Contact Hamiltonian dynamics: Variational principles, invariants, completeness and periodic behavior*

Qihuai Liu
School of Mathematics and Computing Science,
Guangxi Colleges and Universities Key Laboratory of
Data Analysis and Computation
Guilin University of Electronic Technology
Guilin, 541004, China

Pedro J. Torres
Departamento de Matemática Aplicada, Facultad de Ciencias
Universidad de Granada
Granada, 18071, Spain

Chao Wang
School of Mathematical Sciences
Yancheng Teachers University
Yancheng, 224002, China

Abstract

This paper describes the connections between the notions of Hamiltonian system, contact Hamiltonian system and nonholonomic system from the perspective of differential equations and dynamical systems. It shows that action minimizing curves of nonholonomic system satisfy the dissipative Lagrange system, which is equivalent to the contact Hamiltonian system under some generic conditions. As the initial research of contact Hamiltonian dynamics in this direction, we investigate the dynamics of contact Hamiltonian systems in some special cases including invariants, completeness of phase flows and periodic behavior.

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* Email: qhuailiu@gmail.com (Q. Liu), ptorres@ugr.es (P.J. Torres), wangchaosudamath@163.com (C. Wang)

1 Introduction

A Hamiltonian system with Hamiltonian function $H : \mathbb{R} \times T^*\mathbb{R}^n \to \mathbb{R}$ defined by

$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p}(t, q, p), \\ \dot{p} = -\dfrac{\partial H}{\partial q}(t, q, p) \end{cases} \tag{1.1}$$

describes reversible systems such as in Mechanics and Electromagnetism, where dissipation effects are neglected. The remarkable geometrical structure and properties, such as the symplectic structure or pseudo-Poisson bracket and Dirac structure, have aroused the interest of many physicists and mathematicians. In the autonomous case, the Hamiltonian function is invariant, which means that the energy of system is conserved.

In the case when other effects of the environment on the system (including dissipation) are taken into account, Hamiltonian systems have been extended by considering tensor fields which are no more skew-symmetric, replacing the standard Poisson bracket by a so-called Leibniz bracket. One of the proposals is extending symplectic Hamiltonian dynamics to contact Hamiltonian dynamics by adding an extra dimension in a natural way. Correspondingly, Hamiltonian system (1.1) extends to the contact Hamiltonian system on contact manifold with a contact Hamiltonian function $H : \mathbb{R} \times T^*\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ defined by

$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p}(t, q, p, S), \\ \dot{p} = -\dfrac{\partial H}{\partial q}(t, q, p, S) - p\dfrac{\partial H}{\partial S}(t, q, p, S), \\ \dot{S} = p\dfrac{\partial H}{\partial p}(t, q, p, S) - H(t, q, p, S). \end{cases} \tag{1.2}$$

In the coordinates (q, p, S) of phase space, the contact form is globally given by the 1-form

$$\omega = \mathrm{d}S - \sum_{i=1}^{n} p_i \mathrm{d}q_i.$$

Comparing with autonomous Hamiltonian systems, the autonomous contact Hamiltonian function is no more invariant, hence the energy is not a conserved quantity.

It is possible to do the other way round and perform the symplectification of a contact Hamiltonian vector by adding an extra dimension. In this way, any contact Hamiltonian system can be converted into a Hamiltonian system. However, it may cause the loss of important properties such as the convexity and superlinearity of Hamiltonian functions. This direction has been proposed in the classical book [1] by V. I. Arnold and has been applied extensively in some problems from Geometry and Physics, see for instance [2, 3, 4].

It is well known that, under the Legendre transformation, the Lagrange system

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}(t, q(t), \dot{q}(t))\right) = \frac{\partial L}{\partial q}(t, q(t), \dot{q}(t)) \tag{1.3}$$

is equivalent to a Hamiltonian system under generic conditions. The action minimizing curves q of

$$\inf_{q(t_0)=q_0, q(t_1)=q_1} \int_{t_0}^{t_1} L(t, q(\tau), \dot{q}(\tau)) \mathrm{d}\tau \tag{1.4}$$

with respect to L must satisfy Lagrange equation (1.3), where the infimum is taken over all C^2 curves with the fixed endpoints $q(t_0) = q_0, q(t_1) = q_1$. This problem is usually called the Lagrange problem without constraints, also named Hamilton's principle of least action. The existence of action minimizers has established under Tonelli assumptions, see [5, Section 3.1], [6] or [7] for a modern treatment.

A natural question arises on how to connect a contact Hamiltonian system and the corresponding Lagrange problem. In this paper, we have shown that, under generic conditions, the contact Hamiltonian system is equivalent to certain dissipative Lagrange system. The action minimizing curves of the corresponding Lagrange problem with nonholonomic constraints must satisfy the dissipative Lagrange equation, see Theorem 3.3. This fact has guiding significances for the principle of dynamics. On one hand, we shed new light to the implicit variational principle established by J. Yan, K. Wang and L. Wang [8], which is related to the Weak KAM theory for Hamilton-Jacobi equation with unknown functions. On the other hand, it guides us studying the dynamical properties of the contact Hamiltonian system from the point of view of dynamical systems and dissipative differential equations. Specially, when the contact Hamiltonian $H(t, q, p, S)$ is linear with respect to S, the system is decoupled into a second order dissipative ordinary differential equation and a first order ordinary differential equation.

Contact Hamiltonian systems have important applications in contact geometry, weak KAM theory and several physical fields. One of the important background is the description of both the thermodynamic properties of matter [9, 10] and the dissipative systems at the mesoscopic level [11]. For a simple thermodynamic system, the thermodynamic phase space is defined in the following coordinates: U denotes the energy, and the pairs (q_i, p_i) denote the pairs of conjugated extensive variables (the entropy $q_1 = S$, the volume $q_2 = V$, and the number of mole $q_3 = N$) and intensive variables (the temperature $p_1 = T$, minus the pressure $p_2 = -P$, and the chemical potential $p_3 = \mu$). In this case, the contact form is the Gibbs form (see [12])

$$\omega = \mathrm{d}U - T\mathrm{d}S + P\mathrm{d}V - \mu\mathrm{d}N.$$

In a recent work [13], classical Hamiltonian mechanics has been extended to contact Hamiltonian mechanics.

Another background of application is related to the method of characteristics by which one can construct smooth solutions of first order Hamilton-Jacobi equation (see for instance [14]). For example, the characteristics $(q(t), p(t), S(t))$ of the Hamilton-Jacobi equation

$$\partial_t u + H(x, D_x u, u) = 0$$

solve the contact Hamiltonian equation (1.2) with a time-independent contact Hamiltonian $H(q, p, S)$. The recent paper [8] establishes an implicit variational principle for the contact Hamiltonian systems generated by $H(q, p, S)$ with respect to the contact 1-form ω under Tonelli and Lipschitz continuity conditions, which is a stepping stone to make a further exploration on contact Hamiltonian systems and general Hamilton-Jacobi equations depending on unknown functions explicitly by global minimization methods.

In this paper, we focus on the study of the contact Hamiltonian system from the point of view of dynamical systems and differential equations. The structure of this paper is as follows. In Section 2, we introduce the symplectification of a contact Hamiltonian vector and then obtain the asymptotic properties of solutions and invariants for autonomous contact Hamiltonian systems by means of the property of energy conservation of a Hamiltonian system. Section 3 begins with the introduction of some fundamental results of nonholonomic systems, then we give a formal derivation of contact Hamiltonian equation. At the end of this section, we establish the connections between dissipative Lagrange system, contact Hamiltonian system and nonholonomic system from the point of view of differential equations and dynamical systems. Since the completeness is a precondition that ensures that the Lagrange problem with nonholonomic constraints is well defined, in Section 4 we seek for various conditions for the completeness of the phase flow of a contact Hamiltonian system. Finally, as an example of research on contact Hamiltonian mechanics, in the last section we study a time-dependent harmonic oscillator with changing-sign damping, where invariants of system, periodic behavior and stability of solutions are studied and a novel class of three-dimensional nonautonomous isochronous systems has been identified.

2 Symplectification of a contact Hamiltonian vector

The relation between Hamiltonian system and contact Hamiltonian system has been established by symplectification of a contact Hamiltonian vector, although it is known that some important properties such as convexity will be lost [1]. Accordingly, contact geometry can be understood in terms of symplectic geometry through its symplectification [15].

