Existence and stability of periodic solutions of the relativistic oscillator

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Introduction

Classical pendulum equation:

$$x'' + kx' + a\sin x = p(t) \equiv \overline{p} + \tilde{p}(t) \tag{1}$$

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Questions:

- **Degeneracy problem:** is $I_{\tilde{p}}$ nondegenerate?.
- If $k = 0$, it is known that $0 \in I_{\tilde{p}}$ (Hamel, 1922). Is this true for $k > 0$?

Introduction

Theorem (Ortega's counterexample)

Given positive constants a, k *and* T*, there exists* $p \in \tilde{C}_T$ *such that the equation* (1) *with* $\overline{p} = 0$ *has no* T*-periodic solutions.*

Introduction

Relativistic pendulum equation:

$$\phi(x')' + kx' + a\sin x = p(t) \equiv \overline{p} + \tilde{p}(t) \tag{2}$$

where $\phi :]-c, c[\to \mathbb{R}$ is given by

$$\phi(u) = \frac{u}{\sqrt{1 - \frac{u^2}{c^2}}}.$$

$c \equiv$ Speed of light in the vacuum

Main result

$$\big(\phi(x')\big)' + h(x)x' + g(x) = p(t) \tag{3}$$

where h, g are continuous functions, p is continuous and T-periodic and $\phi :]-c, c[\to \mathbb{R}$ is an increasing homeomorphism with $\phi(0) = 0$.

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Theorem (Bereanu-Mawhin, JDE, 2007)

If $g : \mathbb{R} \to \mathbb{R}$ is strictly monotone then $I_{\tilde{p}} = \text{Range}\, g$ for any $\tilde{p} \in \tilde{C}_T$.

Main result

Theorem

Let us assume that there exist real numbers $R_1 < R_2$ such that

(H1) $g \in C^1([R_1, R_2])$ *and* $g'(x) > 0$ *for all* $x \in]R_1, R_2[$.

(H2) $\frac{cT}{\sqrt{3}} \max_{x \in [R_1, R_2]} |g'(x)| \leq |g(R_2) - g(R_1)|$.

Then, for every $\tilde{p} \in \tilde{C}_T$, *the set* $I_{\tilde{p}}$ *of eq.* (3) *contains the closed interval*

$$[g(R_1) + \frac{cT}{2\sqrt{3}} \max_{x \in [R_1, R_2]} |g'(x)|, g(R_2) - \frac{cT}{2\sqrt{3}} \max_{x \in [R_1, R_2]} |g'(x)|]$$

Moreover, if $\overline{p}$ *belongs to the previous interval, then the corresponding* T*-periodic solution belongs to* $[R_1, R_2]$.

Main result

Theorem

Let us assume that there exist real numbers $R_1 < R_2$ such that

(H1) $g \in C^1([R_1, R_2])$ *and* $g'(x) < 0$ *for all* $x \in]R_1, R_2[$.

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Then, for every $\tilde{p} \in \tilde{C}_T$*, the set* $I_{\tilde{p}}$ *of eq.* (3) *contains the closed interval*

$$[g(R_2) + \frac{cT}{2\sqrt{3}} \max_{x\in[R_1,R_2]} |g'(x)|, g(R_1) - \frac{cT}{2\sqrt{3}} \max_{x\in[R_1,R_2]} |g'(x)|]$$

Moreover, if $\overline{p}$ *belongs to the previous interval, then the corresponding* T*-periodic solution belongs to* $[R_1, R_2]$.

Corollary for the relativistic pendulum

Corollary

Let us assume that $cT \leq 2\sqrt{3}$*. Then, for every*

$$|\overline{p}| \leq a\left(1 - \frac{cT}{2\sqrt{3}}\right),$$

eq. (2) possesses two different solutions x_1, x_2 *which verify* $-\frac{\pi}{2} \leq x_1 \leq \frac{\pi}{2} \leq x_2 \leq \frac{3\pi}{2}$.

Proof.

Step 1: change of variables $y = g(x) - \overline{p}$

$$\phi\left(\frac{y'}{g'(g^{-1}(y+\overline{p}))}\right)' + h(g^{-1}(y+\overline{p}))\frac{y'}{g'(g^{-1}(y+\overline{p}))} + y(t) = \tilde{p}(t). \tag{4}$$

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Step 2: Formulation of the fixed point problem.

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Step 2: Formulation of the fixed point problem.
Integrating,

$$\phi\left(\frac{y'}{g'(g^{-1}(y+\overline{p}))}\right) = \int_0^t (p(s) - y(s))ds - H(g^{-1}(y+\overline{p})) + C,$$

where H is a primitive of the function h and C is a constant to be fixed later.
If

$$F[y](t) = \int_0^t (p(s) - y(s))ds - H(g^{-1}(y+\overline{p})).$$

then

$$y(t) = \int_0^t g'(g^{-1}(y+\overline{p}))\phi^{-1}\left(F[y](t) + C\right) ds + D.$$

Proof.

Step 2: Formulation of the fixed point problem.

Lemma

For any $y \in \tilde{C}_T$, there exists a unique choice of C_y, D_y such that

$$\mathcal{T}[y](t) \equiv \int_0^t g'(g^{-1}(y+\overline{p}))\phi^{-1}\left(F[y](t)+C_y\right) ds + D_y \in \tilde{C}_T. \tag{5}$$

The functional $\mathcal{T} : \tilde{C}_T \to \tilde{C}_T$ is continuous and compact.

Proof.

