A Dirichlet problem involving the mean curvature operator in Minkowski space

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On the occasion of the 60th birthday of Fabio Zanolin

The problem

We consider the existence and multiplicity of radial positive solutions of the problem

$$\mathcal{M}v + f(|x|, v) = 0, \qquad x \in B_R, \tag{1}$$

$$v = 0, \qquad x \in \partial B_R, \tag{2}$$

where $B_R = \{x \in \mathbb{R}^N : |x| < R\}$ and $f : [0, R] \times [0, \alpha) \to \mathbb{R}$ is a continuous function, which is positive on $(0, R] \times (0, \alpha)$, and

$$\mathcal{M}v = \operatorname{div}\left(\frac{\nabla v}{\sqrt{1 - |\nabla v|^2}}\right).$$

The problem

Setting, as usual, $r = |x|$ and $v(x) = u(r)$, the Dirichlet problem (1) reduces to the mixed boundary value problem

$$(r^{N-1}\phi(u'))' + r^{N-1} f(r,u) = 0, \qquad u'(0) = 0 = u(R), \tag{3}$$

Standing hypotheses:

(H_ϕ) $\phi : (-a, a) \to \mathbb{R}$ $(0 < a < \infty)$ *is an odd, increasing homeomorphism with* $\phi(0) = 0$;

(H_f) $f : [0, R] \times [0, \alpha) \to [0, \infty)$ *is a continuous function with* $0 < \alpha \leq \infty$ *and* $f(r, s) > 0$ *for all* $(r, s) \in (0, R] \times (0, \alpha)$.

Existence result I

Theorem 1

Assume that

$$\lim_{s\to 0+} \frac{f(r,s)}{\phi(s)} = +\infty \quad \text{uniformly with} \quad r \in [0,R] \tag{4}$$

$$\limsup_{s\to 0} \frac{\phi(\tau s)}{\phi(s)} < +\infty \quad \text{for all} \quad \tau > 0. \tag{5}$$

Then problem (3) has at least one positive solution if either $aR < \alpha$ or $\alpha = a$, $R = 1$ and

$$\lim_{s\to a-} \frac{f(r,s)}{\phi(s)} = 0 \quad \text{uniformly with} \quad r \in [0,1], \tag{6}$$

holds true.

Examples

Fix $0 \leq q < 1$ and let $\mu : [0, R] \to (0, \infty)$, $h : [0, R] \times [0, \infty) \to [0, \infty)$ be continuous functions.

(i) The Dirichlet problem

$$\mathcal{M}v + \mu(|x|)v^q + h(|x|, v) = 0 \quad \text{in} \quad B_R, \quad v = 0 \quad \text{on} \quad \partial B_R,$$

has at least one positive classical radial solution for any $R > 0$.

Examples

According to Theorem 3.2 and Remark 3.1 in

- A. Capietto, W. Dambrosio, F. Zanolin, Infinitely many radial solutions to a boundary value problem in a ball, Ann. Mat. Pura Appl. 179 (2001), 159-188.

the problem

$$\mathcal{M}v + \mu(|x|)|v|^{q-1}v = 0 \quad \text{in} \quad B_R, \quad v = 0 \quad \text{on} \quad \partial B_R$$

has infinitely many radial solutions with prescribed number of nodes (the positive case is not covered), provided that $0 < q < 1$ and μ is continuously differentiable.

Examples

(ii) If $0 \leq q < 1 \leq p$ and $\lambda > 0$, then problem

$$\mathcal{M}v + \lambda v^q + v^p = 0 \quad \text{in} \quad B_R, \quad v = 0 \quad \text{on} \quad \partial B_R$$

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In the classical case, using the upper and lower solutions method, it has been proved by Ambrosetti, Brezis and Cerami that problem

$$\Delta v + \lambda v^q + v^p = 0 \quad \text{in} \quad B_R, \quad v = 0 \quad \text{on} \quad \partial B_R$$

has a positive solution iff $0 < \lambda \leq \Lambda$ for some $\Lambda > 0$ ($0 < q < 1 < p$).

