EXISTENCE AND STABILITY OF PERIODIC OSCILLATIONS OF A RIGID DUMBBELL SATELLITE AROUND ITS CENTER OF MASS

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Abstract. We study the existence of stable and unstable periodic solutions of a differential equation that models the planar oscillations of a satellite in an elliptic orbit around its center of mass. The proof is based on a suitable version of Poincaré-Birkhoff theorem and the third order approximation method.

1. Introduction

If we consider a satellite moving in a central Newtonian gravitational field and suppose that the satellite is a rigid body whose center of mass moves in an elliptical orbit, the motion of the satellite can be described as the equation

$$(1 + e\cos\nu)\frac{d^2\varphi}{d\nu^2} - 2e\sin\nu\frac{d\varphi}{d\nu} + \alpha\sin\varphi\cos\varphi = 2e\sin\nu, \tag{1.1}$$

where φ is the angle between one of the satellites' principal central axes of inertia, lying in the orbit plane, and the radius vector of its center of mass, ν is the true anomaly, $0 \le e \le 1$ is the eccentricity of the ellipse, $\alpha > 0$ is the inertial parameter of the satellite and in general $0 < \alpha \le 3$.

This equation was introduced by Beletskii in 1959. See [1, 2] and the references therein. During the last few decades, such an equation has been studied by many researchers and a wide number of articles can be found in the literature [2, 3, 12, 13, 18, 24, 28, 29, 30]. For example, it is well known [2] that if $\alpha = 6e$, then (1.1) admits the particular solution $\varphi = \frac{\nu}{2}$, which corresponds that the satellite rotates in the orbit plane performing three revolutions in absolute space per two revolutions of the center of mass in the orbit. Petrhysyn and Yu made a major progress in [24] to prove the existence of periodic orbits by using Galerkin type finite-dimensional approximations. Later the existence was solved by Hai in [12] by a variational argument, and additional restrictions over the parameters e, α in [24] were eliminated. The second solution was found in [13] by using the mountain pass theorem.

Compared with the existence of periodic solutions, the analysis of the stability of the solutions is less explored and most known stability results are based

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on numerical calculations or concerned with the linear stability. For example, in [30] the regions of stability for the parameters were computed numerically. Later some explicit criteria for linear stability were derived in [24]. However, due to the symplectic charater of the system the linear stability does not imply directly the nonlinear stability of the periodic solutions, which may depend in general on the higher order terms of the asymptitic expansion. Besides the numerical methods, the first analytical results about the nonlinear stability was proved by Nuñez and Torres in [22], as a natural continuation of [21], in which a region of parameters to ensure the existence of twist periodic solutions was explicitly described. It was proved that if $\alpha \leq 1/18$, (**??**) admits a twist periodic solution. Although numerical experiments show that one can expect stability for $\alpha < 1/2$ except for some strong resonances, the theoretical approach can not arrive at this expected results up to now. In a more recent paper [4], a rigorous analysis of nonlinear stability for resonant rotation and the cases of eccentricity close to 1 are studied.

In this paper, we continue the study of some dynamical aspects of eq. (1.1). The first task is to prove the abundance of periodic solutions with arbitrary minimal period and prescribed winding number. The existence of an infinite number of periodic solutions is often considered as a signature of complex dynamics. A direct outcome of the method is that such solutions come in couples and one of them is always unstable. The proof is based on a suitableversion of Poincaré-Birkhoff theorems, which were originally conjectured by Poincaré in 1912 when he studied the restricted three body problem, and were first proved by Birkhoff in 1913. During the last century, different proofs and developments were given. We refer the reader to [19, Section 2.1] for a short review on Poincaré-Birkhof theorems.

The second objective is to find some new conditions for nonlinear stability. The main tool is the method of third order approximation, which was developed by Ortega [23] and Zhang [26] for general time-periodic Lagrangian equations. During the past few years, there has been considerable progress on this topic, we refer the reader to [5, 6, 7, 8, 11, 16, 17, 22] and references therein. In general, the third order approximation method is applicable for conservative systems, and cannot be applied for damped equations, which are in general dissipative. However, as shown in [10], we can also establish the third order approximation method for damped differential equations when the damping coefficient has zero mean, as in our model (1.1). In this way, we are able to identify a stability region on the parameter plane α, e that is different from that of [22].

The paper is organized as follows. In Section 2, we present some preliminary results, which includes the Poincaré-Birkhoff theorem and some basic facts about the third order approximation method. Moreover, we prove a new stability criterion for damped differential equations. Section 3 is contains the proof of the existence of an infinite number of periodic solutions ecoded by the minimal period and the winding number. Finally, in Section 4 we apply the stability criterion obtained in Section 2 to derive a new region of stability.

2. Preliminaries and a new stability criterion

2.1. **The Poincaré-Birkhoff theorem.** Consider two strips $A = \mathbb{R} \times [-a, a]$ and $B = \mathbb{R} \times [-b, b]$, where $b > a > 0$. We will work with a C^k-diffeomorphism $f : A \to B$ defined by

$$f(\theta, r) = (Q(\theta, r), P(\theta, r)),$$

where $Q, P : \mathbb{R}^2 \to \mathbb{R}$ are functions of class C^k satisfying the periodicity conditions

$$Q(\theta+\pi, r) = Q(\theta, r) + \pi, \quad P(\theta+\pi, r) = P(\theta, r).$$

Such generalized periodicity conditions tell us that the map is the lift to $\mathbb{R}^2$ of the corresponding map $\bar{f} : \bar{A} \to \bar{B}$, where $\bar{A} = \mathbb{R}/\mathbb{Z} \times [-a, a]$ and $\bar{B} = \mathbb{R}/\mathbb{Z} \times [-b, b]$. After the identification $\theta + \pi = \theta$, the domain of f can be interpreted as an annulus or a cylinder. We shall think that it is a cylinder with vertical coordinator r and the variable θ as an angle. We say that f is isotopic to the inclusion, if there exists a function $H : A \times [0, 1] \to B$ such that for every $\lambda \in [0, 1]$, $H_\lambda(x) = H(\lambda, x)$ is a homeomorphism with $H_0(x) = f(x)$ and $H_1(x) = x$. The class of the maps satisfying the above characteristics will be indicated by $\varepsilon^k(A)$.