Assume that $H(t,q,p,S)$ is a C^1 contact Hamiltonian function and (q,p,S) are the coordinates of a point of a contact manifold with the one form $\eta = \mathrm{d}S - p\mathrm{d}q$. Let λ be the number by which we must multiply η to obtain the given point of the symplectified space. In these coordinates, we have

$$\omega = \lambda \mathrm{d}S - \lambda p \mathrm{d}q$$

In the coordinates (x, x_0, y, y_0) of the symplectified space, we have

$$x_0 = \lambda, \quad y_0 = S, \quad x = q, \quad y = \lambda p, \tag{2.1}$$

where the form takes the standard form

$$\mathrm{d}\omega = \mathrm{d}x_0 \wedge \mathrm{d}y_0 + \mathrm{d}x \wedge \mathrm{d}y.$$

The action T_μ of the multiplicative group is the multiplication of y, y_0 by a number

$$T_\mu(x, x_0, y, y_0) = (x, x_0, \mu y, \mu y_0).$$

Consider the Hamiltonian function

$$K(t, x, x_0, y, y_0) = \lambda H(t, q, p, S), \tag{2.2}$$

and the associated Hamiltonian system is given by

$$\begin{cases} \dot{x} = \dfrac{\partial K}{\partial y}, & \dot{x}_0 = \dfrac{\partial K}{\partial y_0}, \\ \dot{y} = -\dfrac{\partial K}{\partial x}, & \dot{y}_0 = -\dfrac{\partial K}{\partial x_0}. \end{cases} \tag{2.3}$$

Taking the derivatives with respect to p, q, S, λ of both sides of (2.2), respectively, we have

$$\begin{aligned} &\frac{\partial K}{\partial x} = \lambda \frac{\partial H}{\partial q}, \quad \frac{\partial K}{\partial y} = \frac{\partial H}{\partial p}, \\ &\frac{\partial K}{\partial x_0} = H - p\frac{\partial H}{\partial p}, \quad \frac{\partial K}{\partial y_0} = \lambda \frac{\partial H}{\partial S}. \end{aligned} \tag{2.4}$$

In view of (2.1), (2.3) and (2.4), we know that

$$-\lambda\frac{\partial H}{\partial q}=-\frac{\partial K}{\partial x}=\dot{y}=\dot{\lambda}p+\lambda\dot{p}=\lambda\frac{\partial H}{\partial S}p+\lambda\dot{p},$$

which yields

$$\dot{p}=-\frac{\partial H}{\partial q}-p\frac{\partial H}{\partial S}.$$

Similarly, we obtain the contact Hamiltonian system

$$\begin{cases}\dot{q}=\dfrac{\partial H}{\partial p}(t,q,p,S),\\ \dot{p}=-\dfrac{\partial H}{\partial q}(t,q,p,S)-p\dfrac{\partial H}{\partial S}(t,q,p,S),\\ \dot{S}=p\dfrac{\partial H}{\partial p}(t,q,p,S)-H(t,q,p,S).\end{cases}\tag{2.5}$$

Moreover,

$$\dot{\lambda}=\dot{x}_0=\frac{\partial K}{\partial y_0}=\lambda\frac{\partial H}{\partial S},$$

which implies

$$\lambda(t)=C_0\exp\left(\int_0^t\frac{\partial H}{\partial S}(\tau,q(\tau),p(\tau),S(\tau))\mathrm{d}\tau\right),\tag{2.6}$$

where C_0 is an nonzero integral constant.

As the energy of the autonomous Hamiltonian system is conservative, we know that F defined as

$$F(q,p,S)=H(q(t),p(t),S(t))\exp\left(\int_0^t\frac{\partial H}{\partial S}(q(\tau),p(\tau),S(\tau))\mathrm{d}\tau\right)\equiv C.\tag{2.7}$$

is constant along the flow of X_H defined by (2.5) with an autonomous contact Hamiltonian $H(q,p,S)$. Then it is easy to obtain the following result.

Theorem 2.1. Assume that $H(q,p,S)$ is of class C^1 and satisfies that $\partial H/\partial S\geq\mu>0$ (resp. $\partial H/\partial S\leq-\mu<0$). If the solutions $(q(t),p(t),S(t))$ of the contact Hamiltonian system (2.5) are defined on $(-\infty,+\infty)$, then the solutions $(q(t),p(t),S(t))$ satisfy that

$$\lim_{t\to+\infty}H(q(t),p(t),S(t))=0\quad(\text{resp.}\lim_{t\to-\infty}H(q(t),p(t),S(t))=0).\tag{2.8}$$

Under the conditions of Theorem 2.1, we know that, if $(q(t),p(t),S(t))$ is a T-periodic solution, then it must be such that

$$\int_0^T\frac{\partial H}{\partial S}(q(\tau),p(\tau),S(\tau))\mathrm{d}\tau=0$$

or satisfies the algebraic equation $H(q(t),p(t),S(t))=0$ for all t.

Particularly, let us consider a special case, in which

$$H=H_{\mathrm{mec}}(q,p)+h(S),$$

where H_{mec} is the mechanical energy of the system and $h(S)$ characterizes effectively the interaction with the environment. The evolution of the contact Hamiltonian H can be formally obtained from (2.7) to be

$$H(q,p,S)\equiv H_0\exp\left(-\int_0^t h'(S)\mathrm{d}\tau\right),$$

where $H_0=H(q(0),p(0),S(0))$, which is identical to (54) in [13].

3 Nonholonomic system and contact Hamiltonian system

3.1 Calculus of variations for nonholonomic systems

In this subsection, we give a brief account of the variational principles involved in the derivation of the Lagrange equations of motion of a nonholonomic system in Classical Mechanics.

By definition, a nonholonomic system is a system being subject to differential constraints, whose state depends on the path taken in order to achieve it. The variational problems that have nonholonomic constraints are also called Lagrange problems. Many books account for much of the developments of the Lagrange problems of nonholonomic systems from a point of view of differential geomety or analysis, see for instance [16, 17, 18].

Denote by $C^2[t_0, t_1]$ the set of all C^2 smooth curves on the interval $[t_0, t_1]$. Let $\mathcal{A}$ be a functional of the form

$$\mathcal{A}(x) = \int_{t_0}^{t_1} L(t, x(\tau), \dot{x}(\tau))\mathrm{d}\tau \tag{3.1}$$

and suppose that $\mathcal{A}$ has an extremum at $x \in C^2[t_0, t_1]$ subject to the boundary conditions $x(t_0) = x_0$, $x(t_1) = x_1$, and the nonholonomic constraint

$$g(t, x, \dot{x}) = 0. \tag{3.2}$$

When the function g does not depend on $\dot{x}$, (3.2) is called a holonomic constraint.

We begin first with a general multiplier rule in the following.

Theorem 3.1. [18, Theorem 6.2.1] Let $\mathcal{A}$ be the functional defined by (3.1) and L is a smooth function. Suppose that $\mathcal{A}$ has an extremum at $\gamma \in C^2[t_0, t_1]$ subject to the boundary conditions $x(t_0) = x_0$, $x(t_1) = x_1$ and the nonholonomic constraint (3.2), where g is a smooth function such that $\partial g/\partial \dot{x}_j \neq 0$ for some $j, 1 \leq j \leq n$. Then there exists a constant λ_0 and a function $\lambda_1(t)$ not both zero such that for

$$K(t, x, \dot{x}) = \lambda_0 L(t, x, \dot{x}) - \lambda_1(t) g(t, x, \dot{x}),$$

γ is a solution to the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial \dot{x}}(t, \gamma(t), \dot{\gamma}(t))\right) = \frac{\partial K}{\partial x}(t, \gamma(t), \dot{\gamma}(t)). \tag{3.3}$$

By definition, a smooth extremal x is called *abnormal* if there exists a nontrivial solution to system

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\lambda_1(t)\frac{\partial g}{\partial \dot{x}}(t, x(t), \dot{x}(t))\right) = \lambda_1(t)\frac{\partial g}{\partial x}(t, x(t), \dot{x}(t)); \tag{3.4}$$

otherwise, it is called *norma*l. We have the following result for normal extremals.

Theorem 3.2. [18, Theorem 6.2.2] Let $\mathcal{A}$, γ, L, and g be as in Theorem 3.1. If γ is a normal extremal then there exists a function λ_1 such that γ is a solution to the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial \dot{x}}(t, \gamma(t), \dot{\gamma}(t))\right) = \frac{\partial F}{\partial x}(t, \gamma(t), \dot{\gamma}(t)). \tag{3.5}$$

where

$$F(t, x, \dot{x}) = L(t, x, \dot{x}) - \lambda_1(t) g(t, x, \dot{x}).$$

Moreover, λ_1 is uniquely determined by γ.