Examples

(iii) The Dirichlet problems

$$\mathcal{M}v + \frac{\mu(|x|)v^q}{\sqrt{\alpha^2 - v^2}} = 0 \quad \text{in} \quad B_R, \quad v = 0 \quad \text{on} \quad \partial B_R,$$

and

$$\mathcal{M}v + \frac{\mu(|x|)v^q}{(\alpha - v)^\gamma} = 0 \quad \text{in} \quad B_R, \quad v = 0 \quad \text{on} \quad \partial B_R,$$

have at least one positive classical radial solution for any $R < \alpha$.
(iv) If, in addition, $\gamma < \frac{1}{2}$, then the Dirichlet problem

$$\mathcal{M}v + \frac{\mu(|x|)v^q}{(1 - v^2)^\gamma} = 0 \quad \text{in} \quad \mathcal{B}(R), \quad v = 0 \quad \text{on} \quad \partial\mathcal{B}(R),$$

has at least one positive classical radial solution for any $R \leq 1$.

Sketch of the proof

We use the compact linear operators

$$S : C \to C, \quad Su(r) = \frac{1}{r^{N-1}} \int_0^r t^{N-1} u(t)dt, \quad Su(0) = 0;$$

$$K : C \to C, \quad Ku(r) = \int_r^R u(t)dt, \quad (r \in [0, R])$$

and the Nemytskii type operator

$$N_f : B_\alpha \to C, \quad N_f(u) = f(\cdot, |u(\cdot)|).$$

Sketch of the proof

Lemma 1

A function $u \in C$ is a solution of (3) if and only if it is a fixed point of the continuous nonlinear operator

$$\mathcal{N} : B_\alpha \to C, \qquad \mathcal{N} = K \circ \phi^{-1} \circ S \circ N_f.$$

Moreover, $\mathcal{N}$ is compact on $\overline{B}_\rho$ for all $\rho \in (0, \alpha)$.

Sketch of the proof

Proposition 1

Assume (4) and (5). Then there exists $0 < \rho_0 < \alpha$ such that

$$d_{LS}[I - \mathcal{N}, B_\rho, 0] = 0 \quad \text{for all} \quad 0 < \rho \leq \rho_0.$$

Sketch of the proof

Let $\mathcal{H}(\lambda, \cdot) : B_\alpha \to C$ be the fixed point operator associated to

$$(r^{N-1}\phi(u'))' + r^{N-1}[f(r,|u|) + \lambda] = 0, \qquad u'(0) = 0 = u(R), \quad (7)$$

where $\lambda \in [0,1]$.

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where $\lambda \in [0, 1]$. We can prove

$$u \neq \mathcal{H}(\lambda, u) \quad \text{for all} \quad (\lambda, u) \in [0, 1] \times \partial B_\rho$$

This implies

$$d_{LS}[I - \mathcal{H}(0, \cdot), B_\rho, 0] = d_{LS}[I - \mathcal{H}(1, \cdot), B_\rho, 0].$$

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Besides

$$u \neq \mathcal{H}(1,u) \quad \text{for all} \quad u \in \overline{B}_\rho,$$

implying that

$$d_{LS}[I-\mathcal{H}(1,\cdot),B_\rho,0] = 0.$$

Consequently,

$$\begin{aligned} d_{LS}[I-\mathcal{N},B_\rho,0] &= d_{LS}[I-\mathcal{H}(0,\cdot),B_\rho,0] \\ &= d_{LS}[I-\mathcal{H}(1,\cdot),B_\rho,0] \\ &= 0. \end{aligned}$$

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Proposition 2

If $aR < \alpha$, then one has

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Let $(\lambda, u) \in [0,1] \times \overline{B}_{aR}$ be such that $\mathcal{H}(\lambda, u) = u$. It follows immediately that $||u'|| < a$, implying that $||u|| < aR$. So,

$$u \neq \mathcal{H}(\lambda, u) \quad \text{for all} \quad (\lambda, u) \in [0,1] \times \partial B_{aR},$$

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which implies that

$$d_{LS}[I - \mathcal{H}(0, \cdot), B_{aR}, 0] = d_{LS}[I - \mathcal{H}(1, \cdot), B_{aR}, 0].$$

Consequently,

$$d_{LS}[I - \mathcal{N}, B_{aR}, 0] = d_{LS}[I, B_{aR}, 0] = 1$$

Sketch of the proof

If $\alpha = a$, then in Proposition 2 one has that $R < 1$. We consider now the case $R = 1$, assuming that f is sublinear with respect to ϕ at a.

