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GREEN'S FUNCTIONS OF PARTIAL DIFFERENTIAL EQUATIONS WITH INVOLUTIONS

F. Adrián F. Tojo[†,1], Pedro Torres[‡,2]

Abstract In this paper we develop a way of obtaining Green's functions of partial differential equations with linear involutions by reducing the equation to a higher-order PDE without involutions. The developed theory is applied to a model of heat transfer in a conducting plate which is bent in half.

Keywords Green's functions, PDEs, Linear involution, Heat equation.

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1. Introduction

The study of differential equations with involutions dates back to the work of Silberstein [10] who, in 1940, obtained the solution of the equation $f(x) = f(1/x)$. In the field of differential equations there has been quite a number of publications (see for instance the monograph on the subject of reducible differential equations of Wiener [11]) but most of them relate to ordinary differential equations (ODEs). There has also been some work in partial differential equations (PDEs), for instance [11] or [2], where they study a PDE with reflection.

In what Green's functions for equations with involutions is concerned, we find in [3] the first Green's function for ODEs with reflection and in [4] we have a framework that allows the reduction of any differential equation with reflection and constant coefficients. This setting is established in a general way, so it can be used as well for other operators (the Hilbert transform, for instance) or in other yet unexplored problems, like PDEs [8]. In this work we take this last approach and find a way of reducing general linear PDEs with linear involutions to usual PDEs.

The paper is structured as follows. In Section 2 we develop an abstract framework, with definitions and adequate notation in order to treat linear PDEs as elements of a vector space consisting of symmetric tensors. This will allow us to systematize the algebraic transformations necessary in order to obtain the desired reduction of the problem. In Section 3 we start providing a simple example that shows how the general process works and then prove the main result of the paper, Theorem 3.1, that permits a general reduction in the case of order two involutions. We end the Section with a problem with an order 3 involution (Example 3.2), illustrating that the same principles could be applied to higher order involutions.

[†]Departamento de Análise Matemática, Facultade de Matemáticas, Universidade de Santiago de Compostela, Spain. Email address: fernandoadrian.fernandez@usc.es
[‡]Departamento de Matemática Aplicada, Facultad de Ciencias, Universidade de Granada, Spain. Email address: ptorres@ugr.es
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Finally, in Section 4, we describe a way to obtain Green's functions for PDEs with linear involutions and apply it to a model of the process of heat transfer in a conducting plate which is bent in half with the two halves separated by some insulating material. We study the problem for different kinds of boundary conditions and a general heat source.

2. Definitions and notation

2.1. Derivatives

Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$, $n \in \mathbb{N}$ and $\Omega \subset V := \mathbb{F}^n$ a connected open subset. For $p \geq 2$, note by $V^{\odot p}$ the space of symmetric tensors or order p, that is, the space of tensors of order p modulus the permutations of their components. We note $V^{\odot 1} = V$ and $V^{\odot 0} = \mathbb{F}$. For the convenience of the reader, we summarize now the properties and operations of the symmetric tensors:

- $V^{\odot p} := \{v_1^1 \odot \cdots \odot v_1^k + \cdots + v_r^1 \odot \cdots \odot v_r^k \ : \ v_j^s \in \mathbb{F}^n;\ j = 1, \dots, r;\ s = 1, \dots, k;\ r, k \in \mathbb{N}\}$.
- $(v_1^1 \odot \cdots \odot v_1^k) \odot (v_1^{k+1} \odot \cdots \odot v_1^p) = v_1^1 \odot \cdots \odot v_1^p;\ v_1^s \in \mathbb{F}^n;\ s = 1, \dots, p;\ p \in \mathbb{N}$.
- $v_1 \odot v_2 = v_2 \odot v_1;\ v_1, v_2 \in \mathbb{F}^n$.
- $\lambda(v_1 \odot v_2) = (\lambda v_1) \odot v_2;\ v_1, v_2 \in \mathbb{F}^n$.
- $(v_1 + v_2) \odot v_3 = v_1 \odot v_3 + v_2 \odot v_3;\ v_1, v_2, v_2 \in \mathbb{F}^n$.
- $0 \odot v_1 = 0;\ v_1 \in \mathbb{F}^n$.

With these properties, $V^{\odot p}$ is an $\mathbb{F}$-vector space of dimension $\binom{n+p-1}{p}$.