3.2 Formal derivation of contact Hamiltonian equation

Consider a C^2 smooth and time-dependent contact Hamiltonian $H(t,q,p,S)$ and a bounded real interval $[t_0,t_1] \subset \mathbb{R}$. The corresponding Lagrange problem consists in determining the extrema for functionals of the form

$$\mathcal{A}(q,p,S) = \int_{t_0}^{t_1} \left(p(\tau)\cdot\dot{q}(\tau) - H(\tau,q(\tau),p(\tau),S(\tau))\right) \mathrm{d}\tau \tag{3.6}$$

subject to the boundary conditions

$$q(t_i) = q_i, \quad p(t_i) = p_i, \quad S(t_i) = S_i, \quad i = 0,1 \tag{3.7}$$

and a nonholonomic constraint of the form

$$g := p\cdot\dot{q} - H(t,q,p,S) - \dot{S} = 0, \tag{3.8}$$

where $(t_i,q_i),(t_i,p_i) \in \mathbb{R}\times\mathbb{R}^n$, $(t_i,S_i) \in \mathbb{R}\times\mathbb{R}$ are the fixed starting points and end points, respectively.

Usually, this optimization problem is not well defined, mainly because the path space may be an empty set. Especially, when the uniqueness of solutions for Cauchy problem (3.8) is satisfied, the solution with the initial value $S(t_0) = S_0$ may not go through the point (t_1,S_1) at time t_1. Therefore, in the following we only give a formal derivation of contact Hamiltonian equation.

In view of (3.8), we prefer to consider the action functional

$$\tilde{\mathcal{A}}(q,p,S) = \int_{t_0}^{t_1} \Big(p(\tau)\cdot\dot{q}(\tau) - H(\tau,q(\tau),p(\tau),S(\tau)) - \dot{S}(\tau)\Big)\,\mathrm{d}\tau, \tag{3.9}$$

since there is only a constant difference between $\tilde{\mathcal{A}}$ and $\mathcal{A}$, that is, $\mathcal{A} = \tilde{\mathcal{A}} + S(t_1) - S(t_0)$. Assume that $\mathcal{A}$ has an extremum at $\gamma = (q,p,S) \in C^2[t_0,t_1]$ subject to the boundary condition (3.7) and the nonholonomic constraint (3.8). Obviously, the curve γ is also an extremum of $\tilde{\mathcal{A}}$. Notice that $\partial g/\partial\dot{S} = -1 \neq 0$. By Theorem 3.1, there exists a nontrivial function λ_1 such that $\gamma = (q,p,S)$ is a solution of system

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial\dot{x}}(t,\gamma(t),\dot{\gamma}(t))\right) = \frac{\partial F}{\partial x}(t,\gamma(t),\dot{\gamma}(t)), \tag{3.10}$$

where

$$F(t,x,\dot{x}) = \lambda_1(t)\left(p\cdot\dot{q} - H(q,p,S,t) - \dot{S}\right).$$

Expand (3.10) into several equations

$$\frac{\mathrm{d}\lambda_1 p}{\mathrm{d}t} = -\lambda_1\frac{\partial H}{\partial q}, \quad \lambda_1\left(\dot{q} - \frac{\partial H}{\partial p}\right) = 0, \quad \frac{\mathrm{d}\lambda_1}{\mathrm{d}t} = \lambda_1\frac{\partial H}{\partial S}. \tag{3.11}$$

The function

$$\lambda_1(t) = C_0\mathrm{exp}\left(\int \frac{\partial H}{\partial S}(\tau,q,p,S)\mathrm{d}\tau\right)$$

is uniquely determined by q,p,S, where C_0 is a nonzero integrate constant. Then we also can see that the extremal γ is normal. These equations and the constraint (3.8) imply

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} - p\frac{\partial H}{\partial S}, \quad \dot{S} = p\cdot\dot{q} - H(t,q,p,S). \tag{3.12}$$

When H is independent of S, $\partial H/\partial S = 0$. From (3.11), we know that λ_1 is a nonzero constant function, which yields the Hamiltonian equation

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}. \tag{3.13}$$

We remark that λ_1 is called a Lagrange multiplier, in view of (2.6) , which is equal to the new canonical variable in symplectification of a contact Hamiltonian vector.

3.3 Action minimizers

Consider a C^2 smooth and time-dependent contact Hamiltonian $H(t, q, p, S)$. We also need the following assumptions:

(H_1) **(Strict convexity)** $H_{pp}(t, q, p, S)$ is positive definite for all $(t, q, p, S) \in \mathbb{R} \times T^*\mathbb{R}^n \times \mathbb{R}$.

(H_2) **(Superlinearity)** the limit

$$\lim_{\|p\| \to +\infty} \frac{H(t, q, p, S)}{\|p\|} = +\infty$$

holds uniformly for $(t, q, S) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$.

The Lagrangian $L(t, q, p, S)$ associated to H is defined by

$$L(t, q, \dot{q}, S) = \sup_{p \in T^*\mathbb{R}^n} \{p \cdot v - H(t, q, p, S)\}.$$

Denote the Legendre transformation by $\mathcal{L} : \mathbb{R} \times T^*\mathbb{R}^n \to \mathbb{R} \times T\mathbb{R}^n$. Let $\bar{\mathcal{L}} = (\mathcal{L}, Id)$, where Id is the identity map. Then under conditions (H_1) and (H_2), we know that $\mathcal{L}$ is a diffeomorphism from $\mathbb{R} \times T^*\mathbb{R}^n \times \mathbb{R}$ to $\mathbb{R} \times T\mathbb{R}^n \times \mathbb{R}$, see [19, Proposition 1.2.1]. In fact, we know that the Lagrangian L is also of class C^2 and strictly convex and the flow $\Psi^t = \bar{\mathcal{L}} \circ \Psi^t \circ \bar{\mathcal{L}}^{-1}$ generated by L is complete, see [8]. Moreover, the following equality holds

$$L(t, q, \dot{q}, S) = p \cdot \dot{q} - H(t, q, p, S), \tag{3.14}$$

where p is locally and uniquely determined by $\dot{q} = \partial H/\partial p(t, q, p, S)$ by using the hypothesis (H_1).

Assume that $\mathcal{A}$ defined by (3.6) has an extremum at $\gamma = (q, p, S) \in C^2[t_0, t_1]$ subject to the boundary condition (3.7) and the nonholonomic constraint (3.8), then γ satisfies (3.12). From the hypothesis (H_1) and (3.7), $p_i = p(t_i)\,(i = 0, 1)$ is locally and uniquely determined by $\dot{q}(t_i) = \partial H/\partial p(q(t_i), p(t_i), S(t_i), t_i)$ and $S_i = S(t_i)$. Furthermore, $S_1 = S(t_1)$ is uniquely determined by the nonholonomic constraint (3.8) and the starting point $S(t_0) = S_0$ by using the existence and uniqueness of solutions of ODE. Then, by necessity, the boundary condition of action function $\mathcal{A}$ achieving an extremum is reduced to

$$q(t_i) = q_i, i = 0, 1, \quad S(t_0) = S_0. \tag{3.15}$$

Now we define the action functional $\mathcal{A}^{t_0,t_1}$ associated to the Lagrangian L as

$$\mathcal{A}^{t_0,t_1}(q) = \int_{t_0}^{t_1} L(\tau, q(\tau), \dot{q}(\tau), S(\tau))\mathrm{d}\tau, \tag{3.16}$$

where q belongs to the space of C^2 curves $q : [t_0, t_1] \to \mathbb{R}^n$ satisfying the boundary condition $q(t_0) = q_0, q(t_1) = q_1$, and S is a solution of the Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = L(t, q(t), \dot{q}(t), S(t)), \\ S(t_0) = S_0. \end{cases} \tag{3.17}$$

Notice that the solution of the Cauchy problem (3.17) depends on q, that is, for each given q of class C^2, we assume that there exists a unique solution $S(t) = S(t; q(t))$ of (3.17). A C^2 curve $q : [t_0, t_1] \to \mathbb{R}^n$ is an action minimizer with respect to L when every other C^2 curve $\zeta : [t_0, t_1] \to \mathbb{R}^n$ with the same endpoints of q satisfies that $\mathcal{A}^{t_0,t_1}(q) \leq \mathcal{A}^{t_0,t_1}(\zeta)$ under the nonholonomic constraint (3.17).

Theorem 3.3. Let $L : \mathbb{R} \times T\mathbb{R}^n \times \mathbb{R}$ be a Lagrangian of Class C^2. Then every action minimizer q with respect to L is a solution of the dissipative Lagrange system

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} - \frac{\partial L}{\partial S}\frac{\partial L}{\partial \dot{q}} = 0, \tag{3.18}$$

where S is a solution of the Cauchy problem (3.17) which depends on q.