Proposition 3

Assume that $a = \alpha$ and $R = 1$. If

$$\lim_{s \to a-} \frac{f(r,s)}{\phi(s)} = 0 \quad \text{uniformly with} \quad r \in [0,1],$$

then there exists $0 < \delta_1 < a$ such that

$$d_{LS}[I - \mathcal{N}, B_\delta, 0] = 1 \quad \text{for all} \quad \delta_1 \le \delta < a.$$

A second existence result

Now, we study the problem

$$\left(r^{N-1}\frac{u'}{\sqrt{1-u'^2}}\right)' + r^{N-1}\mu(r)p(u) = 0, \qquad u'(0) = 0 = u(R), \quad (8)$$

under the standing hypotheses:

(H_μ) $\mu : [0, R] \to \mathbb{R}$ *is continuous and* $\mu(r) > 0$ *for all* $r > 0$;

(H_p) $p : [0, \infty) \to \mathbb{R}$ *is a continuous function such that* $p(0) = 0$ *and* $p(s) > 0$ *for all* $s > 0$.

A second existence result

Theorem 2

Let P be the primitive of p with $P(0) = 0$. If

$$R^N < N \int_0^R r^{N-1} \mu(r) P(R-r) dr, \quad (9)$$

then problem (12) has at least one solution u such that $u > 0$ on $[0, R)$ and u is strictly decreasing.

Example

Given $m \geq 0$ and $q > 0$, let us consider the Hénon type problem

$$\mathcal{M}v + \lambda |x|^m v^q = 0 \quad \text{in} \quad B_R, \qquad v = 0 \quad \text{on} \quad \partial B_R. \tag{10}$$

It is easy to see that in this case inequality (9) becomes

$$1 < \lambda \frac{NR^{m+q+1}\Gamma(q+2)\Gamma(N+m)}{(q+1)\Gamma(N+m+q+2)}. \tag{11}$$

Consequently, if (11) holds then problem (10) has at least one classical positive radial solution.

Sketch of the proof

We follow a VARIATIONAL APPROACH. The problem is

$$\left(r^{N-1}\frac{u'}{\sqrt{1-u'^2}}\right)' + r^{N-1}\mu(r)p(u) = 0, \quad u'(0) = 0 = u(R), \quad (12)$$

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Define $K_0 := \{v \in W^{1,\infty} \ : \ \|v'\| \leq a, v(R) = 0\}$. The associated energy functional $I : C \to (-\infty, +\infty]$ is

$$I(v) = \frac{R^N}{N} - \int_0^R r^{N-1}\sqrt{1-v'^2}\,dr - \int_0^R r^{N-1}\mu(r)P(v)\,dr \quad (v \in K_0)$$

and $I \equiv +\infty$ on $C \setminus K_0$.

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Step 1: Each critical point of I is a solution of (12). Moreover, (12) has a solution which is a minimum point of I on C.

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Step 3: Consider the function $v_R \in K_0$ given by

$$v_R(r) = R - r \quad \text{for all} \quad r \in [0, R].$$

Using (9), one gets

$$I(v_R) = \frac{R^N}{N} - \int_0^R r^{N-1}\mu(r)P(R-r)dr < 0.$$

A multiplicity result for the Hénon problem

Given $m \geq 0$ and $q > 1$, let us consider the Hénon type problem

$$\mathcal{M}v + \lambda|x|^m v^q = 0 \quad \text{in} \quad B_R, \qquad v = 0 \quad \text{on} \quad \partial B_R. \tag{13}$$

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It is interesting to compare with the classical Hénon equation

$$\Delta v + \lambda |x|^m v^q = 0 \quad \text{in} \quad B_R, \quad v = 0 \quad \text{on} \quad \partial B_R.$$

If $N \geq 3$, it has a unique positive radial solution if $1 < q < \frac{N+2m+2}{N-2}$ and no solution if $q \geq \frac{N+2m+2}{N-2}$ (Pohozaev identity)

A multiplicity result for the Hénon problem

Theorem 3

Define $\mu_M := \max_{[0,R]} |\mu|$. There exists $\Lambda > 2N/(\mu_M R^{q+1})$ such that problem (13) has zero, at least one or at least two positive solutions according to $\lambda \in (0, \Lambda)$, $\lambda = \Lambda$ or $\lambda > \Lambda$. Moreover, Λ is strictly decreasing with respect to R.

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Step 4:

$$d_{LS}[I - \mathcal{N}, B(u_0, \rho), 0] = 1 \qquad \text{for all } 0 < \rho \leq \rho_0,$$

where u_0 is the solution found by the variational approach.

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We conclude by a simple excision argument.

THANKS
AND
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