For every $v = (v_1, \dots, v_n) \in V$, we define the directional derivative operator as

$$\mathcal{C}^1(\Omega, \mathbb{F}) \xrightarrow{\quad D_v \quad} \mathcal{C}(\Omega, \mathbb{F})$$

$$y \longmapsto v_1 \frac{\partial y}{\partial x^1} + \cdots + v_n \frac{\partial y}{\partial x^n}$$

If ∇y denotes the gradient vector of y, then $D_v(y) = v^T \nabla y$. Observe that $D_{\lambda u + v} = \lambda D_u + D_v$ for every $u, v \in \mathbb{F}^n$ and $\lambda \in \mathbb{F}$, that is, D_v is linear in v. Also, for $u, v \in \mathbb{F}^n$, if $y \in \mathcal{C}^2(\Omega, \mathbb{F})$, then $D_u(D_v y) = D_v(D_u y)$. Furthermore, $D_u \circ D_v$ is bilinear –that is, linear in both u and v, so we can write the identification $D_v \circ D_u \equiv D^2_{v \odot u}$, where $v \odot u$ denotes de symmetric tensor product of u and v. In the same way, we define the composition of higher order derivatives by $D^p_{\omega_2} \circ D^q_{\omega_1} = D^{p+q}_{\omega_2 \odot \omega_1}$ where $\omega_1 \in V^{\odot q}$ and $\omega_2 \in V^{\odot p}$, $p, q \in \mathbb{N}$.

In this way, a linear partial differential equation is given by

$$Ly := \sum_{k=0}^{m} D^k_{\omega_k} y = 0, \tag{2.1}$$

where $\omega_k \in V^{\odot k}$ for $k = 1, \dots, m$ and $D^0_{\omega_0} u \equiv \omega_0 u$ where $\omega_0 \in \mathbb{F}$ (that is, $V^{\odot 0} := \mathbb{F}$). Now, the operator L can be identified with $\omega_0 + \omega_1 + \cdots + \omega_n$, which is an element of the symmetric tensor algebra

$$S^* V := \bigoplus_{k=0}^{\infty} V^{\odot n} = \mathbb{F} \oplus V \oplus (V \odot V) \oplus (V \odot V \odot V) \oplus \cdots$$

It is interesting to point out the the Hilbert space completion of S^*V, that is, $F_+(V) := \overline{S^*V}$, is called the *symmetric* or *bosonic Fock space*, which is widely used in quantum mechanics [5].

2.2. Involutions

Definition 2.1. Let Ω be a set and $A : \Omega \to \Omega$, $p \in \mathbb{N}$, $p \geq 2$. We say that A is an *order p involution* if

1. $A^p \equiv A \circ \overset{p}{\overbrace{\cdots}} \circ A = \mathrm{Id}$,
2. $A^j \neq \mathrm{Id},\ j = 1, \dots, p-1$.

We will consider linear involutions in $\mathbb{F}^n$. They are characterized by the following theorem.

Theorem 2.1 ([1]). *A necessary and sufficient condition for a linear transformation A on a finite dimensional complex vector space V to be an involution of order p is that $A = \alpha_1 P_1 + \cdots + \alpha_k P_k$ where α_j is a p-th root of the unity, and $P_1, \dots, P_k$ are projections such that $P_j P_l = 0$, $i \neq j$ and $P_1 + \cdots + P_k = \mathrm{Id}$.*

Remark 2.1. As an straightforward consequence of this result we have that there are only order two linear involutions in $\mathbb{R}^n$. This is because the only real p-th roots of the unity are contained in $\{\pm 1\}$.

The characterization provided in Theorem 2.1 can be rewritten in the following way.

Corollary 2.1. *A necessary and sufficient condition for a linear transformation A on V to be an involution of order p is that $A = U^{-1}\Lambda U$ where $\Lambda, U \in \mathcal{M}_n(\mathbb{F})$, U is invertible and Λ is a diagonal matrix where the elements of the diagonal are p-th roots of the unity.*

Proof. Consider the characterization of involutions given by Theorem 2.1. Take the vector subspaces $H_j := P_j V$, $j = 1, \dots, k$. Then, $V = H_1 \oplus \cdots \oplus H_k$. Take U^{-1} to be the matrix of which its columns are, consecutively, a basis of H_k. Hence, $A = U^{-1}\Lambda U$ where Λ is a diagonal matrix of diagonal

$$(\alpha_1, \dots, \alpha_1, \alpha_2, \dots, \alpha_2, \dots, \alpha_k, \dots, \alpha_k),$$

where every α_j is repeated accorollaryding to the dimension of H_k. $\square$

2.3. Pullbacks and equations

Let $\mathcal{F}(\Omega, \mathbb{F})$ be the set of functions from $\Omega \subset \mathbb{F}^n$ to $\mathbb{F}$. We define the pullback operator by a function $\varphi \in \mathcal{F}(\Omega, \Omega)$ as

$$\begin{array}{ccc} \mathcal{F}^1(\Omega, \mathbb{F}) & \xrightarrow{\varphi^*} & \mathcal{F}(\Omega, \mathbb{F}) \\ y & \longmapsto & y \circ \varphi \end{array}$$

Assume A is a linear order p involution on Ω (Ω has to be such that $\Omega = A(\Omega)$). From now on, we will omit the composition signs. Observe that, for $v \in V$, $x \in \Omega$

and $y\in\mathcal{C}^1(\Omega,\mathbb{F})$,

$$\begin{aligned}((D_vA^*)y)(x)&=D_v(y(Ax))=v^T\nabla(y(Ax))=v^TA^T\nabla y(Ax)\\&=(Av)^T\nabla y(Ax)=D_{Av}\nabla y(Ax)=(A^*D_{Av})y(x),\end{aligned}$$

or, written briefly, $D_vA^*=A^*D_{Av}$. All the same, for $v_1,\dots,v_j\in V$,

$$D^j_{v_1\odot\cdots\odot v_j}A^*=A^*D^j_{Av_1\odot\cdots\odot Av_j}.$$

If $\omega_k=v_1\odot\cdots\odot v_k\in V^{\odot k}$, we denote $A\omega_k\equiv Av_1\odot\cdots\odot Av_j$. This way, $D^j_{\omega_k}A^*=A^*D^j_{A\omega_k}$.