We say that $f \in \varepsilon^1(A)$ is exact symplectic if there exists a smooth function $V = V(\theta, r)$ with $V(\theta + \pi, r) = V(\theta, r)$ and such that

$$dV = PdQ - rd\theta. \tag{2.1}$$

The following theorem is a slight modified version of Poincaré-Birkhoff theorem proved by Franks in [15] and the statement on the instability was reproved by Maròin [19]. Here we say that a fixed point p_1 of the one-to-one map $f : U \subset \mathbb{R}^N \to \mathbb{R}^N$ is stable in the sense of Lyapunov if for every neighbourhood U_{p_1} of p_1 there exists another neighbourhood $U^* \subset U_{p_1}$ such that, for each $n > 0$, $f^n(U^*)$ is well defined and $f^n(U^*) \subset U_{p_1}$.

Theorem 2.1. [15, 19] *Let $f : A \to B$ be an exact symplectic diffeomorphism belonging to $\varepsilon^2(A)$ such that $f(A) \subset \mathrm{int}(B)$. Suppose that there exists a constant $\epsilon > 0$ such that*

$$\begin{aligned} &Q(\theta, a) - \theta > \epsilon, && \forall\ \theta \in [0, \pi), \\ &Q(\theta, -a) - \theta < -\epsilon, && \forall\ \theta \in [0, \pi). \end{aligned} \tag{2.2}$$

Then f has at least two distinct fixed points p_1 and p_2 in A such that $p_1 - p_2 \neq (k\pi, 0)$ for every $k \in \mathbb{Z}$. Moreover, at least one of the fixed points is unstable if f is analytic.

2.2. **The third-order approximation method.** Given a function $a(\nu)$, let us denote $a_+(\nu) = \max\{a(\nu), 0\}$ and $a_-(\nu) = \max\{-a(\nu), 0\}$ the positive and the negative parts of $a(\nu)$. Obviously $a(\nu) = a_+(\nu) - a_-(\nu)$.

Consider the damped differential equation

$$\frac{d^2\varphi}{d\nu^2} + h(\nu)\frac{d\varphi}{d\nu} + g(\nu, \varphi) = 0, \tag{2.1}$$

where

$$h \in \tilde{L}^1(\mathbb{R}/2\pi\mathbb{Z}) := \left\{ h \in L^1(\mathbb{R}/2\pi\mathbb{Z}) : \bar{h} = \frac{1}{2\pi}\int_0^{2\pi} h(\nu)d\nu = 0 \right\}.$$

Given a 2π-periodic solution ψ, we can expand (2.1) around ψ in the following way

$$\frac{d^2\varphi}{d\nu^2} + h(\nu)\frac{d\varphi}{d\nu} + a(\nu)\varphi + b(\nu)\varphi^2 + c(\nu)\varphi^3 + \cdots = 0, \tag{2.2}$$

where $a, b, c \in C(\mathbb{R}/2\pi\mathbb{Z})$ are given as

$$a(\nu) = g_\varphi(\nu, \psi(\nu)), \quad b(\nu) = \frac{1}{2}g_{\varphi\varphi}(\nu, \psi(\nu)), \quad c(\nu) = \frac{1}{6}g_{\varphi\varphi\varphi}(\nu, \psi(\nu)). \tag{2.3}$$

The linearized equation for (2.1) is given as

$$\frac{d^2\varphi}{d\nu^2}+h(\nu)\frac{d\varphi}{d\nu}+a(\nu)\varphi=0. \tag{2.4}$$

The Poincaré matrix of (2.4) is

$$M(2\pi)=\left(\begin{array}{cc}\phi_1(2\pi) & \phi_2(2\pi)\\ \phi_1'(2\pi) & \phi_2'(2\pi)\end{array}\right),$$

where $\phi_1(\nu)$ and $\phi_2(\nu)$ are real-valued solutions of (2.4) satisfying

$$\phi_1(0)=1,\quad \phi_1'(0)=0,\quad \phi_2(0)=0,\quad \phi_2'(0)=1.$$

Since $h\in\tilde{L}^1(\mathbb{R}/2\pi\mathbb{Z})$, we know $\det M(2\pi)=1$, which plays a crucial role in establishing the third order approximation method because only in this case (2.1) becomes a conservative system. The eigenvalues $\lambda_{1,2}$ of M are called the Floquet multipliers of (2.4). Obviously $\lambda_1\cdot\lambda_2=1$. We say that (2.4) is elliptic if $\lambda_1=\overline{\lambda}_2$, $|\lambda_1|=1$, $\lambda_1\neq\pm1$.

Define

$$\sigma(h)(\nu)=\exp\left(\int_0^\nu h(s)ds\right),$$

and

$$\tau(\nu)=\int_0^\nu \sigma(-h)(s)ds.$$

Then $\tau(\nu)$ is increasing in ν and $\tau(\nu+2\pi)\equiv\tau(\nu)+T$, where $T=\tau(2\pi)>0$. Using the change of time $\varsigma=\tau(\nu)$, the function

$$\hat{a}(\varsigma)=a(\tau^{-1}(\varsigma))\sigma(2h)(\tau^{-1}(\varsigma))$$

is T-periodic. Then the first twist coefficient β of the nonlinear damped equation (2.2) can be written as, up to a positive factor,

$$\begin{aligned}\beta \;=\; & \iint_{[0,2\pi]^2} b(\nu)b(s)\sigma(h)(\nu)\sigma(h)(s)R^3(\nu)R^3(s)\chi_\theta(|\vartheta(\nu)-\vartheta(s)|)\mathrm{d}\nu\mathrm{d}s\\ & -\frac{3}{8}\int_0^{2\pi} c(\nu)\sigma(h)(\nu)R^4(\nu)\mathrm{d}\nu,\end{aligned} \tag{2.5}$$

where the kernel $\chi(\cdot)$ is given by

$$\chi_\theta(x)=\frac{3\cos(x-\theta/2)}{16\sin(\theta/2)}+\frac{\cos 3(x-\theta/2)}{16\sin(3\theta/2)},\qquad x\in[0,\theta]. \tag{2.6}$$

$R(\nu)=r(\tau(\nu))$, $\vartheta(\nu)=\hat{\vartheta}(\tau(\nu))$, r is the positive T-periodic solution of

$$\frac{d^2r}{d\varsigma^2}+\hat{a}(\varsigma)r(\varsigma)=\frac{1}{r^3(\varsigma)}. \tag{2.7}$$

$\theta=T\rho$, ρ is the rotation number of

$$\frac{d^2y}{d\varsigma^2}+\hat{a}(\varsigma)y(\varsigma)=0, \tag{2.8}$$

and $\hat{\varphi}$ is given by

$$\hat{\varphi}(\varsigma)=\int_0^\varsigma\frac{1}{r^2(\xi)}d\xi.$$

The formula (2.5) was obtained in [10]. We say that the equilibrium of the nonlinear system (2.2) is twist if the linear equation (2.4) is elliptic and the first twist coefficient $\beta \neq 0$. By the Moser twist theorem [25], a twist periodic solution is necessarily stable in the sense of Lyapunov.

