$. Again, we assume that ρ_0 is large enough, so that $-\dot{\vartheta}(t) > 1/2$ on $[t_1,t_2]$, and $t_2 - t_1 < \widetilde{\eta}/4$. Analogously, one can prove that $t_4 - t_3 < \widetilde{\eta}/4$, having thus that

$$\text{meas}(J_d^+) < \frac{k}{2}\frac{\widetilde{\eta}}{2}. \tag{4.29}$$

Thus, in view of (4.25) and (4.28), we deduce that

$$\text{meas}(J_-) = \text{meas}(J_d^+) + \text{meas}(J_d^-) < \frac{k}{2}\widetilde{\eta} < k\,\frac{\pi}{\sqrt{B'_+}},$$

from which

$$\frac{k}{2\sqrt{B'_+}} > \frac{1}{2\pi}\,\text{meas}(J_-).$$

Summing up, from (4.24) we have

$$\frac{k}{2}\left(\frac{1}{\sqrt{B_+}} + \frac{1}{\sqrt{B'_+}}\right) \geq \frac{1}{2\pi}\int_0^{2\pi} \min\left\{\frac{b(1+\cos t) - \delta}{B_+}, 1\right\} dt - \frac{K_\delta}{\rho_1}. \tag{4.30}$$

On the other hand, reasoning on (4.21) with a similar argument and taking (4.17) into account, we have

$$\frac{k}{2} \leq \frac{\mu}{2\pi}\int_{J_+} \frac{\max\{\frac{b(1+\cos t)+\delta}{A_+}, 1\}(u-1)^2 + (u'/\mu)^2}{(u-1)^2 + (u'/\mu)^2} dt + \frac{\mu}{2\pi}\int_{J_+} \frac{K_\delta}{\mu^2(u-1)^2 + {u'}^2} dt.$$

Since the function Ψ defined in (4.22) is non-increasing whenever $\alpha \geq \beta$, choosing $\mu = \sqrt{A_+}$ and taking $\alpha = \max\left\{\frac{b(1+\cos t)+\delta}{A_+}, 1\right\}(u-1)^2$ and $\beta = (u-1)^2$, we obtain

$$\frac{k}{2} \leq \frac{\sqrt{A_+}}{2\pi}\int_{J_+} \max\{\frac{b(1+\cos t)+\delta}{A_+}, 1\}dt + \frac{\sqrt{A_+}}{2\pi}\int_{J_+} \frac{K_\delta}{A_+(u-1)^2 + {u'}^2}dt.$$

Again, we can assume ρ_0 (and thus R_2) so large that

$$\sqrt{A_+}(u(t)-1)^2 + \dot{u}(t)^2 \geq \rho_1, \quad t \in J_+.$$

Hence,

$$\frac{k}{2\sqrt{A_+}} \leq \frac{1}{2\pi}\int_{J_+} \max\left\{\frac{b(1+\cos t)+\delta}{A_+}, 1\right\} dt + \frac{K_\delta}{\rho_1}. \tag{4.31}$$

We are now able to conclude the proof. Assume first that $u(t)-1$ has at most $2n$ zeros. Then $k \leq n$, but this contradicts (4.18) and (4.30). On the contrary, if $u(t)-1$ has at least $2n+2$ zeros, since $k \in \mathbb{N}$ it has to be $k \geq n+1$. However, this contradicts (4.19) and (4.31). The proof is completed.

Remark 4.2.4 In [9], the relationships between conditions (4.5) and (4.6) and the rotation number of "large" solutions of a first order planar system were highlighted. This perfectly agrees with what we have seen in the proof which has just been performed; indeed, conditions (4.5) and (4.6) force the solutions of the Cauchy problems associated with (4.1) not to perform an integer number of turns around $(1, 0)$ in the time interval $[0, 2\pi]$. Thus, they turn to be hypotheses on the number of rotations made by the solutions of equation (4.1) in the phase plane.

Using the previous results, a basic application of the Poincaré-Bohl Theorem allows to prove Theorem 4.2.1.

Proof 61 (Proof of Theorem 4.2.1) Let us take R_2 sufficiently large satisfying Lemma 4.2.4

and set $B = \{(x,y) \in \Lambda : \mathcal{N}(x,y) \leq R_2\}$. In view of Lemma 4.2.1, the Poincaré map

$$P : B \to \mathbb{R}^2, \qquad P(x_0, y_0) = (u(2\pi), u'(2\pi)),$$

where $(u(t), u'(t))$ is the unique solution of the problem

$$u'' + b(1+\cos t)u - \frac{1}{u} = 0, \qquad u(0) = x_0 > 0, \quad u'(0) = y_0,$$

is well defined. Moreover, if (x_0, y_0) is a fixed point of P, then it is $(u(0), u'(0)) = (u(2\pi), u'(2\pi))$, i.e., $u(t)$ is a periodic solution of (4.1). Therefore, to get the conclusion it is sufficient to prove that P has a fixed point. However, if we denote by τ_1 the unitary right translation in the plane (u, u'), the map $\Phi := \tau_{-1} \circ P \circ \tau_1 : \tau_{-1}(B) \to \mathbb{R}^2$ satisfies all the hypotheses of the Poincaré-Bohl fixed point theorem, since $0 \in \tau_{-1}(B)$ and $\Phi(z) \neq \lambda z$ for every $\lambda > 1$, in view of Lemma 4.2.4. Consequently, P has a fixed point and the statement is proved.