Moreover, assume that (H_1) and (H_2) hold. Let $p = \partial L/\partial \dot{q}$. Then (q, p, S) is a solution of contact Hamiltonian system

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} - p\frac{\partial H}{\partial S}, \quad \dot{S} = p \cdot \dot{q} - H(t, q, p, S).$$

Proof. Assume $q : [t_0, t_1] \to \mathbb{R}^n$ is an action minimizer with respect to L, then the Cauchy problem (3.17) has a unique solution $S(t; q(t))$. Let $S_1 = S(t_1; q(t))$. Consider the action functional

$$\tilde{\mathcal{A}}(q, S) = \int_{t_0}^{t_1} \Big(L(\tau, q(\tau), \dot{q}(\tau), S(\tau)) - \dot{S}(\tau)\Big)\, \mathrm{d}\tau \tag{3.19}$$

with the boundary conditions

$$q(t_0) = q_0, \quad q(t_1) = q_1, \quad S(t_0) = S_0, \quad S(t_1) = S_1 \tag{3.20}$$

and a nonholonomic constraint of the form

$$g := L(t, q, \dot{q}, S) - \dot{S} = 0. \tag{3.21}$$

Notice that $\mathcal{A}^{t_0,t_1}(q) = \tilde{\mathcal{A}}(q, S) + S_1 - S_0$. Assume that $\tilde{\mathcal{A}}(q, S)$ has an extremum at $\gamma = (q, S) \in C^2[t_0, t_1]$ subject to the boundary condition (3.20) and the nonholonomic constraint (3.21). Obviously, the curve q is also an extremum of $\mathcal{A}^{t_0,t_1}(q)$. Notice that $\partial g/\partial \dot{S} = -1 \neq 0$. By Theorem 3.2, there exists a nontrivial function λ_1 such that $\gamma = (q, S)$ is a solution of system

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial \dot{x}}(t, \gamma(t), \dot{\gamma}(t))\right) = \frac{\partial F}{\partial x}(t, \gamma(t), \dot{\gamma}(t)), \tag{3.22}$$

where

$$F(t, x, \dot{x}) = \lambda_1(t)\Big(L(q, \dot{q}, S, t) - \dot{S}\Big), \quad x = (q, S), \quad \dot{x} = (\dot{q}, \dot{S}).$$

Then it follows that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial \dot{q}}\right) = \frac{\partial F}{\partial q} \Rightarrow \dot{\lambda}_1\frac{\partial L}{\partial \dot{q}} + \lambda_1\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) = \lambda_1\frac{\partial L}{\partial q},\\
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial \dot{S}}\right) = \frac{\partial F}{\partial S} \Rightarrow -\dot{\lambda}_1 = \lambda_1\frac{\partial L}{\partial S},
\end{aligned}$$

which yields the desired equation (3.18).

In view of (3.14), by a direct computation we have

$$\frac{\partial L}{\partial q} = -\frac{\partial H}{\partial q}, \quad \frac{\partial L}{\partial S} = -\frac{\partial H}{\partial S}.$$

Let $p = \partial L/\partial \dot{q}$, then it follows from (3.18) that

$$\dot{p} = \frac{\partial L}{\partial q} + \frac{\partial L}{\partial \dot{q}}\frac{\partial L}{\partial S} = -\frac{\partial H}{\partial q} - p\frac{\partial H}{\partial S}.$$

Furthermore, from the equality $H(q, p, S, t) = p \cdot \dot{q} - L(t, q, \dot{q}, S)$, we have $\dot{q} = \partial H/\partial p$. □

Define the solution operator $\mathcal{N}$ of (3.18) by $\mathcal{N}(q) = S(t; q(t))$ and denote by $C_0^2[t_0, t_1]$ the space of C^2 curves $q : [t_0, t_1] \to \mathbb{R}^n$ satisfying the boundary condition $q(t_0) = q_0, q(t_1) = q_1$. To ensure the operator is well defined (uniqueness of solutions), we give an additional condition that has used in [8] with a Lipschitz constant not depending on time.

(H_3) **(Lipschitz)** there exists an integrable function K on $[t_0, t_1]$ such that

$$|L(t, q, v, S_1) - L(t, q, v, S_2)| \leq K(t)|S_1 - S_2|,$$

for all $(t, q, v) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$.

We have the following property for the solution operator.

Proposition 3.1. *Assume (H_3) that holds, then the solution operator $\mathcal{N}$ is continuous on the curve space $C_0^2[t_0, t_1]$.*

Proof. From (3.17), the operator $\mathcal{N}$ is defined implicitly by

$$\mathcal{N}(\xi)(t) = S_0 + \int_{t_0}^{t} L(\tau, \xi(\tau), \dot{\xi}(\tau), \mathcal{N}(\xi)(\tau))\mathrm{d}\tau, t \in [t_0, t_1], \tag{3.23}$$

for any $\xi \in C_0^2[t_0, t_1]$. The,n for any $\xi_1, \xi_2 \in C_0^2[t_0, t_1]$, we have

$$\begin{aligned}
|\mathcal{N}(\xi_1)(t) - \mathcal{N}(\xi_2)(t)| &\leq \int_{t_0}^{t} \Big| L(\tau, \xi_1(\tau), \dot{\xi}_1(\tau), \mathcal{N}(\xi_1)(\tau)) - L(\tau, \xi_2(\tau), \dot{\xi}_2(\tau), \mathcal{N}(\xi_2)(\tau)) \Big| \mathrm{d}\tau \\
&\leq \int_{t_0}^{t} \Big| L(\tau, \xi_1(\tau), \dot{\xi}_1(\tau), \mathcal{N}(\xi_1)(\tau)) - L(\tau, \xi_1(\tau), \dot{\xi}_1(\tau), \mathcal{N}(\xi_2)(\tau)) \Big| \mathrm{d}\tau \\
&\quad + \int_{t_0}^{t} \Big| L(\tau, \xi_1(\tau), \dot{\xi}_1(\tau), \mathcal{N}(\xi_2)(\tau)) - L(\tau, \xi_2(\tau), \dot{\xi}_2(\tau), \mathcal{N}(\xi_2)(\tau)) \Big| \mathrm{d}\tau \\
&\leq \int_{t_0}^{t} K(\tau) \cdot |\mathcal{N}(\xi_1)(\tau) - \mathcal{N}(\xi_2)(\tau)| \mathrm{d}\tau \\
&\quad + \int_{t_0}^{t} \left| \frac{\partial L}{\partial q}(\tau, \eta, \dot{\xi}_1, \mathcal{N}(\xi_2))(\xi_1 - \xi_2) \right| + \left| \frac{\partial L}{\partial \dot{q}}(\tau, \xi_1, \zeta, \mathcal{N}(\xi_2))(\dot{\xi}_1 - \dot{\xi}_2) \right| \mathrm{d}\tau,
\end{aligned}$$

where η is between ξ_1 and ξ_2, and ζ is between $\dot{\xi}_1$ and $\dot{\xi}_2$. Therefore, it follows that

$$|\mathcal{N}(\xi_1)(t) - \mathcal{N}(\xi_2)(t)| \leq \int_{t_0}^{t} K(\tau) \cdot |\mathcal{N}(\xi_1)(\tau) - \mathcal{N}(\xi_2)(\tau)| \mathrm{d}\tau + \psi(t)\|\xi_1 - \xi_2\|_{C^1}, \tag{3.24}$$

where

$$\psi(t) = \int_{t_0}^{t} \left| \frac{\partial L}{\partial q}(\tau, \eta, \dot{\xi}_1, \mathcal{N}(\xi_2)) \right| + \left| \frac{\partial L}{\partial \dot{q}}(\tau, \xi_1, \zeta, \mathcal{N}(\xi_2)) \right| \mathrm{d}\tau,$$

which depends on ξ_1 and ξ_2.

By using Grönwall's inequality, we obtain that

$$|\mathcal{N}(\xi_1)(t) - \mathcal{N}(\xi_2)(t)| \leq \|\xi_1 - \xi_2\|_{C^1} \left(\psi(t) + \int_{t_0}^{t} K(\tau) \cdot \psi(\tau) \mathrm{e}^{\int_{t_0}^{\tau} K(s)\mathrm{d}s} \mathrm{d}\tau \right). \tag{3.25}$$

Thus the continuity of the operator $\mathcal{N}$ is proved. □

To finish this section, we remark that through personal communication, we have known that the existence of action minimizers has been proved by Carnnarsa, Cheng and Yan under some generic conditions, see their recent work [20].

4 Completeness of phase flow

The phase flow of a contact Hamiltonian system is a minimal curve of a functional implicitly defined in variational principle, which plays an important role in the weak theory of general Hamilton-Jacobi equations depending on unknown functions [8, 21]. In weak theory, the completeness of the phase flow is a prerequisite for studying the convergence of semigroups.

Completeness of the flow means that every maximal integral curve of the contact Hamiltonian system (2.5) has all of $\mathbb{R}$ as its domain of definition, i.e., for all $(q_0, p_0, S_0) \in T^*M \times \mathbb{R}$, every non-continuable solution $(q(t), p(t), S(t))$ of (2.5) satisfying the initial value

$$q(t_0) = q_0, p(t_0) = p_0, S(t_0) = S_0$$

is defined for all $t \in (-\infty, +\infty)$. Here, M denotes the general configuration space.