2.3. **A new stability criterion.** In this subsection, we prove a novel stability criterion for the nonlinear equation (2.2).

Theorem 2.2. *Assume that $a, b, c \in C(\mathbb{R}/2\pi\mathbb{Z})$ and there exist constants σ_1, σ_2 such that $0 \leq \sigma_1 \leq \sigma_2 \leq \frac{\pi}{2T}(T = \tau(2\pi))$ and*

$$\sigma_1^2 \leq a(t)\sigma(2h)(t) \leq \sigma_2^2, \qquad \text{for all } t. \tag{2.9}$$

Suppose further that

$$\frac{\|B_+\|_1 \|B_-\|_1}{\min\left\{\sin(\frac{3T\sigma_1}{2}), \sin(\frac{3T\sigma_2}{2})\right\}} < \frac{3}{10}\sigma_1^3 \left[\frac{\|C_-\|_1}{\sigma_2^2} - \frac{\|C_+\|_1}{\sigma_1^2}\right], \tag{2.10}$$

where

$$B(\nu) = b(\nu)\sigma(h)(\nu), \quad C(\nu) = c(\nu)\sigma(h)(\nu).$$

Then the trivial solution $x = 0$ of (2.2) is twist and therefore is stable.

Proof. Under the condition (2.9), we know that

$$0 < \sigma_1^2 \leq \hat{a}(\varsigma) = a(t)\sigma(2h)(\nu) \leq \sigma_2^2 \leq \left(\frac{\pi}{2T}\right)^2,$$

therefore equation (2.8) is in the first stability zone and is elliptic [27]. In this case, equation (2.7) has a unique positive T-periodic solution $r(\varsigma)$. Moreover $R(\nu) = r(\varsigma)$. Following from [20, Lemma 4.2], we have the estimates

$$\sigma_2^{-1/2} \leq R(\nu) \leq \sigma_1^{-1/2}, \qquad \text{for all} \quad \nu. \tag{2.11}$$

Moreover, we know that the rotation number ρ of equation (2.8) satisfies

$$0 \leq \sigma_1 \leq \rho \leq \sigma_2 \leq \frac{\pi}{2T},$$

and therefore

$$0 < T\sigma_1 \leq \theta = T\rho \leq T\sigma_2 < \frac{\pi}{2}. \tag{2.12}$$

In this case, the kernel $\chi(x) > 0$ for all $x \in [0, \theta]$. Moreover,

$$\min_{x \in [0,\theta]} \chi_\theta(x) = \chi(0) = \frac{3\cos(\theta/2) + 2\cos(3\theta/2)}{8\sin(3\theta/2)},$$

and

$$\max_{x \in [0,\theta]} \chi_\theta(x) = \chi(\theta/2) = \frac{3\cos\theta + 2}{8\sin(3\theta/2)}.$$

Now we are able to estimate the twist coefficient as follows. The term containing $c(\nu)$ is

$$\begin{aligned}
\beta_1 &= -\frac{3}{8}\int_0^{2\pi} c(\nu)\sigma(h)(\nu)R^4(\nu)\mathrm{d}\nu \\
&= \frac{3}{8}\int_0^{2\pi} c_-(\nu)\sigma(h)(\nu)R^4(\nu)\mathrm{d}\nu - \frac{3}{8}\int_0^{2\pi} c_+(\nu)\sigma(h)(\nu)R^4(\nu)\mathrm{d}\nu \\
&\geq \frac{3}{8}R_{\min}^4\|C_-\|_1 - \frac{3}{8}R_{\max}^4\|C_+\|_1 \\
&\geq \frac{3}{8\sigma_2^2}\|C_-\|_1 - \frac{3}{8\sigma_1^2}\|C_+\|_1. \qquad (2.13)
\end{aligned}$$

The term containing $b(\nu)$ is

$$\begin{aligned}
\beta_2 &= \iint_{[0,2\pi]^2} b(\nu)\sigma(h)(\nu)b(s)\sigma(h)(s)R^3(\nu)R^3(s)\chi_\theta(|\vartheta(\nu)-\vartheta(s)|)\mathrm{d}\nu\mathrm{d}s \\
&= \iint_{[0,2\pi]^2} (B_+(\nu)B_+(s)+B_-(\nu)B_-(s))R^3(\nu)R^3(s)\chi_\theta(|\vartheta(\nu)-\vartheta(s)|)\mathrm{d}\nu\mathrm{d}s \\
&\quad - \iint_{[0,2\pi]^2} (B_+(\nu)B_-(s)+B_-(\nu)B_+(s))R^3(\nu)R^3(s)\chi_\theta(|\vartheta(\nu)-\vartheta(s)|)\mathrm{d}\nu\mathrm{d}s \\
&\geq - \iint_{[0,2\pi]^2} (B_+(\nu)B_-(s)+B_-(\nu)B_+(s))R^3(\nu)R^3(s)\chi_\theta(|\vartheta(\nu)-\vartheta(s)|)\mathrm{d}\nu\mathrm{d}s \\
&\geq -\max_\theta \chi(\theta)R_{\max}^6 \iint_{[0,2\pi]^2} (B_+(\nu)B_-(s)+B_-(\nu)B_+(s))\mathrm{d}\nu\mathrm{d}s \\
&= -\frac{3\cos\theta+2}{4\sin(3\theta/2)}R_{\max}^6\|B_+\|_1\|B_-\|_1 \\
&\geq -\frac{3\cos\theta+2}{4\sin(3\theta/2)}\frac{1}{\sigma_1^3}\|B_+\|_1\|B_-\|_1.
\end{aligned}$$

Using (2.12) and from the monotonicity of sin and cos, we obtain

$$\frac{3\cos\theta+2}{4\sin(3\theta/2)} \leq \frac{5}{4\min\left\{\sin(\frac{3T\sigma_1}{2}),\sin(\frac{3T\sigma_2}{2})\right\}}.$$

Therefore,

$$\beta_2 \geq -\frac{5\|B_+\|_1\|B_-\|_1}{4\sigma_1^3\min\left\{\sin(\frac{3T\sigma_1}{2}),\sin(\frac{3T\sigma_2}{2})\right\}}. \tag{2.14}$$