We can consider now linear partial differential equations with linear involutions of the form

$$Ly:=\sum_{j=0}^{p-1}\sum_{k=0}^{m}(A^*)^jD^k_{\omega^j_k}y=0,$$

where $\omega^j_k\in V^{\odot k}$ for $k=0,\dots,m$; $j=0,\dots,p-1$. This time we can identify L with

$$\left(\omega^0_1+\cdots+\omega^0_m,\,\omega^1_1+\cdots+\omega^1_m,\,\dots,\,\omega^{p-1}_1+\cdots+\omega^{p-1}_m\right)\in(S^*V)^p.$$

The interest in these equations appears when they can be reduced to usual partial differential equations.

Definition 2.2 ([4])**.** If $\mathbb{F}[D]$ is the ring of polynomials on the usual differential operator D and $\mathcal{A}$ is any operator algebra containing $\mathbb{F}[D]$, then an equation $Lx=0$, where $L\in\mathcal{A}$, is said to be a *reducible differential equation* if there exits $R\in\mathcal{A}$ such that $RL\in\mathbb{F}[D]$.

In our present case, the first projection of the algebra $(S^*V)^p$ is precisely the algebra of partial differential operators on n variables $\mathrm{PD}_n[\mathbb{F}]$, so we want to find elements $R\in(S^*V)^p$ such that they nullify the last $p-1$ components of L.

3. Reducing the operators

We start with an illustrative example.

Example 3.1. Let $V=\mathbb{R}^2$, $v=(v_1,v_2)\in V$ and

$$A=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

A is an order 2 involution. Consider the equation

$$v_1\frac{\partial y}{\partial x_1}(x)+v_2\frac{\partial y}{\partial x_2}(x)+y(Ax)=0,\ x=(x_1,x_2)\in\mathbb{R}^2. \tag{3.1}$$

Here we work with the operator $L=D_v+A^*$. Take then $R=D_{-Av}+A^*$ and consider the identity operator Id. We have that

$$\begin{aligned}RL=&(D_{-Av}+A^*)(D_v+A^*)=D_{-Av}D_v+A^*D_v+D_{-Av}A^*+(A^*)^2\\=&D_{-Av\odot v}+A^*D_v+A^*D_{-AAv}+\mathrm{Id}=D_{-Av\odot v}+A^*D_v+A^*D_{-v}+\mathrm{Id}\\=&D_{-Av\odot v}+A^*D_v-A^*D_v+\mathrm{Id}=D_{-Av\odot v}+\mathrm{Id}\,.\end{aligned}$$

Hence, every two-times differentiable solution of equation (3.1) has to be a solution of the partial differential equation

$$-v_1^2\frac{\partial^2 y}{\partial x_1^2}(x)+v_2^2\frac{\partial^2 y}{\partial x_2^2}(x)+y=0,\ x=(x_1,x_2)\in\mathbb{R}^2.$$

Remark 3.1. With the notation we have introduced, it is extremely important the use of parentheses. Observe that every $\omega\in(\mathbb{F}^n)^{\odot k}$ can be expressed as $\omega=v_1^1\odot\cdots\odot v_1^k+\cdots+v_r^1\odot\cdots\odot v_r^k$ for some $v_j^s\in\mathbb{F}^n,\ j=1,\dots,r,\ s=1,\dots,k$; $r,k\in\mathbb{N}$. Hence, for $c\in\mathbb{F}$,

$$\begin{aligned}(cA)\omega =&cAv_1^1\odot\cdots\odot cAv_1^k+\cdots+cAv_r^1\odot\cdots\odot cAv_r^k\\ =&c^k(Av_1^1\odot\cdots\odot Av_1^k)+\cdots+c^k(Av_r^1\odot\cdots\odot Av_r^k)=c^k(A\omega)\equiv c^kA\omega.\end{aligned}$$

Theorem 3.1. *Let A be an order 2 linear involution on $\mathbb{F}^n$. Let $L\in(S^*V)^p$ be defined as in* (2.1)*. Then there exists $R\in(S^*V)^p$ defined as*

$$Ry:=\sum_{j=0}^{p-1}\sum_{k=0}^{m}(A^*)^jD^k_{\xi_k^j}y=0,$$

where $\xi_k^0=-A\omega_k^0$, $\xi_k^1=\omega_k^1$, for $k=0,1,\dots,$ such that $RL\in\mathrm{PD}_n[\mathbb{F}]$. Furthermore, L and R commute.