Firstly, we consider the time-periodic contact Hamiltonian $H(q, p, S, t)$ which satisfies that

(HT) $H(t, q, p, S)$ is of class C^1 and T-periodic in t such that, for some positive constants $C_1, C_2 > 0$,

$$\left|\frac{\partial H}{\partial t}(t, q, p, S)\right| \leq C_1 H(t, q, p, S) + C_2.$$

We also need the following coercive and Lipschitz assumptions.

(HC) **Coercivity**: $H(t, q, p, S)$ is coercive with respect to the variables q, p, S, i.e.,

$$\begin{aligned}
&\lim_{\|p\| \to +\infty} |H(t, q, p, S)| = +\infty, \text{ uniformly for } (t, q, S) \in \mathbb{R} \times M \times \mathbb{R}, \\
&\lim_{\|q\| \to +\infty} |H(t, q, p, S)| = +\infty, \text{ uniformly for } (t, p, S) \in \mathbb{R} \times T_q^*M \times \mathbb{R}, \\
&\lim_{|S| \to +\infty} |H(t, q, p, S)| = +\infty, \text{ uniformly for } (t, q, p) \in \mathbb{R} \times T^*M.
\end{aligned} \tag{4.1}$$

(HL) **Lipschitz continuity**: $H(t, q, p, S)$ is uniformly Lipschitz in S, i.e., there exists $\mu > 0$ such that

$$\left|\frac{\partial H}{\partial S}(t, q, p, S)\right| \leq \mu, \tag{4.2}$$

for all $(t, q, p, S) \in \mathbb{R} \times T^*M \times \mathbb{R}$.

Condition (HL) has been introduced in [8], which includes a special contact Hamiltonian $\lambda S + H(q, p)$ related to the discounted Hamilton-Jacobi equation (see [22]). We have the following theorem.

Theorem 4.1. Assume that conditions (HT), (HC) and (HL) hold, then the phase flow of the contact Hamiltonian system (2.5) is complete.

Proof. By contradiction, we assume $(q(t), p(t), S(t))$ is a non-continuable solution of (2.5) satisfying the initial value

$$q(t_0) = q_0, p(t_0) = p_0, S(t_0) = S_0,$$

which is defined on (α, β). Define the function $\mathfrak{H} : (\alpha, \beta) \to \mathbb{R}$ by

$$\mathfrak{H}(t) = H(t, q(t), p(t), S(t)) \exp\left(\int_{t_0}^{t} \frac{\partial H}{\partial S}(\tau, q(\tau), p(\tau), S(\tau)) \mathrm{d}\tau\right).$$

Then we have

$$\left.\frac{\mathrm{d}\mathfrak{H}}{\mathrm{d}t}\right|_{(2.5)} = \frac{\partial H}{\partial t}(t, q, p, S) \exp\left(\int_{t_0}^{t} \frac{\partial H}{\partial S}(\tau, q(\tau), p(\tau), S(\tau)) \mathrm{d}\tau\right)$$

By using condition (HT), we know that

$$\left|\frac{\mathrm{d}\mathfrak{H}}{\mathrm{d}t}\right| \leq C_1\mathfrak{H} + C_2\mu.$$

Using Grönwall's inequality, we obtain that, for $t \in [t_0, \beta)$,

$$\mathfrak{H}_0 \mathrm{e}^{C_1(t_0-t)} + \mu C_2(t_0-t)\mathrm{e}^{-C_1 t} \leq \mathfrak{H}(t) \leq \mathfrak{H}_0 \mathrm{e}^{C_1(t-t_0)} + \mu C_2(t-t_0)\mathrm{e}^{C_1 t}$$

and for $t \in (\alpha, t_0)$,

$$\mathfrak{H}_0 \mathrm{e}^{C_1(t-t_0)} + \mu C_2(t-t_0)\mathrm{e}^{C_1 t} \leq \mathfrak{H}(t) \leq \mathfrak{H}_0 \mathrm{e}^{C_1(t_0-t)} + \mu C_2(t_0-t)\mathrm{e}^{-C_1 t},$$

where $\mathfrak{H}_0 = H(q_0, p_0, S_0, t_0)$. Therefore, the following estimate holds

$$|\mathfrak{H}(t)| \leq C_0 := |\mathfrak{H}_0|\mathrm{e}^{C_1(\beta-\alpha)} + \mu C_2(\beta-\alpha)\mathrm{e}^{C_1(|\alpha|+|\beta|)}, \quad t \in (\alpha, \beta),$$

by the Lipschitz condition (HL), which implies

$$\begin{aligned} |H(t, q(t), p(t), S(t))| \leq & C_0 \exp\left(\int_{t_0}^{t} \left|\frac{\partial H}{\partial S}(\tau, q(\tau), p(\tau), S(\tau))\right| \mathrm{d}\tau\right) \\ \leq & C_0 \exp\left(\mu(\beta-\alpha)\right), \quad t \in (\alpha, \beta). \end{aligned}$$

By the coercive conditions (HC), there exists a positive constant $R_0 > 0$ such that

$$\|q(t)\| \leq R_0, \quad \|p(t)\| \leq R_0, \quad |S(t)| \leq R_0, \text{ for all } t \in (\alpha, \beta).$$

Therefore, $(q(t), p(t), S(t))$ is a continuable solution, which is a contradiction. $\square$

In the following, we will give another set of conditions for completeness of autonomous contact Hamiltonian systems, which has been used frequently in the context of variational calculus. Assume that $H(q, p, S)$ is a C^2 function satisfying the following assumptions

(H_1) **Convexity:** For every $(q, p, S) \in T^*M \times \mathbb{R}$, the Hessian $\partial^2 H/\partial p_i \partial p_j(q, p, S)$, calculated in linear coordinates on the fiber T^*M, is positive definite as a quadratic form.

(H_2) **Superlinearity:**

$$\lim_{\|p\| \to +\infty} \frac{H(q, p, S)}{\|p\|} = \infty, \text{ uniformly for } (q, S) \in M \times \mathbb{R}. \tag{4.3}$$

(H_3) **Boundedness:** For every $K > 0$,

$$A(K) := \sup_{(q,S) \in M \times \mathbb{R}} \Big\{ H(q, p, S) : \|p\| \leq K \Big\} < \infty. \tag{4.4}$$

Hypotheses (H_1) and (H_2) are called Tonelli conditions, and (H_3) is usually introduced when the manifold M is noncompact [5, 23].

Firstly, let us recall the Fenchel transforms. Given a convex function $H(q, p, S)$, the Fenchel transform (or the convex dual) is the function $L : TM \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ defined by

$$L(q, v, S) = \max_{p \in T_q M} \left(p \cdot v - H(q, p, S)\right), \tag{4.5}$$

for every $(q, S) \in M \times \mathbb{R}$. Since H is superlinear and convex on p, L is convex on v and finite everywhere. Moreover, we have the Fenchel equality

$$p \cdot v = L(q, v, S) + H(q, p, S) \tag{4.6}$$

if and only if $p = \partial L/\partial v(q, v, S)$, or $v = \partial H/\partial p(q, v, S)$.

Lemma 4.1. Assume that conditions (H_1)-(H_3) and (HL) hold. For every $K > 0$, if $\|p\| \leq K$, then $\|v\| \leq C(K)$ with $C(K)$ a constant depending on K, where v is defined by $v = \partial H/\partial p(q, v, S)$.

Proof. For any given $R > 0$ and $(q, v) \in TM$, there exists $p_q \in T_q^*M$ such that $\|p_q\| = R$ and $p_q \cdot v = R\|v\|$. Then, by (4.4), it follows that

$$\begin{aligned} L(q, v, S) = & \max_{p \in T_qM} (p \cdot v - H(q, p, S)) \\ \geq & p_q \cdot v - H(q, p_q, S) \\ \geq & R\|v\| - A(R). \end{aligned}$$

Equivalently, it shows that L is superlinear with respect to v.

If $\|p\| \leq K$ and $(q, p, S) \in T^*M \times \mathbb{R}$, by (4.4), $H(q, p, S) \leq A(K)$, which implies that

$$L(q, v, S) = p \cdot v - H(q, p, S) \leq K\|v\| - A(K), \tag{4.7}$$

where v is given by $v = \partial H/\partial p(q, v, S)$. From the superlinearity of L, for every $(q, v, S) \in TM \times \mathbb{R}$ and $K + 1 > 0$, there exists a constant $A(K + 1)$ such that

$$L(q, v, S) \geq (K + 1)\|v\| - A(K + 1). \tag{4.8}$$

Together with (4.7) and (4.8), we have

$$\|v\| \leq C(K) := A(K + 1) - A(K).$$

□

Theorem 4.2. Assume that conditions (H_1)-(H_3) and (HL) hold, then the phase flow of the contact Hamiltonian system (2.5) is complete.

