Periodic solutions of differential equations with weak singularities

Pedro J. Torres

Departamento de Matemática Aplicada,
Universidad de Granada (Spain)

AIMS' Sixth International Conference on Dyn. Systems, Diff. Equations and Applications,
June 2006

We look for positive T-periodic solutions of the model equation

$$x'' + a(t)x = \frac{b(t)}{x^{\lambda}} + c(t), \qquad (1)$$

with $a, b, c \in L^1[0, T]$ and $\lambda > 0$.

- Lazer-Solimini [Proc. A.M.S. 1987]

$$x'' = \frac{1}{x^{\lambda}} + c(t) \tag{2}$$

Summary of known results

- Lazer-Solimini [Proc. A.M.S. 1987]

$$x'' = \frac{1}{x^\lambda} + c(t) \qquad (2)$$

  - If $\lambda \geq 1$ (*strong force condition*), $\bar{c} < 0$ is a necessary and sufficient condition.

Summary of known results

- Lazer-Solimini [Proc. A.M.S. 1987]

$$x'' = \frac{1}{x^{\lambda}} + c(t) \tag{2}$$

  - If $\lambda \geq 1$ (*strong force condition*), $\bar{c} < 0$ is a necessary and sufficient condition.
  - If $0 < \lambda < 1$ (weak force condition): $\exists\, c$ with $\bar{c} < 0$ such that $\nexists$ periodic sol.

Summary of known results

- Lazer-Solimini [Proc. A.M.S. 1987]

$$x'' = \frac{1}{x^{\lambda}} + c(t) \qquad (2)$$

  - If $\lambda \geq 1$ (*strong force condition*), $\bar{c} < 0$ is a necessary and sufficient condition.
  - If $0 < \lambda < 1$ (weak force condition): $\exists$ c with $\bar{c} < 0$ such that $\nexists$ periodic sol.
- Since then, the strong force condition became standard in the related references:

Summary of known results

- Lazer-Solimini [Proc. A.M.S. 1987]

$$x'' = \frac{1}{x^\lambda} + c(t) \tag{2}$$

  - If $\lambda \geq 1$ (*strong force condition*), $\bar{c} < 0$ is a necessary and sufficient condition.
  - If $0 < \lambda < 1$ (weak force condition): $\exists$ c with $\bar{c} < 0$ such that $\nexists$ periodic sol.
- Since then, the strong force condition became standard in the related references:
  - Topological degree

Summary of known results

- Lazer-Solimini [Proc. A.M.S. 1987]

$$x'' = \frac{1}{x^\lambda} + c(t) \tag{2}$$

  - If $\lambda \geq 1$ (*strong force condition*), $\overline{c} < 0$ is a necessary and sufficient condition.
  - If $0 < \lambda < 1$ (weak force condition): $\exists$ c with $\overline{c} < 0$ such that $\nexists$ periodic sol.
- Since then, the strong force condition became standard in the related references:
  - Topological degree
  - Poincaré-Birkhoff Theorem

Summary of known results

- I. Rachunková, M. Tvrdý, I. Vrkoč [J. Differential Equations (2001)]

$$x'' + k^2 x = \frac{b}{x^\lambda} + c(t) \tag{3}$$

Theorem

For $0 < k^2 \leq \mu_1 := \left(\frac{\pi}{T}\right)^2$ and $\lambda, b > 0$, eq.(3) has a T-periodic solution if

$$c_* > -\left(\frac{\pi^2 - T^2 k^2}{T^2 \lambda b}\right)^{\frac{\lambda}{\lambda+1}} (\lambda + 1) b \tag{4}$$

Summary of known results

- I. Rachunková, M. Tvrdý, I. Vrkoč [J. Differential Equations (2001)]

$$x'' + k^2 x = \frac{b}{x^\lambda} + c(t) \qquad (3)$$

Theorem

For $0 < k^2 \leq \mu_1 := \left(\frac{\pi}{T}\right)^2$ *and* $\lambda, b > 0$*, eq.(3) has a* T*-periodic solution if*

$$c_* > -\left(\frac{\pi^2 - T^2k^2}{T^2\lambda b}\right)^{\frac{\lambda}{\lambda+1}} (\lambda+1)b \qquad (4)$$

$$k^2 = \mu_1 \implies c_* > 0$$

Summary of known results

At least for strong potentials, this result is optimal:
Counterexample by D. Bonheure, C. Fabry, D. Smets [Discrete Contin. Dyn. Syst.(2002)]

$$x'' + \mu_1 x = \frac{b}{x^3} + \epsilon \sin(\frac{2\pi}{T} t)$$

has no T-periodic solutions for $\epsilon > 0$ sufficiently small.