Combined the inequalities (2.13) and (2.14), we obtain

$$\beta \geq \frac{3}{8\sigma_2^2}\|C_-\|_1 - \frac{3}{8\sigma_1^2}\|C_+\|_1 - \frac{5\|B_+\|_1\|B_-\|_1}{4\sigma_1^3\min\left\{\sin(\frac{3T\sigma_1}{2}),\sin(\frac{3T\sigma_2}{2})\right\}}. \tag{2.15}$$

It is easy to verify that $\beta > 0$ under the condition (2.10). $\square$

Remark 2.3. *In Theorem* 2.2*, if we assume that all conditions holds with* $0 \leq \sigma_1 \leq \sigma_2 \leq \frac{\pi}{3T}$*, then* $\sin\frac{3T\sigma_1}{2} < \sin\frac{3T\sigma_2}{2}$*. In this case, the trivial solution* $x = 0$ *of* (2.2) *is twist if*

$$\frac{\|B_+\|_1\|B_-\|_1}{\sin(\frac{3T\sigma_1}{2})} < \frac{3}{10}\sigma_1^3\left[\frac{\|C_-\|_1}{\sigma_2^2} - \frac{\|C_+\|_1}{\sigma_1^2}\right]. \tag{2.16}$$

3. Multiplicity of periodic solutions

The objective is to prove our model equation (1.1) fits the hypotheses of the Poincaré-Birkhoff Theorem exposed in Subsection 2.1. One can easily verify that eq. (1.1) can be rewriten as the following Euler-Lagrange equation

$$\frac{\partial}{\partial\nu}\left(\frac{\partial L}{\partial\varphi'}\right) - \frac{\partial L}{\partial\varphi} = 0,$$

where the Lagrangian is given as

$$L(\varphi,\varphi',\nu) = \frac{(1+e\cos\nu)^2(\varphi')^2}{2} + \frac{\alpha(1+e\cos\nu)}{4}\cos 2\varphi + 2e(1+e\cos\nu)\sin\nu\ \varphi.$$

On the other hand, using the change of variables given by the Legendre transform

$$\begin{cases} \varphi = \varphi, \\ \phi = \frac{\partial L}{\partial \varphi'} = (1 + e\cos\nu)^2 \varphi', \end{cases}$$

we can get the Hamiltonian

$$H(\varphi, \phi, \nu) = \frac{\phi^2}{2(1+e\cos\nu)^2} - \frac{\alpha(1+e\cos\nu)}{4}\cos 2\varphi - 2e(1+e\cos\nu)\sin\nu\ \varphi,$$

and the new Hamiltonian system

$$\begin{cases} \varphi' = \frac{\partial H}{\partial \phi} = \frac{\phi}{(1+e\cos\nu)^2}, \\ \phi' = -\frac{\partial H}{\partial \varphi} = -\alpha(1+e\cos\nu)\sin\varphi\cos\varphi + 2e(1+e\cos\nu)\sin\nu. \end{cases} \tag{4.1}$$

Denote by $(\varphi(\nu), \phi(\nu)) = (\varphi(\nu, \theta, r), \phi(\nu, \theta, r))$ the solution of the system (4.1) satisfying the initial condition

$$(\varphi(0), \phi(0)) = (\theta, r). \tag{4.2}$$

It is easy to see that there exist positive constants a_1, a_2 such that

$$|\nabla H(\varphi, \phi, \nu)| \leq a_1\sqrt{\varphi^2 + \phi^2} + a_2,$$

which is the sufficient condition for the following result, see [14].

Lemma 3.1. *The solution* (φ, ϕ) *of problem* (4.1)-(4.2) *is unique and globally defined.*

Let us define the Poincaré map associated to system (4.1) as

$$\begin{aligned} S(\theta, r) &= (Q(\theta, r), P(\theta, r)) \\ &= (\varphi(2\pi, \theta, r), \phi(2\pi, \theta, r)). \end{aligned}$$

Clearly, the fixed points of the Poincaré map S correspond to the 2π-periodic solutions of (4.1). It follows from π-periodicity of the function $\sin\varphi\cos\varphi$ and Lemma 3.1 that

$$\varphi(\nu, \theta + \pi, r) = \varphi(\nu, \theta, r) + \pi, \quad \phi(\nu, \theta + \pi, r) = \phi(\nu, \theta, r). \tag{4.3}$$

Thus

$$Q(\theta + \pi, r) = Q(\theta, r) + \pi, \quad P(\theta + \pi, r) = P(\theta, r),$$

which implies that the map S is defined on the cylinder.

Obviously, the partial derivatives of $H(\varphi, \phi, \nu)$ with respect to the variables (φ, ϕ) of order equal to 2 are continuous in the variables (φ, ϕ, ν). Using the theorem of differentiability with respect to the initial conditions, we know that $S \in C^2(A)$. By the uniqueness of $(\varphi(\nu, \theta, r), \phi(\nu, \theta, r))$, S is a diffeomorphism of A. The isotopy to the identity is given by the flow

$$\begin{aligned} \Psi_\lambda(\theta, r) &= \Psi((1-\lambda)2\pi, \theta, r) \\ &= (\varphi((1-\lambda)2\pi, \theta, r), \phi((1-\lambda)2\pi, \theta, r)),\ \lambda \in [0, 1]. \end{aligned}$$

Notice that $\Psi_0(\theta, r) = S(\theta, r)$, $\Psi_1(\theta, r) = (\theta, r)$ and this isotopy is valid on the cylinder. Now we can assert that the map $S \in \varepsilon^2(A)$.