4.2.2 Main result

In order to prove our Theorem it will be convenient, for any $n \in \mathbb{N}$, to define the absolutely continuous functions $F_n, G_n : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by

$$F_n(b,x) = \frac{1}{2\pi}\int_0^{2\pi} \min\left\{\frac{b(1+\cos t)}{\sqrt{x}}, \sqrt{x}\right\} dt - \frac{n}{2},$$
$$G_n(b,x) = \frac{1}{2\pi}\int_0^{2\pi} \max\left\{\frac{b(1+\cos t)}{\sqrt{x}}, \sqrt{x}\right\} dt - \frac{n+1}{2}.$$

Both functions are non-decreasing with respect to the variable b. Moreover, if there exists $n \in \mathbb{N}$ such that $\inf_{x>0} G_n(b,x) < 0$ and $\sup_{x>0} F_n(b,x) > 0$, then Theorem 4.2.1 implies that (4.1) has at least one periodic solution. Therefore, we have the following proposition.

Proposition 4.2.1 *Assume that there exists $n \in \mathbb{N}$ such that*

$$b \in \left(\inf \left\{ b > 0 : \sup_{x>0} F_n(b,x) > 0 \right\}, \sup \left\{ b > 0 : \inf_{x>0} G_n(b,x) < 0 \right\} \right). \tag{4.32}$$

Then, (4.1) has at least one 2π-periodic solution.

Let us first observe that, in view of the continuity and the monotonicity of the functions F_n, G_n in the variable b, there exist b_0^n and b_1^n such that

$$\left\{ b > 0 : \sup_{x>0} F_n(b,x) > 0 \right\} = (b_0^n, +\infty),$$

and

$$\left\{ b > 0 : \inf_{x>0} G_n(b,x) < 0 \right\} = (0, b_1^n).$$

The point is to prove that these two intervals contain common points, i.e., $b_0^n < b_1^n$. We will show this in the case when $n = 0$ and $n = 1$, and the estimates performed in this last case will allow to achieve the new result consisting in the following Theorem.

Theorem 4.2.2 *If $b \in [0.4705, 0.59165]$, then (4.1) has at least one 2π-periodic solution.*

In particular, a gross estimation of the interval in (4.32) would lead to prove existence for

$$b \in \left(\frac{n^2}{2}, \frac{(n+1)^2}{4} \left(\frac{\pi}{1+\pi} \right)^2 \right). \tag{4.33}$$

Indeed, setting $B_+ = 2b$, since $b > n^2/2$ we have

$$F_n(b, B_+) = \frac{1}{2\pi}\sqrt{b} \int_0^{2\pi} \min \left\{ \frac{1+\cos t}{\sqrt{2}}, \sqrt{2} \right\} dt - \frac{n}{2} = \sqrt{\frac{b}{2}} - \frac{n}{2} > 0.$$

On the other hand, we choose

$$A_+ = \frac{4b^2}{(n+1)^2}\Big(\frac{\pi+1}{\pi}\Big)^2,$$

so that, since $b < \frac{1}{4}(n+1)^2(\pi/(1+\pi))^2$,

$$\begin{aligned} G_n(b, A_+) &= \frac{1}{2\pi}\int_0^{2\pi} \max\left\{\frac{(n+1)\pi(1+\cos t)}{2(\pi+1)}, \frac{2b}{n+1}\frac{\pi+1}{\pi}\right\} dt - \frac{n+1}{2} \\ &< \frac{1}{2\pi}\frac{(n+1)\pi}{2(\pi+1)}\int_0^{2\pi} \max\{1+\cos t, 1\} dt - \frac{n+1}{2} = 0. \end{aligned}$$

Now, in order for the interval in (4.33) to be nonempty, we need

$$\frac{n^2}{2} < \frac{(n+1)^2}{4}\left(\frac{\pi}{1+\pi}\right)^2,$$

which approximately requires $n < 1.1$. Since $n \in \mathbb{N}$, we can take either $n = 0$ or $n = 1$, so that the 2π-periodic solvability of (4.1) is guaranteed whenever

$$b \in \left(0, \frac{1}{4}\left(\frac{\pi}{1+\pi}\right)^2\right) \cup \left(\frac{1}{2}, \left(\frac{\pi}{1+\pi}\right)^2\right).$$

However, taking into account that F_1, G_1 are non-decreasing, we can use a numerical approach to estimate the interval in (4.32) and try to compute approximately, by means of a numerical software, its endpoints, obtaining

$$\sup_{x>0} F_1(0.4705, x) > 0 \qquad (\text{but } \sup_{x>0} F_1(0.47, x) < 0)$$

and

$$\inf_{x>0} G_1(0.59165, x) < 0 \qquad (\text{but } \inf_{x>0} G_1(0.591, x) > 0)\ ,$$

whence the statement of Theorem 4.2.2.

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