Summary of known results

- P.J.T. [J. Differential Equations (2003)]

Theorem

For $0 < k^2 < \mu_1 := \left(\frac{\pi}{T}\right)^2$ *and* $\lambda, b > 0$, *eq.(3) has a* T*-periodic solution if*

$$\begin{aligned} c_* &< 0, \\ c^* &\leq \frac{c_*}{\cos^\lambda\left(\frac{kT}{2}\right)} + \frac{k}{T}\sin kT \left(\frac{b}{|c_*|}\right)^{\frac{1}{\lambda}}. \end{aligned} \tag{5}$$

Summary of known results

- D. Bonheure, C. De Coster [Topol. Methods Nonlinear Anal. (2003)]

Theorem

Let be $k^2 = \mu_1$ *and* $\lambda, b > 0$. *If*

$$\gamma(t) = \int_t^{t+T} c(s) \sin\left(\pi \frac{s-t}{T}\right) ds > 0, \qquad \forall t, \tag{6}$$

eq.(3) has a T-periodic solution.

Summary of known results

- D. Bonheure, C. De Coster [Topol. Methods Nonlinear Anal. (2003)]

Theorem

Let be $k^2 = \mu_1$ and $\lambda, b > 0$. If

$$\gamma(t) = \int_t^{t+T} c(s) \sin\left(\pi \frac{s-t}{T}\right) ds > 0, \qquad \forall t, \tag{6}$$

eq.(3) has a T-periodic solution.

Note: $\gamma(t)$ is the unique T-periodic solution of the linear equation $x'' + \mu_1 x = c(t)$.

Work to be done

Work to be done

- The results of Rachunková et al. and Bonheure-deCoster do not cover important cases

$$x'' + a(t)x = \frac{b(t)}{x^\lambda} + c(t)$$

Work to be done

- The results of Rachunková et al. and Bonheure-deCoster do not cover important cases

$$x'' + a(t)x = \frac{1}{x^{\lambda}}$$

"Brillouin equation"

Work to be done

- The results of Rachunková et al. and Bonheure-deCoster do not cover important cases

$$x'' + a(t)x = \frac{1}{x^\lambda}$$

- The results of P.J.T. do not cover the "critical" value μ_1.

The general equation.

Let us consider

$$x'' + a(t)x = f(t,x) + c(t), \qquad (7)$$

with $a, c \in L^1[0,T]$ and $f \in Car([0,T] \times \mathbb{R}^+, \mathbb{R})$.

The general equation.

Let us consider

$$x'' + a(t)x = f(t,x) + c(t), \qquad (7)$$

with $a, c \in L^1[0,T]$ and $f \in Car([0,T] \times \mathbb{R}^+, \mathbb{R})$.

STANDING HYPOTHESIS:

(H1) The Hill's equation $x'' + a(t)x = 0$ is non-resonant and the corresponding Green's function $G(t,s)$ is non-negative for every $(t,s) \in [0,T] \times [0,T]$.

Note: If $a(t) \equiv k^2$, $(H1) \Leftrightarrow 0 < k^2 \leq \mu_1$

Main result.

Define

$$\gamma(t) = \int_0^T G(t,s)c(s)ds,$$

Theorem

Let us assume that there exist $b \succ 0$ and $\lambda > 0$ such that

$$0 \leq f(t,x) \leq \frac{b(t)}{x^\lambda}, \qquad \text{for all } x > 0, \text{ for a.e. } t$$

If $\gamma_* > 0$, then there exists a T-periodic solution of (7).

Proof.

Schauder's fixed point theorem to

$$\mathcal{F}[x](t) := \int_0^T G(t,s)\left[f(s,x(s)) + c(s)\right] ds =$$

$$= \int_0^T G(t,s) f(s,x(s)) ds + \gamma(t)$$

Proof.

Schauder's fixed point theorem to

$$\mathcal{F}[x](t) := \int_0^T G(t,s)\,[f(s,x(s)) + c(s)]\,ds =$$
$$= \int_0^T G(t,s) f(s,x(s))ds + \gamma(t)$$

Define

$$K = \{x \in C_T \ : \ r \leq x(t) \leq R \text{ for all } t\}$$

then

$$\mathcal{F}(K) \subset K$$

by taking

$$r := \gamma_*, \quad R = \frac{\beta^*}{\gamma_*^\lambda} + \gamma^*.$$

□

The case $\gamma_* = 0$.

The case $\gamma_* = 0$.

Theorem

Let us assume $(H1)$ *and that there exist* $b, \hat{b} \succ 0$ *and* $0 < \lambda < 1$ *such that*

$$0 \leq \frac{\hat{b}(t)}{x^\lambda} \leq f(t,x) \leq \frac{b(t)}{x^\lambda}, \qquad \textit{for all } x > 0, \ \textit{for a.e. } t.$$

If $\gamma_* = 0$*, then there exists a* T*-periodic solution of* (7)*.*

The case $\gamma_* = 0$.