Proof. For convenience, define ξ_k^j and ω_k^j outside the index range $j=0,\dots,p-1$, $k=0,\dots m$ to be zero. In general,

$$\begin{aligned}RL=&\sum_{l=0}^{p-1}\sum_{r=0}^{m}(A^*)^lD^r_{\xi_r^l}\left(\sum_{j=0}^{p-1}\sum_{k=0}^{m}(A^*)^jD^k_{\omega_k^j}\right)=\sum_{l,j=0}^{p-1}\sum_{r,k=0}^{m}(A^*)^lD^r_{\xi_r^l}(A^*)^jD^k_{\omega_k^j}\\ =&\sum_{l,j=0}^{p-1}\sum_{r,k=0}^{m}(A^*)^{l+j}D^r_{A^j\xi_r^l}D^k_{\omega_k^j}=\sum_{l,j=0}^{p-1}\sum_{r,k=0}^{m}(A^*)^{l+j}D^{r+k}_{A^j\xi_r^l\odot\omega_k^j}\\ =&\sum_{l,j=0}^{p-1}(A^*)^{l+j}\left(\sum_{s=0}^{2m}\sum_{k=0}^{s}D^s_{A^j\xi_{s-k}^l\odot\omega_k^j}\right)=\sum_{l,j=0}^{p-1}(A^*)^{l+j}\left(\sum_{s=0}^{2m}D^s_{\sum_{k=0}^{s}A^j\xi_{s-k}^l\odot\omega_k^j}\right).\end{aligned}$$

In the particular case $p=2$, we have that

$$\begin{aligned}RL=&\sum_{s=0}^{2m}D^s_{\sum_{k=0}^{s}\xi_{s-k}^0\odot\omega_k^0}+\sum_{s=0}^{2m}D^s_{\sum_{k=0}^{s}A\xi_{s-k}^1\odot\omega_k^1}\\ &+A^*\left(\sum_{s=0}^{2m}D^s_{\sum_{k=0}^{s}\xi_{s-k}^1\odot\omega_k^0}+\sum_{s=0}^{2m}D^s_{\sum_{k=0}^{s}A\xi_{s-k}^0\odot\omega_k^1}\right)\\ =&\sum_{s=0}^{2m}D^s_{\sum_{k=0}^{s}(\xi_{s-k}^0\odot\omega_k^0+A\xi_{s-k}^1\odot\omega_k^1)}+A^*\left(\sum_{s=0}^{2m}D^s_{\sum_{k=0}^{s}(\xi_{s-k}^1\odot\omega_k^0+A\xi_{s-k}^0\odot\omega_k^1)}\right).\end{aligned}$$

So it is enough to check that, for $s=0,\dots,2m$,

$$\sum_{k=0}^{s}\left(\xi_{s-k}^1\odot\omega_k^0+A\xi_{s-k}^0\odot\omega_k^1\right)=0.$$

Substituting the ξ_k^j by their given values,

$$\begin{aligned}&\sum_{k=0}^{s}\left(\xi_{s-k}^1\odot\omega_k^0+A\xi_{s-k}^0\odot\omega_k^1\right)=\sum_{k=0}^{s}\left(\omega_{s-k}^1\odot\omega_k^0-A^2\omega_{s-k}^0\odot\omega_k^1\right)\\&=\sum_{k=0}^{s}\left(\omega_{s-k}^1\odot\omega_k^0-\omega_{s-k}^0\odot\omega_k^1\right)=\sum_{k=0}^{s}\omega_{s-k}^1\odot\omega_k^0-\sum_{k=0}^{s}\omega_{s-k}^0\odot\omega_k^1\\&=\sum_{k=0}^{s}\omega_{s-k}^1\odot\omega_k^0-\sum_{k=0}^{s}\omega_k^0\odot\omega_{s-k}^1=0.\end{aligned}$$

Let us see that L and R commute.

$$LR=\sum_{s=0}^{2m}D^s_{\sum_{k=0}^{s}\left(\omega_{s-k}^0\odot\xi_k^0+A\omega_{s-k}^1\odot\xi_k^1\right)}+A^*\left(\sum_{s=0}^{2m}D^s_{\sum_{k=0}^{s}\left(\omega_{s-k}^1\odot\xi_k^0+A\omega_{s-k}^0\odot\xi_k^1\right)}\right).$$

Now,

$$\begin{aligned}&\sum_{k=0}^{s}\left(\omega_{s-k}^0\odot\xi_k^0+A\omega_{s-k}^1\odot\xi_k^1\right)=\sum_{k=0}^{s}\omega_k^0\odot\xi_{s-k}^0+\sum_{k=0}^{s}A\omega_{s-k}^1\odot\omega_k^1\\&=\sum_{k=0}^{s}\xi_{s-k}^0\odot\omega_k^0+\sum_{k=0}^{s}A\xi_{s-k}^1\odot\omega_k^1.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\sum_{k=0}^{s}\left(\omega_{s-k}^1\odot\xi_k^0+A\omega_{s-k}^0\odot\xi_k^1\right)=\sum_{k=0}^{s}\left(\omega_{s-k}^1\odot(-A\omega_k^0)+A\omega_{s-k}^0\odot\omega_k^1\right)\\&=\sum_{k=0}^{s}\left(-\omega_{s-k}^1\odot A\omega_k^0+A\omega_{s-k}^0\odot\omega_k^1\right)=\sum_{k=0}^{s}\left(-\omega_k^1\odot A\omega_{s-k}^0+A\omega_{s-k}^0\odot\omega_k^1\right)=0.\end{aligned}$$

Hence, the result is proven. □

Similar reductions can be found for higher order involutions, although the coefficients may have a much more complex expression.

