Recent advances on mathematical models involving singular nonlinearities

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Model I:

A mass-spring model of electrostatically actuated micro-electro-mechanical system

The model

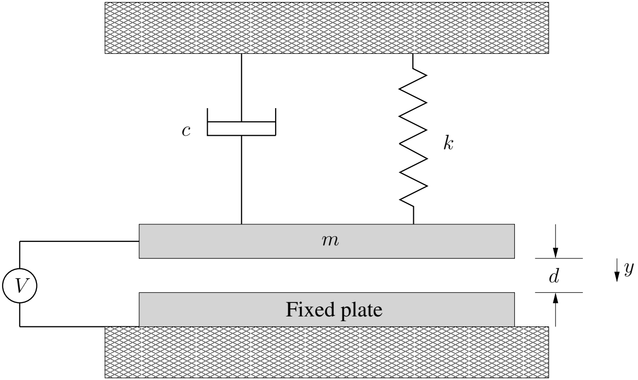

Figure: Mass-spring model of electrostatically actuated MEMS

The model

$$my'' + cy' + ky = \frac{\varepsilon_0 A}{2} \frac{V^2(t)}{(d-y)^2}, \tag{1}$$

with $V(t) = v_{dc} + v_{ac} cos(\omega t)$ (AC-DC voltage)

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- M.I. Younis: MEMS Linear and Nonlinear Statics and Dynamics. Springer, NewYork (2011)

Static pull-in

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For a DC voltage $V(t) = v_{dc} > 0$, equilibria are the roots of

$$y(d - y^2) = \frac{\varepsilon_0 A v_{dc}^2}{2k},$$

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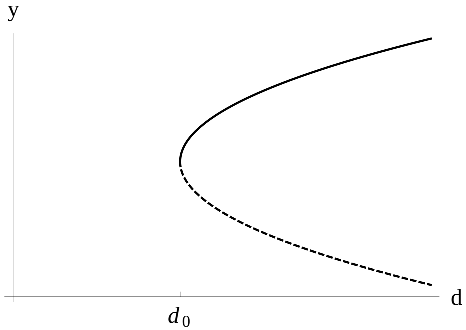

Figure: Saddle-node bifurcation at $d_0 = \frac{3}{2}\left(\frac{\varepsilon_0 A v_{dc}^2}{k}\right)^{1/3}$.

Dynamic pull-in

For an AC-DC voltage $V(t) = v_{dc} + v_{ac} cos(\omega t)$,

Dynamic pull-in$\equiv$ non-autonomous saddle-node bifurcation

Main result

Let $V(t)$ be a continuous, positive, T-periodic function with $T = \frac{2\pi}{\omega}$. By convenience, we call $V_m = \min_{[0,T]} V(t)$, $V_M = \max_{[0,T]} V(t)$.

Theorem

There exists $d_0 > 0$ such that

1. If $d < d_0$, (1) has no T-periodic solutions.
2. If $d = d_0$, (1) has at least one T-periodic solution.
3. If $d > d_0$, (1) has at least two T-periodic solutions.

Besides, d_0 admits the following quantitative estimate

$$\frac{3}{2}\left(\frac{\varepsilon_0 A V_m^2}{k}\right)^{1/3} \leq d_0 \leq \frac{3}{2}\left(\frac{\varepsilon_0 A V_M^2}{k}\right)^{1/3}. \qquad (2)$$

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- Gutiérrez, A., Torres, P.J.: Non-autonomous saddle-node bifurcation in a canonical electrostatic MEMS, International J. Bifurcation and Chaos **23**, No. 5, 1350088 (9 pages), (2013)

The model as a singular equation

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leads to

$$u'' + c\,u' + ku + \frac{a(t)}{u^2} = s, \qquad (3)$$

with c, $k > 0$, $s := kd/m$ and $a(t) := \frac{\varepsilon_0 A}{2m} V^2(t)$.