Theorem

Let us assume $(H1)$ *and that there exist* $b, \hat{b} \succ 0$ *and* $0 < \lambda < 1$ *such that*

$$0 \leq \frac{\hat{b}(t)}{x^\lambda} \leq f(t,x) \leq \frac{b(t)}{x^\lambda}, \qquad \text{for all } x > 0, \text{ for a.e. } t.$$

If $\gamma_* = 0$*, then there exists a* T*-periodic solution of* (7)*.*

Sometimes, the presence of a weak nonlinearity is an ADVANTAGE.

The case $\gamma_* = 0$.

Theorem

Let us assume $(H1)$ *and that there exist* $b, \hat{b} \succ 0$ *and* $0 < \lambda < 1$ *such that*

$$0 \leq \frac{\hat{b}(t)}{x^\lambda} \leq f(t,x) \leq \frac{b(t)}{x^\lambda}, \qquad \text{for all } x > 0, \text{ for a.e. } t.$$

If $\gamma_* = 0$*, then there exists a* T*-periodic solution of* (7)*.*

Sometimes, the presence of a weak nonlinearity is an ADVANTAGE.

Open problem for strong singularities!!

The particular case $c(t) \equiv 0$.

$$x'' + a(t)x = \frac{b(t)}{x^\lambda}$$

The particular case $c(t) \equiv 0$.

$$x'' + a(t)x = \frac{b(t)}{x^\lambda}$$

Define

$$\beta(t) = \int_0^T G(t,s)b(s)ds$$

Theorem

If $b \succ 0$ and $0 < \lambda < 1$, then there exists a T-periodic solution such that

$$\left(\frac{\beta_*}{\beta^{*\lambda}}\right)^{\frac{1}{1-\lambda^2}} \le x(t) \le \left(\frac{\beta^*}{\beta_*^{\lambda}}\right)^{\frac{1}{1-\lambda^2}}$$

The particular case $c(t) \equiv 0$.

$$x'' + a(t)x = \frac{b(t)}{x^\lambda}$$

Define

$$\beta(t) = \int_0^T G(t,s)b(s)ds$$

Theorem

If $b \succ 0$ and $0 < \lambda < 1$, then there exists a T-periodic solution such that

$$\left(\frac{\beta_*}{{\beta^*}^\lambda}\right)^{\frac{1}{1-\lambda^2}} \leq x(t) \leq \left(\frac{\beta^*}{{\beta_*}^\lambda}\right)^{\frac{1}{1-\lambda^2}}$$

Optimal bounds:

The particular case $c(t) \equiv 0$.

$$x'' + a(t)x = \frac{a(t)}{x^\lambda}$$

Define

$$\beta(t) = \int_0^T G(t,s)a(s)ds$$

Theorem

If $b \succ 0$ and $0 < \lambda < 1$, then there exists a T-periodic solution such that

$$\left(\frac{\beta_*}{\beta^{*\lambda}}\right)^{\frac{1}{1-\lambda^2}} \leq x(t) \leq \left(\frac{\beta^*}{\beta_*^{\lambda}}\right)^{\frac{1}{1-\lambda^2}}$$

Optimal bounds:if $a(t) \equiv b(t)$,

The particular case $c(t) \equiv 0$.

$$x'' + a(t)x = \frac{a(t)}{x^\lambda}$$

Define

$$\beta(t) = \int_0^T G(t,s)a(s)ds$$

Theorem

If $b \succ 0$ and $0 < \lambda < 1$, then there exists a T-periodic solution such that

$$\left(\frac{\beta_*}{\beta^{*\lambda}}\right)^{\frac{1}{1-\lambda^2}} \leq x(t) \leq \left(\frac{\beta^*}{\beta_*^{\lambda}}\right)^{\frac{1}{1-\lambda^2}}$$

Optimal bounds:if $a(t) \equiv b(t)$, then $\beta_* = \beta^* = 1$ and we get the exact solution $x(t) = 1$.

The case $\gamma^* \leq 0$.

$$x'' + a(t)x = \frac{b(t)}{x^\lambda} + c(t),$$

The case $\gamma^* \leq 0$.

$$x'' + a(t)x = \frac{b(t)}{x^\lambda} + c(t),$$

Theorem

Let us assume that $b \succ 0$ and $0 < \lambda < 1$. If $\gamma^* \leq 0$ and

$$\gamma_* \geq \left[\frac{\beta_*}{\beta^{*\lambda}}\lambda^2\right]^{\frac{1}{1-\lambda^2}}\left(1 - \frac{1}{\lambda^2}\right) \tag{8}$$

then there exists a positive T-periodic solution.