Example 3.2. Let A be and order 3 linear involution in $\mathbb{C}^n$, $v\in\mathbb{C}^n\backslash\{0\}$ and consider the operator $L=D_v+A^*$. Define now

$$R:=D_{v\odot A^2v}-A^*D_{A^2v}+(A^*)^2.$$

Observe that second derivatives occur in R but not in L. We have that

$$\begin{aligned}RL=&D_{v\odot A^2v}D_v-A^*D_{A^2v}D_v+(A^*)^2D_v+D_{v\odot A^2v}A^*-A^*D_{A^2v}A^*+(A^*)^2A^*\\=&D_{v\odot v\odot A^2v}-A^*D_{v\odot A^2v}+(A^*)^2D_v+A^*D_{v\odot A^2v}-(A^*)^2D_v+\operatorname{Id}\\=&D_{v\odot v\odot A^2v}+\operatorname{Id}.\end{aligned}$$

Unfortunately, we do not have commutativity in general:

$$\begin{aligned}LR=&D_vD_{v\odot A^2v}-D_vA^*D_{A^2v}+D_v(A^*)^2+A^*D_{v\odot A^2v}-(A^*)^2D_{A^2v}+\operatorname{Id}\\=&D_{v\odot v\odot A^2v}-A^*D_{A^*v\odot A^2v}+(A^*)^2D_{A^2v}+A^*D_{v\odot A^2v}-(A^*)^2D_{A^2v}+\operatorname{Id}\\=&D_{v\odot v\odot A^2v}+A^*D_{(v-A^*v)\odot A^2v}+\operatorname{Id}.\end{aligned}$$

In the particular case v is a fixed point of A, $RL = LR$.

The obtaining of a general expression for associated operators in the case of order 3 involutions and the conditions under which such operators commute is an interesting open problem.

4. Green's functions

Consider now the following problem

$$Lu = h;\ B_\lambda u = 0,\ \lambda \in \Lambda, \tag{4.1}$$

where $L \in (S^*V)^p$, $h \in \mathrm{L}^1(\mathbb{F}^n, \mathbb{F})$, the $B_\lambda : \mathcal{C}(\mathbb{F}^n, \mathbb{F}) \to \mathbb{F}$ are linear functionals, $\lambda \in \Lambda$ and Λ is an arbitrary set.

Let $R \in (S^*V)^p$, $f \in \mathrm{L}^1(\mathbb{F}^n, \mathbb{F})$ and consider the problem

$$RLv = f;\ B_\lambda v = 0,\ B_\lambda Rv = 0,\ \lambda \in \Lambda. \tag{4.2}$$

Given a function $G : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}$, we define the operator H_G such that $H_G(h)|_x := \int_{\mathbb{F}^n} G(x, s)h(s)\,\mathrm{d}\,s$ for every $h \in \mathrm{L}^1(\mathbb{F}^n, \mathbb{F})$, assuming such an integral is well defined. Also, given an operator R for functions of one variable, define the operator $R_\vdash$ as $R_\vdash G(t, s) := R(G(\cdot, s))|_t$ for every s, that is, the operator acts on G as a function of its first variable.

We have now the following theorem relating problems (4.1) and (4.2). The proof for the case of ordinary differential equations can be found in [4]. The case of PDEs is analogous.

Theorem 4.1. *Let L, $R \in (S^*V)^p$, $h \in \mathrm{L}^1(\mathbb{F}^n, \mathbb{F})$. Assume L commutes with R and that there exists G such that H_G is well defined satisfying*

(I) $(RL)_\vdash G = 0$,

(II) $B_{\lambda\vdash} G = 0,\ \lambda \in \Lambda$,

(III) $(B_\lambda R)_\vdash G = 0,\ \lambda \in \Lambda$,

(IV) $RLH_G h = H_{(RL)_\vdash G} h + h$,

(V) $LH_{R_\vdash G} h = H_{L_\vdash R_\vdash G} h + h$,

(VI) $B_\lambda H_G = H_{B_{\lambda\vdash} G},\ \lambda \in \Lambda$,

(VII) $B_\lambda R H_G = B_\lambda H_{R_\vdash G} = H_{(B_\lambda R)_\vdash G},\ \lambda \in \Lambda$.