Sketch of the proof

A T-periodic solution of

$$u'' + c\,u' + ku + \frac{a(t)}{u^2} = s$$

is a fixed point of the functional

$$\Phi[u] := L^{-1}\left[s - (k+1)u + \frac{a(t)}{u^2}\right]$$

where $Lu := u'' + c\,u' - u$.
Φ is a compact operator on the Banach space of the T-periodic continuous functions and the Leray-Schauder degree $\deg_{LS}(I - \Phi, \Omega)$ is well-defined whenever Φ has no fixed points in the boundary of Ω.

Sketch of the proof

- Existence of the unstable branch: lower and upper solution method

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- Multiplicity (second branch): excision of the degree

Stability

Theorem

Assume that

$$4k < \frac{\varepsilon_0 A V_m^2}{2} \left(\frac{\omega c V_m^2}{\pi k d V_M^2} \right)^3 + \omega^2 + \frac{c^2}{m}. \tag{4}$$

Then, if $d > d_0$, there exist exactly two T-periodic solutions, one asymptotically stable and another unstable.

Stability

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- S. Ai, J. A. Pelesko, Dynamics of a canonical electrostatic MEMS/NEMS system, J. Dyn. Differ. Eqns., 20 (2007), 609–641. ("viscosity dominated regime")

Stability

Example

For the physical parameters: $m = 3.5 \times 10^{-11}$ Kg, $k = 0.17$ N/m, $c = 1.78 \times 10^{-6}$ Kg/s, $A = 1.6 \times 10^{-9}\ m^2$, $\varepsilon_0 = 8.85 \times 10^{-12}$ F/m. If $V(t) = 10 + 2\cos(\omega t)$ V, then the bifurcation value is bounded by $2.62033\,\mu m < d_0 < 3.4336\,\mu m$.If $d > d_0$ and $\omega \geq 0.76772 s^{-1}$ then there are exactly two periodic solutions, one asymptotically stable and the other unstable.

Open problem I

To identify the dynamic pull-in (non-autonomous saddle-node bifurcation) when $V(t)$ change its sign.

Model II:

Motion of fluid particles induced by a prescribed vortex path in a circular domain

The model

A fixed point vortex on the unbounded plane:

$$\dot{\zeta} = \frac{\Gamma}{2\pi i}\left(\frac{1}{\zeta - z}\right)$$

where the complex variable ζ represents the evolution on the position of a particle transport induced by the flux generated by a fixed vortex placed at z.

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where the complex variable ζ represents the evolution on the position of a particle transport induced by the flux generated by a fixed vortex placed at z.

This is a planar system with hamiltonian structure, where the stream function

$$\Psi(\zeta) = \frac{\Gamma}{2\pi}\ln|\zeta - z|$$

plays the role of the hamiltonian.

The model

Influence of a circular domain of radius R:

$$\dot{\bar{\zeta}} = \frac{\Gamma}{2\pi i}\left(\frac{1}{\zeta - z} - \frac{1}{\zeta - \frac{R^2}{|z|^2}z}\right).$$

The first term models the action of the vortex whereas the second term corresponds to the wall influence on the flow.

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Now the hamiltonian is

$$\Psi(\zeta) = \frac{\Gamma}{2\pi} \ln \left| \frac{\zeta - z}{\overline{z}\zeta - R^2} \right|$$

The model

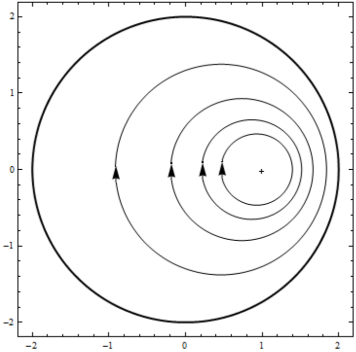

Figure: Stream lines of a fixed vortex located at $(1, 0)$ in the circular domain of radius $R = 2$.