The case $\gamma^* \leq 0$.

$$x'' + a(t)x = \frac{b(t)}{x^\lambda} + c(t),$$

Theorem

Let us assume that $b \succ 0$ *and* $0 < \lambda < 1$. *If* $\gamma^* \leq 0$ *and*

$$\gamma_* \geq \left[\frac{\beta_*}{\beta^{*\lambda}}\lambda^2\right]^{\frac{1}{1-\lambda^2}} \left(1 - \frac{1}{\lambda^2}\right) \tag{8}$$

then there exists a positive T*-periodic solution.*

Note: The bound goes to $-\beta_*$ when $\lambda \to 0^+$

Back to the equation with fixed coefficients.

$$x'' + k^2 x = \frac{b}{x^\lambda} + c(t)$$

Back to the equation with fixed coefficients.

$$x'' + k^2 x = \frac{b}{x^\lambda} + c(t)$$

Corollary

Let us assume that $0 < \lambda < 1$ *and* $0 < k^2 \leq \mu_1 := \left(\frac{\pi}{T}\right)^2$. *Then, there exists a positive* T*-periodic solution if* $c(t) < 0$ *for a.e.* t *and*

$$c_* \geq \left[b\, k^{2\lambda}\, \lambda^{\frac{2\lambda^2}{1-\lambda}}\right]^{\frac{1}{1+\lambda}} (\lambda^2 - 1). \qquad (9)$$

Back to the equation with fixed coefficients.

$$x'' + k^2 x = \frac{b}{x^\lambda} + c(t)$$

Corollary

Let us assume that $0 < \lambda < 1$ *and* $0 < k^2 \leq \mu_1 := \left(\frac{\pi}{T}\right)^2$*. Then, there exists a positive* T*-periodic solution if* $c(t) < 0$ *for a.e.* t *and*

$$c_* \geq \left[b\,k^{2\lambda}\,\lambda^{\frac{2\lambda^2}{1-\lambda}}\right]^{\frac{1}{1+\lambda}}(\lambda^2 - 1). \qquad (9)$$

Note: Now, the bound goes to $-b$ when $\lambda \to 0^+$.

Existence beyond μ_1.

$$x'' + k^2 x = \frac{b(t)}{x^\lambda} + c(t)$$

Existence beyond μ_1.

$$x'' + k^2 x = \frac{b(t)}{x^\lambda} + \overline{c} + \tilde{c}(t)$$

Existence beyond μ_1.

$$x'' + k^2 x = \frac{b(t)}{x^\lambda} + \overline{c} + \tilde{c}(t)$$

Define the sequence

$$\mu_n = \left(\frac{n\pi}{T}\right)^2$$

$\mu_{2k+1} \equiv$ eigenvalues of the Dirichlet problem
$\mu_{2k} \equiv$ eigenvalues of the periodic problem

Existence beyond μ_1.

$$x'' + k^2 x = \frac{b(t)}{x^\lambda} + \overline{c} + \tilde{c}(t)$$

Define the sequence

$$\mu_n = \left(\frac{n\pi}{T}\right)^2$$

$\mu_{2k+1} \equiv$ eigenvalues of the Dirichlet problem
$\mu_{2k} \equiv$ eigenvalues of the periodic problem

Theorem

Let us assume that $k^2 \neq \mu_{2n}$, $n \in \mathbb{N}^*$. Then, for any $\tilde{c} \in L^1[0, T]$ there exists $C_0 > 0$ such that the eq. possesses a unique positive T-periodic for any $\overline{c} > C_0$.

Existence beyond μ_1.

$$x'' + k^2 x = \frac{b(t)}{x^\lambda} + \bar{c} + \tilde{c}(t)$$

Define the sequence

$$\mu_n = \left(\frac{n\pi}{T}\right)^2$$

$\mu_{2k+1} \equiv$ eigenvalues of the Dirichlet problem
$\mu_{2k} \equiv$ eigenvalues of the periodic problem

Theorem

Let us assume that $k^2 \neq \mu_{2n}$, $n \in \mathbb{N}^*$. Then, for any $\tilde{c} \in L^1[0, T]$ there exists $C_0 > 0$ such that the eq. possesses a unique positive T-periodic for any $\bar{c} > C_0$.

Note: No sign condition over *b*!!

Stability beyond μ_1.

$$x'' + k^2 x = \frac{b(t)}{x^\lambda} + \overline{c} + \tilde{c}(t)$$

Stability beyond μ_1.

$$x'' + k^2 x = \frac{b(t)}{x^\lambda} + \overline{c} + \tilde{c}(t)$$

Theorem

Let us assume that $k^2 \neq \left(\frac{n\pi}{mT}\right)^2$ *for all* $n, m \in \mathbb{N}^*$ *with* $1 \leq m \leq 4$ *and* $b(t) > 0$ *for a.e.* t*. Then, for any* $\tilde{c} \in L^1[0, T]$ *there exists* $C_1 > C_0 > 0$ *such that for any* $\overline{c} > C_1$ *the unique* T*-periodic solution is Lyapunov stable.*