Proof. By contradiction, we assume $(q(t), p(t), S(t))$ is a non-continuable solution of (2.5) satisfying the initial value

$$q(t_0) = q_0, p(t_0) = p_0, S(t_0) = S_0,$$

which is defined on (α, β). Using (2.7), we know that

$$H(q(t), p(t), S(t)) \equiv H_0 \exp\left(-\int_{t_0}^{t} \frac{\partial H}{\partial S}(q(\tau), p(\tau), S(\tau))\mathrm{d}\tau\right), \quad t \in (\alpha, \beta),$$

where $H_0 = H(q_0, p_0, S_0)$. Then it follows that

$$|H(q(t), p(t), S(t))| \leq H_0 \exp\left(\mu(\beta - \alpha)\right), \quad t \in (\alpha, \beta).$$

By the superlinear condition (H_2) of H, there exists a positive constant $K > 0$ such that

$$\|p(t)\| \leq K, \text{ for all } t \in (\alpha, \beta).$$

On one hand, from Lemma 4.1, we know that there exists a constant $C(K) > 0$ such that

$$\|v(t)\| = \|\partial H/\partial p(q(t), v(t), S(t))\| = \|q'(t)\| \leq C(K).$$

Then it follows that $\|q(t)\| \leq \|q_0\| + C(K)(\beta - \alpha)$, for all $t \in (\alpha, \beta)$. One the other hand, by (4.4) we have

$$\begin{aligned} |\dot{S}| = & |p\partial H/\partial p - H(q, p, S)| \\ \leq & \|p\| \cdot \|v\| + |H(q, p, S)| \leq KC(K) + A(K), \end{aligned}$$

which yields that $|S(t)| \leq |S_0| + (KC(K) + A(K))(\beta - \alpha)$, $t \in (\alpha, \beta)$. Therefore, $(q(t), p(t), S(t))$ is a continuable solution, which is a contradiction. □

5 Time-dependent Harmonic oscillators with changing-sign damping

Consider the one-dimensional damped oscillator with changing-sign damping coefficient $\gamma(t)$, mass m and time-dependent frequency $\omega(t)$, whose contact Hamiltonian is

$$H(t,q,p,S)=\frac{p^2}{2m}+\frac{1}{2}m\omega^2(t)q^2+\gamma(t)S. \tag{5.1}$$

The contact Hamiltonian system of motions reads

$$\begin{cases} \dot{q}=\dfrac{p}{m}, \\ \dot{p}=-m\omega^2(t)q-\gamma(t)p, \\ \dot{S}=\dfrac{p^2}{2m}-\dfrac{1}{2}m\omega^2(t)q^2-\gamma(t)S, \end{cases} \tag{5.2}$$

where the dot denotes differentiation with respect to the time t.

5.1 Invariants

The contact Hamiltonian vector field X_H takes the form

$$X_H=\left(p\frac{\partial H}{\partial p}-H\right)\frac{\partial}{\partial S}-\left(p\frac{\partial H}{\partial S}+\frac{\partial H}{\partial q}\right)\frac{\partial}{\partial p}+\frac{\partial H}{\partial p}\frac{\partial}{\partial q}. \tag{5.3}$$

For any given function $F(q,p,S,t)\in C^1(\mathbb{R}^4)$, its evolution along the flow of the contact Hamiltonian vector field X_H is given by

$$\begin{aligned} \left.\frac{\mathrm{d}F}{\mathrm{d}t}\right|_{(5.2)} &= X_H(F) \\ &= -H\frac{\partial F}{\partial S}+p\,\{F,H\}_{(S,p)}+\{F,H\}_{(q,p)}, \end{aligned}$$

where $\{\,,\,\}_{(q,p)}$ is the standard Poisson bracket.

Definition 5.1. A non-constant function F is an invariant of the contact system given by X_H if F is constant along the flow of X_H, that is, $X_H(F)=0$.

The canonical invariants for the linear oscillator with constant damping coefficient were identified by H. R. Lewis Jr. [24], and recently a new invariant defined on contact manifold has been obtained by A. Bravetti, H. Cruz and D. Tapias [13]. These invariants are easy to extend to the case of the linear oscillator with changing-sign damping by proposing the ansatz

$$F(t,q,p,S)=\beta(t)p^2-2\xi(t)qp+\eta(t)q^2+\varsigma(t)S. \tag{5.4}$$

Inserting (5.4) into equation $X_H(F)=0$, we can get a kind of first order linear differential equations with respect to the undetermined coefficients β,ξ,η and ς, see [13, Appendix A] for the details. Then by solving these equations the coefficients are determined as follows

$$\begin{aligned}
\beta(t) =& \frac{1}{2m}\alpha^2(t)\mathrm{e}^{\Upsilon(t)},\\
\xi(t) =& \left[\frac{\alpha(t)}{2}\left(\dot\alpha(t) - \frac{\gamma(t)}{2}\alpha(t)\right) + \frac{\varsigma_0}{4}\right]\mathrm{e}^{\Upsilon(t)},\\
\eta(t) =& \frac{m}{2}\left[\left(\dot\alpha(t) - \frac{\gamma(t)}{2}\alpha(t)\right)^2 + \frac{1}{\alpha^2(t)}\right]\mathrm{e}^{\Upsilon(t)},\\
\varsigma(t) =& \varsigma_0\mathrm{e}^{\Upsilon(t)},
\end{aligned}$$

where the function $\alpha(t)$ satisfies the Ermakov-Pinney equation

$$\ddot\alpha(t) + \left(\omega^2(t) - \frac{\gamma^2(t)}{4} - \frac{\dot\gamma(t)}{2}\right)\alpha(t) = \frac{1}{\alpha^3(t)}, \tag{5.5}$$

and

$$\Upsilon(t) = \int_0^t \gamma(s)\,\mathrm{d}s$$

is the prime function of $\gamma(t)$ and ς_0 is the arbitrary integral constant. Then we obtain the invariants of contact system (5.2) as

$$F(t,q,p,S) = \frac{m\mathrm{e}^{\Upsilon(t)}}{2}\left[\left(\alpha(t)\frac{p}{m} - \left[\dot\alpha(t) - \frac{\gamma(t)}{2}\alpha(t)\right]q\right)^2 + \left(\frac{q}{\alpha(t)}\right)^2\right] + \varsigma_0\mathrm{e}^{\Upsilon(t)}\left[S - \frac{qp}{2}\right]. \tag{5.6}$$

Each particular solution of the Ermakov-Pinney equation for $\alpha(t)$ and each integral constant ς_0 determine an invariant of contact system (5.2). Since the difference between two different invariants is also an invariant, we obtain two particular invariants of the system as follows

$$F_0(t,q,p,S) = \mathrm{e}^{\Upsilon(t)}\left(S - \frac{qp}{2}\right), \tag{5.7}$$

$$F_1(t,q,p,S) = \mathrm{e}^{\Upsilon(t)}\left[\left(\alpha(t)\frac{p}{m} - \left[\dot\alpha(t) - \frac{\gamma(t)}{2}\alpha(t)\right]q\right)^2 + \left(\frac{q}{\alpha(t)}\right)^2\right]. \tag{5.8}$$

In fact, the invariants $F_0(t,q,p,S)$ and $F_1(t,q,p,S)$ completely determine the solution of contact system (5.2), which depends on $\alpha(t)$. Since $F_1 \geq 0$ is constant along the flow of X_H, let

$$\left(\alpha(t)\frac{p}{m} - \left[\dot\alpha(t) - \frac{\gamma(t)}{2}\alpha(t)\right]q\right)\mathrm{e}^{\frac{\Upsilon(t)}{2}} = h_0\cos\phi(t), \quad \left(\frac{q}{\alpha(t)}\right)\mathrm{e}^{\frac{\Upsilon(t)}{2}} = h_0\sin\phi(t).$$

where $h_0 = F_1(0, q(0), p(0), S(0))$ is a constant and $\phi(t)$ is the corresponding polar coordinate. Solving the algebraic equations above and using the invariant $F_0(q,p,S)$, we obtain that

$$q(t) = h_0\alpha(t)e^{-\frac{\Upsilon(t)}{2}}\sin\phi(t), \tag{5.9}$$

$$p(t) = mh_0\left(\frac{\cos\phi(t)}{\alpha(t)} + \sin\phi(t)\left[\dot\alpha(t) - \frac{\gamma(t)}{2}\alpha(t)\right]\right)e^{-\frac{\Upsilon(t)}{2}}, \tag{5.10}$$

$$S(t) = \frac{1}{4}e^{-\Upsilon(t)}\left(4h_1 + h_0^2 m\sin(2\phi(t)) + 2\alpha(t)\left[\dot\alpha(t) - \frac{\alpha(t)}{2}\gamma(t)\right]h_0^2 m\sin^2\phi(t)\right), \tag{5.11}$$

where $h_1 = F_0(q(0), p(0), S(0), 0)$. From $\dot{q} = p/m$, we have

$$\dot{\phi}(t)\alpha(t)e^{-\Upsilon(t)}\cos\phi(t) = \frac{\cos\phi(t)}{\alpha(t)}e^{-\Upsilon(t)},$$

which leads to that

$$\dot{\phi}(t) = \frac{1}{\alpha^2(t)}$$

or $\cos\phi(t) \equiv 0$, for all $t \in (-\infty, +\infty)$.