Theorem 3.2. *Equation* (1.1) *has at least two geometrically distinct* 2π*-periodic solutions and at least one of them is unstable.*

Proof. Consider the C^1 function

$$\begin{aligned}V(\theta,r)=&\int_0^{2\pi}L(\varphi,\varphi',\nu)\\=&\int_0^{2\pi}[\frac{(1+e\cos\nu)^2(\varphi'(\nu,\theta,r))^2}{2}+\frac{\alpha(1+e\cos\nu)}{4}\cos 2(\varphi(\nu,\theta,r))\\&+2e(1+e\cos\nu)\sin\nu\ \varphi(\nu,\theta,r)]d\nu.\end{aligned}$$

By the fact (4.3), we obtain

$$\begin{aligned}V(\theta+\pi,r)=&\int_0^{2\pi}[\frac{(1+e\cos\nu)^2(\varphi'(\nu,\theta+\pi,r))^2}{2}\\&+\frac{\alpha(1+e\cos\nu)}{4}\cos(2\varphi(\nu,\theta+\pi,r))\\&+2e(1+e\cos\nu)\sin\nu\ \varphi(\nu,\theta+\pi,r)]d\nu.\\=&\int_0^{2\pi}[\frac{(1+e\cos\nu)^2(\varphi'(\nu,\theta,r))^2}{2}\\&+\frac{\alpha(1+e\cos\nu)}{4}\cos(2\varphi(\nu,\theta,r))\\&+2e(1+e\cos\nu)\sin\nu\ (\varphi(\nu,\theta,r)+\pi)]d\nu\\=&V(\theta,r)+\int_0^{2\pi}2e(1+e\cos\nu)\sin\nu d\nu\\=&V(\theta,r).\end{aligned}$$

The partial derivatives of $V(\theta,r)$ is computed as

$$\begin{aligned}V_\theta(\theta,r)=&\int_0^{2\pi}[(1+e\cos\nu)^2\varphi'(\nu,\theta,r)\frac{\partial\varphi'}{\partial\theta}\\&-\alpha(1+e\cos\nu)\sin(\varphi(\nu,\theta,r))\cos(\varphi(\nu,\theta,r))\frac{\partial\varphi}{\partial\theta}\\&+2e(1+e\cos\nu)\sin\nu\frac{\partial\varphi}{\partial\theta}]d\nu.\end{aligned}$$

It follows from the second equation of (4.1) that

$$V_\theta(\theta,r)=\int_0^{2\pi}(1+e\cos\nu)^2\varphi'(\nu,\theta,r)\frac{\partial\varphi'}{\partial\theta}+\frac{\partial\varphi}{\partial\theta}\phi'd\nu. \tag{4.4}$$

Integrating by part and using the first equation in (4.1) we get

$$\begin{aligned}\int_0^{2\pi}\frac{\partial\varphi}{\partial\theta}\phi'd\nu=&(\frac{\partial\varphi}{\partial\theta}\phi)|_0^{2\pi}-\int_0^{2\pi}\frac{\partial\varphi'}{\partial\theta}\phi d\nu\\=&\phi(2\pi,\theta,r)\frac{\partial\varphi(2\pi,\theta,r)}{\partial\theta}-\phi(0,\theta,r)\frac{\partial\varphi(0,\theta,r)}{\partial\theta}\\&-\int_0^{2\pi}\frac{\partial\varphi'}{\partial\theta}(1+e\cos\nu)^2\varphi'(\nu,\theta,r)d\nu.\end{aligned}$$

Substituting the above equation into (4.4) gives

$$V_\theta(\theta,r)=\phi(2\pi,\theta,r)\frac{\partial\varphi(2\pi,\theta,r)}{\partial\theta}-\phi(0,\theta,r)\frac{\partial\varphi(0,\theta,r)}{\partial\theta}. \tag{4.5}$$

Analogously, we can get

$$V_r(\theta, r) = \phi(2\pi, \theta, r)\frac{\partial\varphi(2\pi, \theta, r)}{\partial r} - \phi(0, \theta, r)\frac{\partial\varphi(0, \theta, r)}{\partial r}. \tag{4.6}$$

By (4.5) and (4.6), we have

$$\begin{aligned}
dV =& V_\theta d\theta + V_r dr \\
=& [\phi(2\pi, \theta, r)\frac{\partial\varphi(2\pi, \theta, r)}{\partial\theta} - \phi(0, \theta, r)\frac{\partial\varphi(0, \theta, r)}{\partial\theta}]d\theta \\
&+ [\phi(2\pi, \theta, r)\frac{\partial\varphi(2\pi, \theta, r)}{\partial r} - \phi(0, \theta, r)\frac{\partial\varphi(0, \theta, r)}{\partial r}]dr \\
=& \phi(2\pi, \theta, r)[\frac{\partial\varphi(2\pi, \theta, r)}{\partial\theta}d\theta + \frac{\partial\varphi(2\pi, \theta, r)}{\partial r}dr] \\
&- \phi(0, \theta, r)[\frac{\partial\varphi(0, \theta, r)}{\partial\theta}d\theta + \frac{\partial\varphi(0, \theta, r)}{\partial r}dr] \\
=& \phi(2\pi, \theta, r)d\varphi(2\pi, \theta, r) - \phi(0, \theta, r)d\varphi(0, \theta, r) \\
=& PdQ - rd\theta,
\end{aligned}$$

which means that the map $S(\theta, r)$ is exact symplectic.

Next we prove that the Poincaré map S satisfies the boundary twist condition (2.2), i,e, there exist $\rho > 0$ and $\epsilon > 0$ such that

$$Q(\theta, \rho) - \theta > \epsilon, \quad Q(\theta, -\rho) - \theta < -\epsilon, \quad \theta \in [0, \pi).$$

Integrating the second equation of (4.1) from 0 to $\nu \in [0, 2\pi]$, we get

$$\begin{aligned}
\phi(\nu) &= r + \int_0^\nu -\alpha(1 + e\cos s)\sin\varphi\cos\varphi + 2e(1 + e\cos s)\sin s ds \\
&\geq r - \nu(\alpha + 2e)(1 + e) \\
&\geq r - 2\pi(\alpha + 2e)(1 + e).
\end{aligned}$$

We can find a constant $\rho_1 \geq 2\pi(\alpha + 2e)(1 + e) > 0$ so that if $r > \rho_1$ then $\phi(\nu) > 0$. It follows from the first equation of (4.1) that

$$\varphi' = \frac{\phi}{(1 + e\cos\nu)^2} > 0,$$

which means that $\varphi(\nu)$ is increasing for $\nu \in [0, 2\pi]$. Take $\rho = \rho_1 + 1$ and we have

$$Q(\theta, \rho) - \theta = \varphi(2\pi, \theta, \rho) - \varphi(0, \theta, \rho) > 0.$$

By a standard compactness argument, we can conclude that there exists $\epsilon > 0$ such that

$$Q(\theta, \rho) - \theta > \epsilon, \ \theta \in [0, \pi).$$

Analogously, we can conclude that

$$Q(\theta, -\rho) - \theta < -\epsilon, \ \theta \in [0, \pi).$$

In order to apply Theorem 2.1, we take $A = \mathbb{R} \times [-\rho, \rho]$. Since the solutions of (4.1) are globally defined, one can find a larger B such that $S(A) \subset \mathrm{int}B$.

Since the right-hand side of (4.1) is analytic with respect to the variables (φ, ϕ), the Poincaré map S is also analytic following from the analytic dependence on initial conditions.