Then, $v := H_G f$ is a solution of problem (4.2) *and $u := H_{R_\vdash G} h$ is a solution of problem* (4.1)*.*

4.1. A model of stationary heat transfer in a bent plate

We now consider a circular plate which is bent in half, with each of the two distinct halves separated by a very small distance which may be filled with some kind of (imperfect) heat insulating material (see Figure 4.1).

The heat equation which determines the temperature u on the plate for this situation is given by

$$\frac{\partial u}{\partial t}(t, x, y) = \alpha \left[\frac{\partial^2 u}{\partial x^2}(t, x, y) + \frac{\partial^2 u}{\partial y^2}(t, x, y)\right] + \beta[u(t, x, -y) - u(t, x, y)],$$

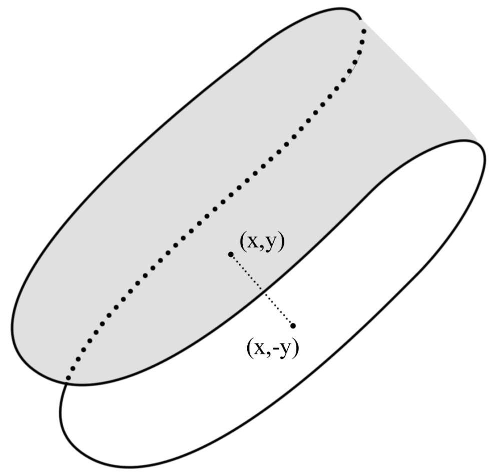

Figure 1. A section of the plate bent in half.

where

$$\frac{\partial u}{\partial t}(t,x,y) = \alpha \left[\frac{\partial^2 u}{\partial x^2}(t,x,y) + \frac{\partial^2 u}{\partial y^2}(t,x,y) \right],$$

is the usual heat equation with heat transfer coefficient $\alpha > 0$ and the term that goes with $\beta > 0$ relates to the heat transfer from the corollaryresponding point in the other half of the plate due to Newton's law of cooling.

If we consider the associated stationary problem

$$\alpha \left[\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) \right] + \beta [u(x,-y) - u(x,y)] = 0,$$

it can be rewritten in a convenient way as

$$Lu := \alpha \Delta u + \beta (A^* - \mathrm{Id}) u = 0,$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \text{ and } A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If we think of a circular plate in which the boundary is constantly cooled and the surface has a constant heat source given by a function h, we are imposing Dirichlet boundary conditions in the ball B of radius $\rho \in \mathbb{R}^+$ and considering the problem

$$Lu = h, \ u|_{\partial B} = 0. \tag{4.3}$$

Observe that, Δ, expressed in tensor notation, is $\Delta = D_{\omega_2^0}$ where

$$\omega_2^0 = \frac{1}{2} \left[(1,1) \odot (1,1) + (1,-1) \odot (1,-1) \right].$$

Besides, $A\omega_2^0 = \omega_2^0$ and, thus, $\Delta A^* = A^* \Delta$. Hence, using Theorem 3.1, we have to take $R = -\alpha\Delta + \beta A^* + \beta \,\mathrm{Id}$ and thus

$$\begin{aligned} RL = & -\alpha^2\Delta^2 - \alpha\beta A^*\Delta + \alpha\beta\Delta + \alpha\beta A^*\Delta + \beta^2 \,\mathrm{Id} - \beta^2 A^* + \alpha\beta\Delta + \beta^2 A^* - \beta^2 \,\mathrm{Id} \\ = & -\alpha^2\Delta^2 + 2\alpha\beta\Delta = (-\alpha^2\Delta + 2\alpha\beta \,\mathrm{Id})\Delta. \end{aligned}$$

Now, the boundary conditions transformed by R are

$$0 = Ru = -\alpha\Delta u + \beta A^* u + \beta u = -\alpha\Delta u,$$

that is, the reduced problem becomes

$$RLu = Rh =: f,\ u|_{\partial B} = 0,\ \Delta u|_{\partial B} = 0, \tag{4.4}$$

which is equivalent to the sequence of problems

$$\Delta u = v,\ u|_{\partial B} = 0, \tag{4.5}$$

$$(-\alpha^2\Delta + 2\alpha\beta\,\mathrm{Id})v = f,\ v|_{\partial B} = 0. \tag{4.6}$$

Problem (4.5) is the well-known Poisson equation with Dirichlet conditions on the circle of radius ρ. The Green's function can be written in polar coordinates as

$$G_1(r,\varphi,\tilde r,\tilde\varphi) = \frac{-1}{4\pi}\ln\left[\frac{r^2\tilde r^2 - 2\rho^2 r\tilde r\cos(\varphi-\tilde\varphi)+\rho^4}{\rho^2 r^2 - 2\rho^2 r\tilde r\cos(\varphi-\tilde\varphi)+\rho^2\tilde r^2}\right].$$