5.2 Periodic motions

In Subsection 5.1, we have seen that the solutions of contact system (5.2) are completely determined by the solutions of the Ermakov-Pinney equation (5.5). In this subsection, we assume that the coefficients $\gamma(t)$ and $\omega(t)$ are C^2 smooth and 2π-periodic. Define the function $I(t)$ by

$$I(t) := \omega^2(t) - \frac{\gamma^2(t)}{4} - \frac{\dot{\gamma}(t)}{2}. \tag{5.12}$$

Definition 5.2. A rest point of system (5.2) is called a *global isochronous center* if any neighborhood of the rest point consists entirely of closed trajectories surrounding that point, and the periodic solutions associated to these closed trajectories have the same period.

When the function I is always positive and constant, that is, $I(t) \equiv \mu^2, \mu > 0$, for all $t \in \mathbb{R}$, the solution of the contact system (5.2) can be expressed explicitly. In this case, the Ermakov-Pinney equation reads

$$\ddot{\alpha}(t) + \mu^2\alpha(t) = \frac{1}{\alpha^3(t)}. \tag{5.13}$$

The solutions of (5.13) can be solved as

$$\alpha(t) = \rho\left(\cos^2(\mu t + \theta) + \frac{1}{\mu^2\rho^4}\sin^2(\mu t + \theta)\right)^{\frac{1}{2}}, \tag{5.14}$$

which are all π/μ-periodic, where $\rho\,(\rho \neq 0), \theta$ are arbitrary integral constants. In view of (5.9)-(5.11), the form of the solutions, we conclude the following theorem.

Theorem 5.1. Assume that $\gamma(t)$ and $\omega(t)$ are C^2 smooth and 2π-periodic functions satisfying

$$I(t) = \omega^2(t) - \frac{\gamma^2(t)}{4} - \frac{\dot{\gamma}(t)}{2} \equiv \frac{n^2}{4}, \quad n \in \mathbb{Z}^+$$

and

$$\bar{\gamma} = \frac{1}{2\pi}\int_0^{2\pi} \gamma(s)\mathrm{d}s = 0.$$

Then the origin O is a global isochronous center of contact system (5.2).

Proof. Recall that (5.14) are all π/μ-periodic solutions of (5.13).

To prove q, p, S defined by (5.9)-(5.11) are $4nl\pi$-periodic functions, we only need to prove that $\cos\phi(t)$ and $\sin\phi(t)$ are $4nl\pi$-periodic functions. Let

$$\begin{aligned}
\bar{\phi} &= \frac{1}{T}\int_0^T \frac{1}{\alpha^2(t)}\mathrm{d}t \\
&= \frac{\mu}{\pi}\int_0^{\frac{\pi}{\mu}} \frac{2\mu^2\rho^2}{1 + \mu^2\rho^4 + (-1 + \mu^2\rho^4)\cos[2(\theta + t\mu)]}\,\mathrm{d}t \ (\text{let } s = 2(\theta + t\mu)) \\
&= \frac{\mu}{\pi}\int_{2\theta}^{2\theta+2\pi} \frac{\mu\rho^2}{1 + \mu^2\rho^4 + (-1 + \mu^2\rho^4)\cos[s]}\,\mathrm{d}s.
\end{aligned}$$

We will apply the following formula of integration

$$\int \frac{1}{a+b\cos s}\mathrm{d}s = \frac{2}{a+b}\sqrt{\frac{a+b}{a-b}}\arctan\left[\sqrt{\frac{a-b}{a+b}}\tan\left[\frac{s}{2}\right]\right] + C,$$

where $a^2 > b^2$. Let

$$a = \frac{1+\mu^2\rho^4}{\mu\rho^2}, b = \frac{-1+\mu^2\rho^4}{\mu\rho^2}.$$

Therefore, we obtain $\bar{\phi} = \mu$. Other way to deduce this fact is by using the relation between the Ermakov-Pinney equation (5.13) and the Hill's equation $x'' + \mu^2 x = 0$. It is known (see for instance [25]), that for any solution $\alpha(t)$ of the Ermakov-Pinney equation, $\bar{\phi}$ as defined above is just the rotation number of Hill's equation $x'' + \mu^2 x = 0$, which turns out to be μ.

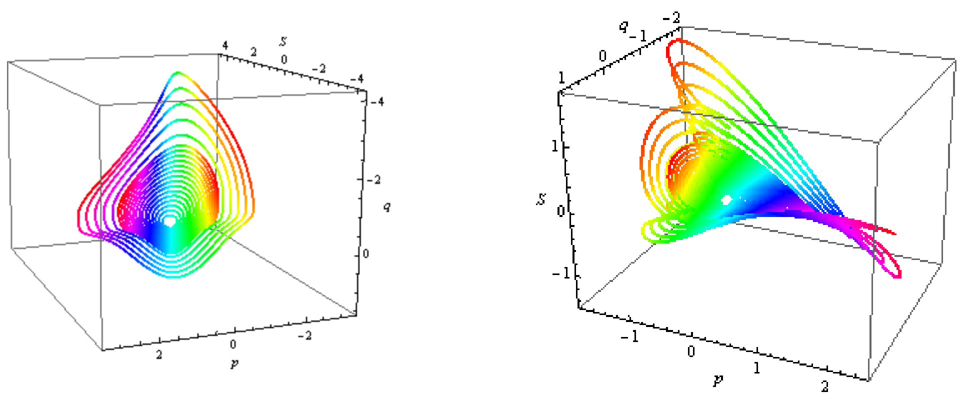
Notice that

$$\tilde{\phi}(t) = \int_0^t \left[\frac{1}{\alpha^2(t)} - \bar{\phi}\right]\mathrm{d}t$$

is $\dfrac{\pi}{\mu} = \dfrac{l\pi}{n}$-periodic, which is also $2nl\pi$-periodic. Since $\phi(t) = \phi_0 + \tilde{\phi}(t) + \bar{\phi}t = \phi_0 + \tilde{\phi}(t) + nt/l$, from (5.9)-(5.11) we know that q, p, S are all $4nl\pi$-periodic.

The case that $\cos\phi(t) \equiv 0$ holds obviously. $\square$

Theorem 5.1 provides a new method to construct isochronous systems in the three dimension space.



(b) Quasi-periodic orbits.

(a) Periodic orbits.

Figure 1: Trajectories surrounding the origin in the phase space.

Example 5.1. *Consider the following system*

$$\begin{cases} \dot{q}(t) = p(t), \\ \dot{p}(t) = -\left(4+\cos^2 t - 2\sin t\right)\dfrac{q(t)}{4} - \cos t p(t), \\ \dot{S}(t) = \dfrac{p^2(t)}{2} - \left(4+\cos^2 t - 2\sin t\right)\dfrac{q^2(t)}{8} - \cos t S(t), \end{cases} \tag{5.15}$$

where the function $I(x) \equiv 1$. By using Theorem 5.1, we know that the origin O is a global isochronous center. We plot some orbits near the origin O, see Figure 1(a). If we set $\omega^2(t) = \left(8 + \cos^2 t - 2\sin t\right)/4$, $\gamma(t) = \cos t$ and $m = 1$ in (5.2), we will see that there are infinitely many quasi-periodic solutions with the frequencies $\omega_1 = 1, \omega_2 = 2\sqrt{2}$ and 2π-periodic solutions surrounding around the origin. The periodic solutions are corresponding to the constant solution of the associated Ermakov-Pinney equation, see Figure 1(b).