Up to now, all the conditions of Theorem 2.1 are satisfied, thus we get that the Poincaré map $S(\theta, r) = (Q(\theta, r), P(\theta, r)) = (\varphi(2\pi, \theta, r), \phi(2\pi, \theta, r))$ has at least two fixed points and one of them is unstable. Therefore, (4.1) has at least two geometrically distinct 2π-periodic solutions and at least one of them is unstable. $\square$

Now we consider the existence of the so-called 2π-periodic solutions of (1.1) with winding number $N \in \mathbb{Z}$, that is

$$\varphi(\nu + 2\pi) = \varphi(\nu) + N\pi, \ \forall \nu \in \mathbb{R}.$$

Such solutions are also called rotating or running solutions. Of course, one has the usual 2π-periodic solutions when the winding number is zero.

Let $\varphi(\nu)$ be a 2π-periodic solution of (1.1) with winding number N. Taking the change of variables

$$\omega(\nu) = \varphi(\nu) - \frac{N}{2}\nu,$$

we obtain

$$\begin{aligned}\omega(\nu + 2\pi) &= \varphi(\nu + 2\pi) - \frac{N}{2}(\nu + 2\pi)\\ &= \varphi(\nu) - \frac{N}{2}\nu\\ &= \omega(\nu),\end{aligned}$$

which implies that the 2π-periodic solutions of (1.1) with winding number N correspond to the usual 2π-periodic solutions of the equation

$$(1 + e\cos\nu)\frac{d^2\omega}{d\nu^2} - 2e\sin\nu\frac{d\omega}{d\nu} + \alpha\sin(\omega + \frac{N\nu}{2})\cos(\omega + \frac{N\nu}{2}) = (N+2)e\sin\nu. \tag{4.7}$$

Proceeding as in the proof of Theorem 3.2, we can prove the following result.

Theorem 3.3. *For every integer N, equation* (1.1) *has at least two geometrically distinct 2π-periodic solutions with winding number N and at least one of them is unstable.*

Finally we study the existence of k-order subharmonic solutions of (1.1) with winding number N, that is,

$$\varphi(\nu + 2k\pi) = \varphi(\nu) + N\pi, \ \forall \nu \in \mathbb{R}. \tag{4.8}$$

Theorem 3.4. *For each couple of relatively prime natural numbers N, k, equation* (1.1) *has at least two geometrically distinct k-order subharmonic solutions with winding number N and $2k\pi$ is the minimal period. Moreover, at least one of them is unstable.*

Proof. The proof is a straightforward modification of the proof of Theorem 3.2, with 2π replaced by $2k\pi$. We only need to prove that $2k\pi$ is the minimal period of subharmonic solutions $\varphi(\nu)$ with winding number N.

Assume by contradiction that $2l\pi$ is the minimal period, where $l \in \{1, 2, ..., k-1\}$, which means that there exist a nonzero integer j such that

$$\varphi(\nu + 2l\pi) = \varphi(\nu) + j\pi, \ \forall \nu \in \mathbb{R}. \tag{4.9}$$

Let n_1 and n_2 be positive integers such that

$$n_1 l = n_2 k. \tag{4.10}$$

By (4.8), we have

$$\varphi(\nu + 2n_2k\pi) = \varphi(\nu) + n_2N\pi, \ \forall \nu \in \mathbb{R},$$

and by (4.9), we have

$$\varphi(\nu + 2n_1 l\pi) = \varphi(\nu) + n_1 j\pi, \ \forall \nu \in \mathbb{R}.$$

By the above two equalities and the uniqueness of the solution $\varphi(\nu)$, we can get that $n_2 N = n_1 j$, i.e.,

$$\frac{n_2}{n_1} = \frac{j}{N}.$$

From (4.10), we know that

$$\frac{n_2}{n_1} = \frac{l}{k},$$

which implies that

$$\frac{N}{k} = \frac{j}{l}, \tag{4.11}$$

which is impossible because N and k are relatively prime and j is a nonzero integer and $l \in \{1, 2, ..., k-1\}$. $\square$

4. Stable periodic solutions

In this section, we prove that when the parameters α, e are in a concrete region, the satellite equation (1.1) admits a twist 2π-periodic solution $\varphi(\nu)$, which has the smallest L^∞ norm among all of 2π-periodic solutions of (1.1).

First we rewrite equation (1.1) as the equivalent form

$$\frac{d^2\varphi}{d\nu^2} - \frac{2e\sin\nu}{1+e\cos\nu}\frac{d\varphi}{d\nu} + \frac{\alpha\sin\varphi\cos\varphi}{1+e\cos\nu} = \frac{2e\sin\nu}{1+e\cos\nu}, \tag{3.1}$$

which is of the form (2.1) with

$$h(\nu) = -\frac{2e\sin\nu}{1+e\cos\nu}$$

and

$$g(\nu, \varphi) = \frac{\alpha\sin\varphi\cos\varphi}{1+e\cos\nu} - \frac{2e\sin\nu}{1+e\cos\nu}.$$

Then

$$\begin{aligned}\sigma(h)(\nu) &= \exp\left(\int_0^\nu -\frac{2e\sin s}{1+e\cos s}ds\right)\\ &= \left(\frac{1+e\cos\nu}{1+e}\right)^2.\end{aligned}$$

Define

$$\tau(\nu) = \int_0^\nu \sigma(-h)(s)ds = \int_0^\nu \left(\frac{1+e}{1+e\cos s}\right)^2 ds,$$

and let

$$T = \tau(2\pi) = \frac{2\pi\sqrt{1+e}}{(1-e)^{\frac{3}{2}}}.$$

Now we consider the following linear damped equation

$$\frac{d^2\varphi}{d\nu^2} - \frac{2e\sin\nu}{1+e\cos\nu}\frac{d\varphi}{d\nu} + \frac{\alpha}{1+e\cos\nu}\varphi = 0 \tag{3.2}$$

associated to periodic boundary conditions

$$\varphi(2\pi) = \varphi(0), \ \varphi'(2\pi) = \varphi'(0). \tag{3.3}$$

When the anti-maximum principle holds for (3.2)-(3.3), the non-homogeneous equation

$$\frac{d^2\varphi}{d\nu^2} - \frac{2e\sin\nu}{1+e\cos\nu}\frac{d\varphi}{d\nu} + \frac{\alpha}{1+e\cos\nu}\varphi = f(\nu)$$

has a unique 2π-periodic solution, which can be written as

$$\varphi(\nu) = \int_0^{2\pi} G(\nu,s)f(s)ds,$$

where $G(\nu,s)$ is called the Green function of (3.2)-(3.3).