The model

If the vertex is moving following a prescribed path z(t):

$$\dot{\zeta} = \frac{\Gamma}{2\pi i}\left(\frac{1}{\zeta - z(t)} - \frac{1}{\zeta - \frac{R^2}{|z(t)|^2} z(t)}\right). \tag{5}$$

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$$\Psi(t,\zeta) = \frac{\Gamma}{2\pi}\ln\left|\frac{\zeta - z(t)}{\overline{z(t)}\zeta - R^2}\right|$$

is no more a conserved quantity.

The main result

Theorem 1

Let $z : \mathbb{R} \to \mathbb{C}$ be a T-periodic function of class C^1, such that $|z(t)| < R$ for all t. Then, for every integer $k \geq 1$, system (5) has infinitely many kT-periodic solutions lying in the disk $\mathcal{B}_R(0)$. More precisely, for every integer $k \geq 1$, there exists an integer j_k^* such that, for every integer $j \geq j_k^*$, system (5) has two kT-periodic solutions $\zeta_{k,j}^{(1)}(t)$, $\zeta_{k,j}^{(2)}(t)$ such that, for $i = 1, 2$,

$$\|\zeta_{k,j}^{(i)}\|_\infty \leq R \quad \text{and} \quad \mathrm{rot}_{kT}(\zeta_{k,j}^{(i)}) = j. \tag{6}$$

Moreover, for every $k \geq 1$, $j \geq j_k^*$ and $i = 1, 2$,

$$\lim_{j \to +\infty} |\zeta_{k,j}^{(i)}(t) - z(t)| = 0, \quad \text{uniformly in } t \in [0, kT]. \tag{7}$$

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- A. Boscaggin, P.J. Torres, Periodic motions of fluid particles induced by a prescribed vortex path in a circular domain, Physica D 261 (2013) 81-84

Open problem II

Existence of periodic solutions with rotation number equal to zero.

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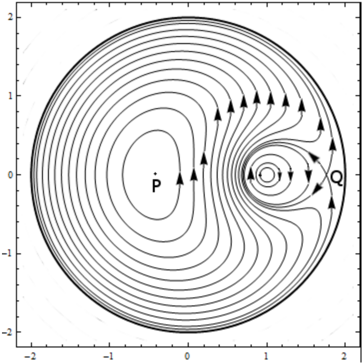

Figure: Stream lines induced by a vortex path $z(t) = \exp(it)$ in the circular domain of radius $R = 2$.

Model III:

Water transport across a cell membrane with fluctuating environmental conditions

The model

$$\begin{aligned} \dot{w}_1 &= \frac{x_{np}}{w_1} + \sum_{j=2}^{n} \frac{w_j}{w_1} - \sum_{i=1}^{n} M_i(t), \\ \dot{w}_k &= b_k \left(M_k(t) - \frac{w_k}{w_1} \right), \qquad k = 2, \ldots, n. \end{aligned} \tag{8}$$

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$w_1(t) \equiv$ intracellular water volume
$w_k(t),\ k = 2, \ldots, n \equiv$ amount of permeating intracellular solute species
$x_{np} \geq 0 \equiv$ amount of non-permeating intracellular solute species (salts)
$M_1 : \mathbb{R} \to [0, +\infty) \equiv$ extracellular concentration of non–permeating solute
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- Benson, J.D., Chicone, C.C., Critser, J.K.: A general model for the dynamics of cell volume, global stability and optimal control, J. Mathematical Biology, **63** (2), 339–359 (2011)

Main results

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(H_1) *there exists* $b > 0$ *such that* $b = b_k$ *for every* $k = 2, \ldots, n$.

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- P.J. Torres, Periodic oscillations of a model for membrane permeability with fluctuating environmental conditions, to appear in Journal of Mathematical Biology

Open problem III

$x_{np} > 0$: conditions for uniqueness and asymptotic stability

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$x_{np} > 0$: conditions for uniqueness and asymptotic stability

$x_{np} = 0$: existence without condition (H1)

THANK YOU FOR YOUR ATTENTION!!