In general, the function $I(t)$ is not always constant. The solutions of contact Hamiltonian system (5.2) is determined by the solutions of the Ermakov-Pinney equation

$$\ddot{\alpha}(t) + I(t)\alpha(t) = \frac{1}{\alpha^3(t)}. \tag{5.16}$$

The general solution of Ermakov-Pinney equation can be written explicitly in terms of a fundamental system of the associated Hill's equation

$$\ddot{\alpha}(t) + I(t)\alpha(t) = 0. \tag{5.17}$$

through a nonlinear superposition principle [25]. We denote the Poincaré matrix of (5.17) by

$$\Phi(t) = \begin{pmatrix} \alpha_1(t) & \alpha_2(t) \\ \dot{\alpha}_1(t) & \dot{\alpha}_2(t) \end{pmatrix},$$

where $\alpha_i(t)$ are solutions of (5.17) satisfying $\alpha_1(0) = \dot{\alpha}_2(0) = 1$ and $\dot{\alpha}_1(0) = \alpha_2(0) = 0$, respectively. The matrix $\Phi(2\pi)$ is called the monodromy matrix of (5.17).

Definition 5.3. Let λ_1, λ_2 be the Floquet multipliers of (5.17) (that is, the eigenvalues of its monodromy matrix). Equation (5.17) is called *elliptic, hyperbolic* or *parabolic*, if $|\lambda_{1,2}| = 1$ but $\lambda_{1,2} \neq \pm 1$, $|\lambda_{1,2}| \neq 1$ or $\lambda_1 = \lambda_2 = \pm 1$, respectively.

Although the relation between Hill's and Ermakov-Pinney equation is known long ago, a key connection between the stability of Hill's equation and the existence of periodic solutions of Ermakov-Pinney equation has been established recently by M. Zhang, see [26, Theorem 2.1]. We extend this relation to contact Hamiltonian system (5.2).

Theorem 5.2. Assume that $\gamma(t)$ and $\omega(t)$ are C^2 smooth and 2π-periodic functions satisfying the average value $\bar{\gamma} = 0$. If one of the following conclusions hold:

(i) The Ermakov-Pinney equation (5.16) has a positive 2π-periodic solution;

(ii) Hill's equation (5.17) is either elliptic or parabolic. In the second case, the solutions are all 2π-periodic or all 2π-anti-periodic.

(iii) Hill's equation (5.17) is stable in the sense of Lyapunov,

then contact Hamiltonian system (5.2) has infinitely many periodic solutions or quasi-periodic solutions.

Proof. The main result in [26] proves the equivalence of conditions (i)-(ii)-(iii). Notice that $I(t)$ is also a 2π-periodic function. Suppose that the associated Ermakov-Pinney equation (5.16) has a positive 2π-periodic solution. Then, recalling that

$$\dot{\phi}(t) = \frac{1}{\alpha^2(t)},$$

it turns out that $\cos\phi(t), \sin\phi(t)$ are quasiperiodic with two natural periods, 2π and $T = \int_0^{2\pi} \frac{1}{\alpha^2(t)} dt$. In view of (5.9)-(5.11), the contact Hamiltonian system (5.2) has infinitely many quasiperiodic solutions by varying h_0. Such solutions will be periodic only when the natural periods are commensurable. $\square$

As is shown in Example 5.15, even if the associated Hill's equation is elliptic or parabolic, the origin is not necessarily a global isochronous center. According to (5.9)-(5.11), the origin is a global isochronous center if and only if the Ermakov-Pinney equation (5.16) has two different independent positive 2π-periodic solutions α_1, α_2 with their mean values

$$\bar{\alpha}_i = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\alpha_i(t)}\mathrm{d}t \in \mathbb{Z}^+, \ i = 1, 2.$$

5.3 Asymptotic stability

Theorem 5.3. Assume that $\gamma(t)$ and $\omega(t)$ are C^2 smooth and 2π-periodic functions satisfying the average value

$$\bar{\gamma} = \frac{1}{2\pi}\int_0^{2\pi} \gamma(s)\mathrm{d}s > 0.$$

If the associated Hill's equation (5.17) is elliptic, then the zero solution of (5.2) is asymptotically stable.

Proof. Let λ_1, λ_2 be the Floquet multipliers of (5.17), then both Floquet multipliers lie on the unit circle in the complex plane. Because both λ_1 and λ_2 have nonzero imaginary parts, one of these Floquet multipliers, say λ_1, lies in the upper half plane. Therefore, there is a real number θ with $0 < 2\theta\pi < \pi$ and $\mathrm{e}^{2\theta\pi i} = \lambda_1$. Then there is a solution of the form $\mathrm{e}^{i\theta t}(r(t) + is(t))$ with r and s both 2π-periodic functions. Hence, the fundamental set of solutions of (5.17) has the form

$$r(t)\cos\theta t - s(t)\sin\theta t, \quad r(t)\sin\theta t + s(t)\cos\theta t. \tag{5.18}$$

By the nonlinear superposition principle [25] (also see [27]), the solutions of the Ermakov-Pinney equation (5.16) has the form

$$\alpha(t) = \left(Au^2(t) + 2Bu(t)v(t) + cv^2(t)\right)^{\frac{1}{2}},$$

where $u(t)$ and $v(t)$ are any two linearly independent solutions of the equation (5.17) and the constants A, B and C are related according to $B^2 - AC = 1/W^2$ with W being the constant Wronskian of the two linearly independent solutions. Therefore, the solutions $\alpha(t)$ of (5.16) are bounded. Rewrite

$$\mathrm{e}^{\Upsilon(t)} = \mathrm{e}^{\int_0^t[\gamma(s)-\bar{\gamma}]\mathrm{d}s}\mathrm{e}^{\bar{\gamma}t} := \psi(t)\mathrm{e}^{\bar{\gamma}t}$$

with $\psi(t)$ a 2π-periodic function. According to (5.9)-(5.11), we know that the solutions of (5.2) has the form

$$q(t) = P_1(t)\mathrm{e}^{-\dfrac{\bar{\gamma}t}{2}}, p(t) = P_2(t)\mathrm{e}^{-\dfrac{\bar{\gamma}t}{2}}, S(t) = P_3(t)\mathrm{e}^{-\bar{\gamma}t}, \tag{5.19}$$

where $P_i(t, \theta)$, $i = 1, 2, 3$, are bounded functions. Therefore, the zero solution is asymptotically stable. □

Theorem 5.4. Assume that $\gamma(t)$ and $\omega(t)$ are C^2 smooth and 2π-periodic functions satisfying the average value

$$\bar{\gamma} = \frac{1}{2\pi}\int_0^{2\pi} \gamma(s)\mathrm{d}s > 0.$$

If the associated Hill's equation (5.17) is parabolic, then the zero solution of (5.2) is asymptotically stable.

Proof. Since Hill's equation (5.17) is parabolic, there are two cases for the Floquet multipliers of (5.17).

Case I. $\lambda_1 = \lambda_2 = 1$. The nature of the solutions depends on the canonical form of $\Phi(2\pi)$. If $\Phi(2\pi) = E$ where E is the identity matrix, then we know that there is a Floquet normal from $\Phi(t) = P(t)$, and $P(t)$ is 2π-periodic, of C^1 and invertible. This means there is a fundamental set of periodic solutions for (5.17). By the nonlinear superposition principle [25], the solutions $\alpha(t)$ of (5.16) is also 2π-periodic, which leads to that zero solution is asymptotically stable.

If $\Phi(2\pi)$ is not the identity matrix, that is, the geometric multiplicity of eigenvalues of $\Phi(2\pi)$ is one, there is a nonsingular matrix C such that

$$C\Phi(2\pi)C^{-1} = E + N = \mathrm{e}^N,$$

where $N \neq 0$ is nilpotent. Therefore, there is a Floquet normal from $\Phi(t) = P(t)\mathrm{e}^{tB}$ with

$$B = C^{-1}\left(\frac{1}{2\pi}N\right)C,$$

which implies

$$\Phi(t) = P(t)\mathrm{e}^{tB} = P(t)C^{-1}\left(E + \frac{t}{2\pi}N\right)C,$$

where $P(t)$ is 2π-periodic, of C^1 and invertible. Again, by using the nonlinear superposition principle, the solutions of the Ermakov-Pinney equation (5.16) has the form

$$\alpha(t) = \left(r(t) + s(t)t + \kappa(t)t^2\right)^{\frac{1}{2}},$$

where r, s and κ are 2π-periodic functions. According to (5.9)-(5.11), the solutions of (5.2) satisfy that

$$|q(t)| \leq (C_1 + C_2 t)e^{-\frac{\Upsilon(t)}{2}}, \quad |p(t)| \leq (C_1 + C_2 t)e^{-\frac{\Upsilon(t)}{2}}, \quad |S(t)| \leq (C_1 + C_2 t^2)e^{-\Upsilon(t)}.$$

Therefore, zero solution is asymptotically stable.

Case II. $\lambda_1 = \lambda_2 = -1$. This situation is similar to Case I, except that the fundamental matrix is represented by $Q(x)\mathrm{e}^{tB}$ where $Q(x)$ is a 4π-periodic matrix function. □

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