It was proved in [9, Theorem 2.7] that, if $\int_0^{2\pi} a(\nu)\sigma(h)(\nu)d\nu > 0$ and

$$\int_0^{2\pi} \sigma(-h)(\nu)d\nu \int_0^{2\pi} a(\nu)\sigma(h)(\nu)d\nu < 4,$$

then the Green's function $G(\nu,s)$ of (2.4)-(3.3) is positive for all $(\nu,s) \in [0,2\pi] \times [0,2\pi]$. Applying this fact to (3.2)-(3.3), we have the following result.

Lemma 4.1. *Assume that*

$$0 < \alpha < \frac{(1-e^2)^{3/2}}{\pi^2}. \tag{3.4}$$

Then the Green function $G(\nu,s)$ of (3.2)-(3.3) *is positive for all $(\nu,s) \in [0,2\pi] \times [0,2\pi]$.*

Proof. Let $a(\nu) = \dfrac{\alpha}{1+e\cos\nu}$. By a direct computation, we know that

$$\int_0^{2\pi} a(\nu)\sigma(h)(\nu)d\nu = \int_0^{2\pi} \frac{\alpha}{1+e\cos\nu}\left(\frac{1+e\cos\nu}{1+e}\right)^2 d\nu = \frac{2\alpha\pi}{(1+e)^2} > 0.$$

Moreover,

$$\begin{aligned}
&\int_0^{2\pi} \sigma(-h)(\nu)d\nu \int_0^{2\pi} a(\nu)\sigma(h)(\nu)d\nu \\
&= \int_0^{2\pi}\left(\frac{1+e}{1+e\cos\nu}\right)^2 d\nu \cdot \frac{2\alpha\pi}{(1+e)^2} \\
&= 2\alpha\pi \int_0^{2\pi} \frac{1}{(1+\cos\nu)^2} d\nu \\
&= \frac{4\pi^2\alpha}{(1-e^2)^{\frac{3}{2}}}.
\end{aligned}$$

Now the result is a direct consequence since the condition (3.4) holds. □

Lemma 4.2. [16, Lemma 2.1] *Let a and b be positive numbers. Then the cubic equation*

$$ax^3 + b = x$$

has a positive root if and only if $27ab^2 \le 4$. In this case, the minimal positive root is given by

$$x = \Phi(a,b) = \frac{2}{\sqrt{3a}}\cos\frac{\pi+y}{y}, \quad \left(y = \arccos\left(\frac{3\sqrt{3a}b}{2}\right) \in (0,\frac{\pi}{2})\right),$$

which satisfies

$$\Phi(a,b) \le \frac{3b}{2}.$$

Theorem 4.3. *Assume that*

$$3\sqrt{2}e \le \alpha < \frac{(1-e^2)^{3/2}}{\pi^2}. \tag{3.5}$$

Then (1.1) *has a unique* 2π*-periodic solution* $\varphi(\nu)$ *such that its* L^∞ *norm* $\|\varphi\|$ *is the smallest among all of* 2π*-periodic solutions of* (1.1)*. Moreover,* $\varphi(\nu)$ *satisfies*

$$\|\varphi\| \le \Phi\left(\frac{2}{3}, \frac{2e}{\alpha}\right) \le \frac{3e}{\alpha}. \tag{3.6}$$

Proof. It is obvious that φ is a 2π-periodic solution of (3.1) if and only if φ is a fixed point of the following operator

$$\begin{aligned}(\mathbb{T}\varphi)(\nu) = &\int_0^{2\pi} G(\nu,s)\frac{\alpha}{2(1+e\cos s)}\left(2\varphi(s) - \sin 2\varphi(s)\right) ds \\ &+ \int_0^{2\pi} G(\nu,s)\frac{2e\sin s}{1+e\cos s}\ ds.\end{aligned}$$

Obviously, $\mathbb{T}$ is a completely continuous operator from $C(\mathbb{R}/2\pi\mathbb{Z})$ to itself. By Lemma 4.1 and the condition (3.5), we know that $G(\nu,s) > 0$. For any $\varphi \in C(\mathbb{R}/2\pi\mathbb{Z})$, it follows from the basic estimate

$$|\varphi - \sin\varphi| \le \frac{|x|^3}{6}$$

that

$$\begin{aligned}|(\mathbb{T}\varphi)(\nu)| \le &\int_0^{2\pi} G(\nu,s)\frac{\alpha}{2(1+e\cos s)}\left|2\varphi(s) - \sin 2\varphi(s)\right| ds \\ &+ \int_0^{2\pi} G(\nu,s)\left|\frac{2e\sin s}{1+e\cos s}\right| ds \\ \le &\int_0^{2\pi} G(\nu,s)\frac{\alpha}{2(1+e\cos s)}\left|2\varphi(s) - \sin 2\varphi(s)\right| ds \\ &+ \int_0^{2\pi} G(\nu,s)\frac{2e}{1+e\cos s} ds \\ \le &\left[\frac{2}{3}\|\varphi\|^3 + \frac{2e}{\alpha}\right]\int_0^{2\pi} G(\nu,s)\frac{\alpha}{1+e\cos s} ds \\ = &\frac{2}{3}\|\varphi\|^3 + \frac{2e}{\alpha},\end{aligned}$$

which yields

$$\|\mathbb{T}\varphi\| \le \frac{2}{3}\|\varphi\|^3 + \frac{2e}{\alpha}.$$

Under the condition (3.5), we know that

$$27 \cdot \frac{2}{3} \cdot (\frac{2e}{\alpha})^2 \le 4.$$

Define

$$\Omega = \left\{\varphi \in C(\mathbb{R}/2\pi\mathbb{Z}) : \|\varphi\| \le \Phi\left(\frac{2}{3}, \frac{2e}{\alpha}\right)\right\}.$$

Then $\mathbb{T}(\Omega) \subset \Omega$. Moreover, by Lemma 4.2, T has a fixed point $\varphi \in \Omega$, which is a 2π-periodic solution of (1.1).