See [9, Section 7.2.3]. On the other hand, problem (4.6) is a Helmholtz equation, and the Green's function can be described in terms of the eigenfunctions of the associated homogeneous problem (see [9, Section 7.3.3]). More concretely, the associated Green's function in polar coordinates is written as

$$\begin{aligned} &G_2(r,\varphi,\tilde r,\tilde\varphi)\\ =&\frac{1}{\alpha^2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{1}{\left(\frac{\mu_{nm}^2}{\rho^2}+\frac{2\beta}{\alpha}\right)\|w_{nm}^{(1)}\|^2}\left[w_{nm}^{(1)}(r,\varphi)w_{nm}^{(1)}(\tilde r,\tilde\varphi)+w_{nm}^{(2)}(r,\varphi)w_{nm}^{(2)}(\tilde r,\tilde\varphi)\right],\end{aligned}$$

where μ_{nm} are the positive zeroes of the Bessel functions J_n, the eigenfunctions are given by

$$w_{nm}^{(1)} = J_n\left(\frac{\mu_{nm}}{\rho}r\right)\cos n\varphi,\quad w_{nm}^{(2)} = J_n\left(\frac{\mu_{nm}}{\rho}r\right)\sin n\varphi,$$

and

$$\|w_{nm}^{(1)}\|^2 = \frac{1}{2}\pi\rho^2(1+\delta_{n\,0})\left[J_n'(\mu_{nm})\right]^2,$$

where $\delta_{ij}=1$ if $i=j$ and 0 if $i\neq j$.

Now, the Green's function associated to problem (4.4) is given by

$$G_3(r,\varphi,\tilde r,\tilde\varphi) = \int_0^\rho\int_0^{2\pi} G_2(r,\varphi,\hat r,\hat\varphi)G_1(\hat r,\hat\varphi,\tilde r,\tilde\varphi)\,\mathrm{d}\hat\varphi\,\mathrm{d}\hat r.$$

In conclusion, the Green's function related to problem (4.3) is

$$G_4(\eta,\xi) = R_\vdash G_3(\eta,\xi) = \int_0^\rho\int_0^{2\pi} R_\vdash G_2(r,\varphi,\hat r,\hat\varphi)G_1(\hat r,\hat\varphi,\tilde r,\tilde\varphi)\,\mathrm{d}\hat\varphi\,\mathrm{d}\hat r,$$

where $R_\vdash$ has to be expressed in polar coordinates in order to act in the first two variables of G_3:

$$R = -\alpha\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right]+\beta A^*+\beta\,\mathrm{Id}.$$

Also, it is known that $J_n'(z)=(n/z)J_n(z)-J_{n+1}(z)$, so

$$\begin{aligned}&R_{\vdash}G_2(r,\varphi,\hat r,\hat\varphi)\\ =&\frac{1}{\alpha^2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{1}{\left(\frac{\mu_{nm}^2}{\rho^2}+\frac{2\beta}{\alpha}\right)\|w_{nm}^{(1)}\|^2}\left[\tilde w_{nm}^{(1)}(r,\varphi)w_{nm}^{(1)}(\tilde r,\tilde\varphi)+\tilde w_{nm}^{(2)}(r,\varphi)w_{nm}^{(2)}(\tilde r,\tilde\varphi)\right],\end{aligned}$$

where

$$\begin{aligned}\tilde w_{nm}^{(1)}=&\left(\left(\frac{\mu_{nm}}{\rho}\right)^2\left[\left(\frac{n\rho}{\mu_{nm}r}\right)^2J_n-\left(1+\frac{(n+1)\rho}{\mu_{nm}r}\right)J_{n+1}+J_{n+2}\right]\right.\\ &\left.\left.+\frac{n}{r}\left[\frac{\rho}{\mu_{nm}r}J_n-J_{n+1}\right]-n^2\left(\frac{\mu_{nm}}{\rho}r\right)^{-2}J_n\right)\right|_{\left(\frac{\mu_{nm}}{\rho}r\right)}\cos n\varphi,\\ \tilde w_{nm}^{(2)}=&\left(\left(\frac{\mu_{nm}}{\rho}\right)^2\left[\left(\frac{n\rho}{\mu_{nm}r}\right)^2J_n-\left(1+\frac{(n+1)\rho}{\mu_{nm}r}\right)J_{n+1}+J_{n+2}\right]\right.\\ &\left.\left.+\frac{n}{r}\left[\frac{\rho}{\mu_{nm}r}J_n-J_{n+1}\right]-n^2\left(\frac{\mu_{nm}}{\rho}r\right)^{-2}J_n\right)\right|_{\left(\frac{\mu_{nm}}{\rho}r\right)}\sin n\varphi.\end{aligned}$$

Example 4.1. Inspired by the previous problem, we now change the term due to Newton's law of cooling by a diffusion term in the following way.

$$\frac{\partial K}{\partial t}(t,x,y)=\alpha\left[\frac{\partial^2K}{\partial x^2}(t,x,y)+\frac{\partial^2K}{\partial y^2}(t,x,y)\right]+\beta\left[\frac{\partial^2K}{\partial x^2}(t,x,-y)+\frac{\partial^2K}{\partial y^2}(t,x,-y)\right],$$

where $\alpha,\beta>0$, $\beta\neq\alpha$.