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$$K = \{y \in \tilde{C}_T : y(t) \in [g(R_1) - \overline{p}, g(R_2) - \overline{p}]\}.$$

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Step 3: Application of the Schauder fixed point problem.

$$K = \{y \in \tilde{C}_T : y(t) \in [g(R_1) - \overline{p}, g(R_2) - \overline{p}]\}.$$

$$\|\mathcal{T}[y]\|_\infty \leq \frac{T}{2\sqrt{3}} \left\| T[y]' \right\|_\infty < \frac{cT}{2\sqrt{3}} \max_{x \in [R_1, R_2]} \left| g'(x) \right| .$$

By using condition $(H2)$, $\mathcal{T}[y] \in K$.

Open problems

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- Prove or disprove existence for $T >>$.
- Stability

Stability

The relativistic (an)harmonic oscillator

$$\left(\frac{mx'}{\sqrt{1-\frac{x'^2}{c^2}}}\right)' + kx = -F_0 \cos \omega t \tag{6}$$

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Chaotic behavior shown numerically in

- J.H. Kim, H.W. Lee, Relativistic chaos in the driven harmonic oscillator, Physical Rev. E, **51** n.2 (1995), pp. 1579-1581.

Stability: Main result

Theorem
Assume that the parameters m, k, ω *and* F_0 *satisfy the following conditions*

(H1) $\frac{k}{m} < \frac{\omega^2}{16}$,

(H2) $F_0 < \frac{1}{4} mc\omega$

(H3) $\frac{(mc\omega)^{19}[(mc\omega)^2 - 16F_0^2]}{120\pi F_0^2 (c^2 m^2 \omega^2 + 4F_0^2)^{19/2}} \sin\left(\frac{6\pi\omega^{1/2} mc^{3/2} k^{1/2}}{(c^2 m^2 \omega^2 + 4F_0{}^2)^{3/4}}\right) > 1$,

then the driven relativistic harmonic oscillator (6) has a unique stable $2\pi/\omega$*-periodic solution with a twist dynamics around it(* $\Longrightarrow$ *Lyapunov stability+ generically KAM dynamics).*

Stability: Corollaries

Corollary

Fixed ω, k, F_0, c *in (6), there exists* $M_0 \equiv M_0(\omega, k, F_0, c) > 0$ *such that if* $m > M_0$ *then the conclusion of Theorem 7 holds.*

Corollary

Fixed m, k, F_0, c *in (6), there exists a critical frequency* $\omega_0 \equiv \omega_0(m, k, F_0, c)$ *such that if* $\omega > \omega_0$ *then the conclusion of Theorem 7 holds.*

Existence: an equivalent Newtonian oscillator

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(6) can be written as

$$x' = \frac{c^2 y}{\sqrt{y^2 c^2 + m^2 c^4}}, \qquad y' = -kx - F_0 \cos \omega t. \tag{7}$$

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By deriving in the second equation, this system is equivalent to the second order equation

$$y'' + f(y) = \omega F_0 \sin \omega t, \tag{8}$$

with

$$f(y) = \frac{kc^2 y}{\sqrt{y^2 c^2 + m^2 c^4}}. \tag{9}$$

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Proposition

Let us assume that $4k < m\omega^2$*. Then, the equation* (8) *has a* T*-periodic solution* φ *such that*

$$-\frac{F_0}{\omega}(1+\sin\omega t) < \varphi(t) < \frac{F_0}{\omega}(1-\sin\omega t) \qquad (10)$$

for all t*.*

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for all t.

Proof.

- $\alpha(t) = \frac{F_0}{\omega}(1-\sin\omega t)$ lower solution
- $\beta(t) = -\frac{F_0}{\omega}(1+\sin\omega t)$ upper solution
- $\beta(t) < \alpha(t)$
- $|f'(s)| \leq \frac{k}{m} < \left(\frac{\pi}{T}\right)^2$

Uniqueness and linear stability.

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Proposition

Assume that

$$4k < m\omega^2 \tag{11}$$

Then, (8) has a unique a T-periodic solution φ which is elliptic.

Lyapunov stability.

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The periodic solution $\varphi(t)$ of a general equation $y'' + g(t, y) = 0$ is translated to the origin by the canonical change $x = y - \varphi(t)$. For the equivalent equation

$$x'' + g(t, x + \varphi(t)) - g(t, \varphi(t)) = 0, \tag{12}$$

the equilibrium $x \equiv 0$ is a solution. By developing the nonlinearity up to the third order,

$$x'' + a(t)x + b(t)x^2 + c(t)x^3 + R(t, x) = 0. \tag{13}$$

where
$a(t) = g_x(t, \varphi(t)), b(t) = g_{xx}(t, \varphi(t))2, c(t) = g_{xxx}(t, \varphi(t))6.$

Definition

We say that the equilibrium $x \equiv 0$ of (13) is of twist type if it is elliptic, not strongly resonant and the associated first twist coefficient $\beta \neq 0$.

Lyapunov stability.

Theorem

Assume that for the equation (13) the following conditions holds:

i) $-c(t) > c_* > 0, \ \forall t \in \mathbb{R}$

ii) $|b(t)| \leq b^*, \forall t \in \mathbb{R}$

iii) $0 < \sigma_1^2 < a(t) < \sigma_2^2 \leq (\frac{\pi}{2T})^2, \ \forall t \in \mathbb{R},$

iv) $c_* > \frac{10T\sigma_2^7}{3\sin(\frac{3T}{2}\sigma_1)\sigma_1^8} b^{*2}$ *(twist condition)*

Then the equilibrium $x \equiv 0$ *of (13) is of twist type.*