Now we prove the uniqueness. Let $\varphi, \varphi_1 \in \Omega$, then

$$\begin{aligned} &|(\mathbb{T}\varphi)(\nu) - (\mathbb{T}\varphi_1)(\nu)| \\ &\le \int_0^{2\pi} \frac{\alpha G(\nu,s)}{2(1+e\cos s)} |(2\varphi(s) - \sin 2\varphi(s)) - (2\varphi_1(s) - \sin 2\varphi_1(s))| \, ds. \end{aligned} \tag{3.7}$$

By using the estimate (3.6), we have

$$\begin{aligned} &|(2\varphi(s) - \sin 2\varphi(s)) - (2\varphi_1(s) - \sin 2\varphi_1(s))| \\ &\le 4\Phi^2 \left(\frac{2}{3}, \frac{2e}{\alpha}\right) |\varphi(s) - \varphi_1(s)| \\ &\le \frac{36e^2}{\alpha^2} |\varphi(s) - \varphi_1(s)|. \end{aligned} \tag{3.8}$$

Using (3.5), (3.7) and (3.8), for all $\varphi, \varphi_1 \in \Omega$, we obtain

$$\begin{aligned} \|\mathbb{T}\varphi - \mathbb{T}\varphi_1)\| \le & \frac{36e^2}{\alpha^2} \int_0^{2\pi} \frac{\alpha G(\nu,s)}{2(1+e\cos s)} ds \|\varphi - \varphi_1\| \\ = & \frac{18e^2}{\alpha^2} \|\varphi - \varphi_1\| \\ \le & \|\varphi - \varphi_1\|. \end{aligned}$$

Thus, if the strict inequality in condition (3.5) is satisfied, we know that $\mathbb{T} : \Omega \to \Omega$ is actually a strict contraction. So $\mathbb{T}$ has a unique fixed point φ in Ω. If the equality in (3.5) holds, one can also obtain the uniqueness from the proof above, although $\mathbb{T}$ may not be a strict contraction.

By the uniqueness of the 2π-periodic solution of (1.1) in Ω, we know that $\|\varphi\|$ is smaller than the norms of other possible 2π-periodic solutions of (1.1). □

Now, we are ready to state the main result of this section.

Theorem 4.4. *Assume that* (3.5) *is satisfied. Then there exists a region* Δ*, which is constructed below (see Fig. 1) , such that if* $(\alpha, e) \in \Delta$*, the* 2π*-periodic solution* φ *of* (1.1) *obtained in Theorem* 4.3 *is twist and therefore is stable.*

Proof. By Theorem 4.3, we know

$$\|\varphi\| \le \Phi\left(\frac{2}{3}, \frac{2e}{\alpha}\right) \le \frac{3e}{\alpha}. \tag{3.9}$$

A computation of the coefficients in (2.3) for equation (3.1) gives

$$a(\nu) = \frac{\alpha \cos 2\varphi(\nu)}{1 + e\cos\nu}, \quad b(\nu) = -\frac{\alpha \sin 2\varphi(\nu)}{1+e\cos\nu}, \quad c(\nu) = -\frac{2\alpha\cos 2\varphi(\nu)}{3(1+e\cos\nu)}.$$

Using (3.9), we have the following estimates

$$\frac{\alpha(1-e)^3}{(1+e)^4} \cos(\frac{6e}{\alpha}) \le a(\nu)\sigma(2h)(\nu) \le \frac{\alpha}{1+e}.$$

Thus (2.9) is satisfied if we can take

$$\sigma_1 = \sqrt{\frac{\alpha(1-e)^3}{(1+e)^4}\cos(\frac{6e}{\alpha})}, \quad \sigma_2 = \sqrt{\frac{\alpha}{1+e}}.$$

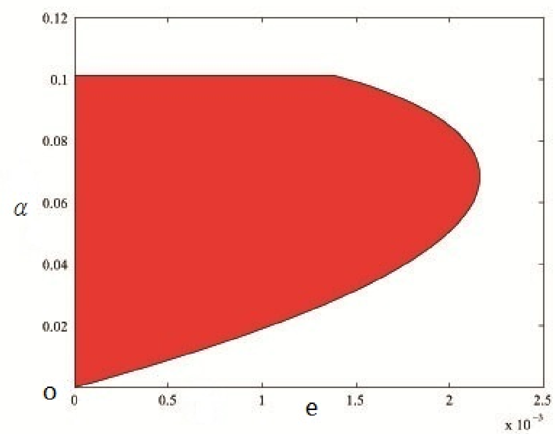

FIGURE 1. The region of stability Δ.

By (3.9), we obtain $C_+ = 0$ and

$$\begin{aligned} C_- &= \int_0^{2\pi} \frac{2\alpha \cos 2\varphi(\nu)}{3(1+e\cos\nu)} \cdot \left(\frac{1+e\cos\nu}{1+e}\right)^2 d\nu \\ &= \frac{2\alpha}{3(1+e)^2} \int_0^{2\pi} \cos 2\varphi(\nu) \cdot (1+e\cos\nu) d\nu \\ &\geq \frac{4\alpha\pi\cos(\frac{6e}{\alpha})}{3(1+e)^2}. \end{aligned}$$

Therefore,

$$\frac{3}{10}\sigma_1^3 \left[\frac{\|C_-\|_1}{\sigma_2^2} - \frac{\|C_+\|_1}{\sigma_1^2}\right] \geq \frac{2\alpha\pi\cos(\frac{6e}{\alpha})}{5(1+e)^2} \cdot \frac{\sigma_1^3}{\sigma_2^2} = G_1(\alpha, e).$$

We note that

$$\begin{aligned} \|B_+\|_1 \|B_-\|_1 &\leq \left(\int_0^{2\pi} \left|\frac{\alpha \sin 2\varphi(\nu)}{1+e\cos\nu}\right| \left(\frac{1+e\cos\nu}{1+e}\right)^2 d\nu\right)^2 \\ &\leq \frac{\alpha^2 \sin^2(\frac{6e}{\alpha})}{(1+e)^4} \left(\int_0^{2\pi} (1+e\cos\nu) d\nu\right)^2 \\ &= \frac{4\pi^2\alpha^2\sin^2(\frac{6e}{\alpha})}{(1+e)^4} = G_2(\alpha, e). \end{aligned}$$

Let

$$\Delta = \left\{(\alpha, e) | (3.5) \text{ is satisfied and } G_1(\alpha, e) > \frac{G_2(\alpha, e)}{\min\left\{\sin(\frac{3T\sigma_1}{2}), \sin(\frac{3T\sigma_2}{2})\right\}}\right\}.$$

Up to now, all conditions of Theorem 2.2 are satisfied if $(\alpha, e) \in \Delta$, thus we get that the least amplitude 2π-periodic solution $\varphi(\nu)$ of (1.1) obtained in Theorem 4.3 is of twist type. $\square$

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