If we consider the associated stationary problem

$$\alpha\left[\frac{\partial^2K}{\partial x^2}(x,y)+\frac{\partial^2K}{\partial y^2}(x,y)\right]+\beta\left[\frac{\partial^2K}{\partial x^2}(x,-y)+\frac{\partial^2K}{\partial y^2}(x,-y)\right]=0,$$

it can be rewritten as

$$LK:=\alpha\Delta K+\beta A^*\Delta K=0,$$

Using Theorem 3.1, we take $R=-\alpha\Delta+\beta A^*\Delta$ and then

$$RL=-\alpha^2\Delta^2-\alpha\beta\Delta A^*\Delta+\beta\alpha A^*\Delta^2+\beta^2(A^*\Delta)^2=\beta^2\Delta^2-\alpha^2\Delta^2=(\beta^2-\alpha^2)\Delta^2.$$

Now, if we consider the fundamental solution of the bi-Laplacian Δ^2 [6, equation (2.61)] we obtain a Green's function given by

$$G_1(\eta,\xi)=\frac{1}{8\pi}\|\eta-\xi\|^2\ln\|\eta-\xi\|,\ \eta,\xi\in\mathbb{R}^2.$$

Hence, in that case, the Green's function associated to L is given by

$$G_2(\eta,\xi)=R_{\vdash}G_1(\eta,\xi)=(\beta-\alpha)\frac{\ln\|\eta-\xi\|+1}{2\pi},\ \eta,\xi\in\mathbb{R}^2.$$

If we consider the problem

$$LK=h,\ u|_{\partial B}=0,$$

the reduced problem becomes

$$(\beta^2-\alpha^2)\Delta^2 K = h,\ u|_{\partial B}=0,\ Ru|_{\partial B}=0. \tag{4.7}$$

Now, the condition $Ru=-\alpha\Delta u+\beta A^*\Delta u=0$ is satisfied if we can guarantee that $\Delta u=0$, so we can consider the problem

$$(\beta^2-\alpha^2)\Delta^2 K = h,\ u|_{\partial B}=0,\ \Delta u|_{\partial B}=0. \tag{4.8}$$

For problem (4.8) we have that the Green's function is given by

$$G_3(\eta,\xi)=\frac{1}{8\pi}\|\eta-\xi\|^2\left(\ln\rho-1+\ln\|\eta-\xi\|\right)+\frac{\rho^2}{8\pi},\ \eta,\xi\in\mathbb{R}^2.$$

Hence, the Green's function related to problem (4.7) is

$$G_4(\eta,\xi)=\frac{\ln\rho+\ln\|\eta-\xi\|}{2\pi}.$$

In general, the functions

$$G_5(\eta,\xi)=\frac{1}{8\pi}\|\eta-\xi\|^2\left(\mu+\ln\|\eta-\xi\|\right)+\frac{\nu}{8\pi},\ \eta,\xi\in\mathbb{R}^2,$$

with $\mu,\nu\in\mathbb{R}$, are Green's functions related to the operator Δ^2 with different boundary conditions. The associated function for the operator L is given by

$$G_6(\eta,\xi)=\frac{1+\mu+\log\|\eta-\xi\|}{2\pi}.$$

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References

[1] A. R. Amir-Moéz and D. W. Palmer, *Characterization of linear involutions*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat., 1993, 4, 23–24.

[2] M. S. Burlutskaya and A. Khromov, *Initial-boundary value problems for first-order hyperbolic equations with involution*, Dokl. Math., 2011, 84(3), 783–786.

[3] A. Cabada and F. A. F. Tojo, *Comparison results for first order linear operators with reflection and periodic boundary value conditions*, Nonlinear Anal., 2013, 78, 32–46.

[4] A. Cabada and F. A. F. Tojo, *Green's functions for reducible functional differential equations*, Bull. Malays. Math. Sci. Soc., 2016.

[5] V. Fock, *Konfigurationsraum und zweite quantelung*, Zeitschrift für Physik, 1932, 75(9-10), 622–647.

[6] F. Gazzola, H.-C. Grunau and G. Sweers, *Polyharmonic boundary value problems: positivity preserving and nonlinear higher order elliptic equations in bounded domains*, 1991, Springer Science & Business Media, 2010.

[7] J. W. Helton, M. de Oliveira, M. Stankus and R. L. Miller, *Ncalgebra - version 4.0.6*, 2015.

[8] M. Kiranea and N. Al-Saltic, *Inverse problems for a nonlocal wave equation with an involution perturbation*, Journal of Nonlinear Sciences and Applications, 2016, 9(3), 1243–1251.

[9] A. Polyanin, *Handbook of linear partial differential equations for engineers and scientists*, CRC Press Company, 2002.

[10] L. Silberstein, *Solution of the equation* $f(x) = f(1/x)$, Lond. Edinb. Dubl. Phil. Mag., 1940, 30(200), 185–186.

[11] J. Wiener, *Generalized solutions of functional differential equations*, World Scientific, 1993.