Guided waves in a nonlinear optical medium: a topological approach

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Physical background

Maxwell's equations:

$$\nabla \wedge E = -\frac{1}{c}\frac{\partial B}{\partial t} \qquad \nabla \wedge H = \frac{1}{c}\frac{\partial D}{\partial t}$$

$$\nabla \cdot D = 0 = \nabla \cdot B$$

$c \equiv$ speed of light in the vacuum
$E \equiv$ electric field
$H \equiv$ magnetic field
$D \equiv$ electric flux density
$B \equiv$ magnetic flux density

Physical background

Maxwell's equations:

$$\nabla \wedge E = -\frac{1}{c}\frac{\partial B}{\partial t} \qquad \nabla \wedge H = \frac{1}{c}\frac{\partial D}{\partial t}$$

$$\nabla \cdot D = 0 = \nabla \cdot B$$

It is assumed:

$$H \equiv B \text{ (non-magnetic medium)}$$

$$E(x, y, z, t) = u(x) e_2 \cos(kz - \omega t)$$

$$D(x, y, z, t) = \varepsilon(x, \tfrac{1}{2}u(x)^2) E(x, y, z, t)$$

where $\varepsilon(x, s)$ is called *dielectric function*.

Physical background

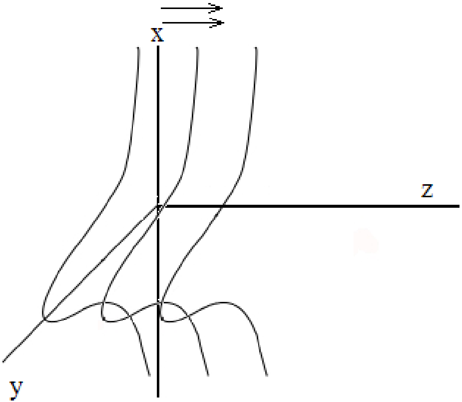


Figura: Wave propagation

Physical background

$$-\ddot{u}(x) + k^2 u(x) = \frac{\omega^2}{c^2}\varepsilon(x, \tfrac{1}{2}u(x)^2)u(x)$$

Guidance conditions:

$$\lim_{|x|\to+\infty} u(x) = \lim_{|x|\to+\infty} \dot{u}(x) = 0$$

$$\int_{\mathbb{R}} u^2(x)dx + \int_{\mathbb{R}} \dot{u}^2(x)dx < +\infty$$

Physical background

$$-\ddot{u}(x) + k^2 u(x) = \frac{\omega^2}{c^2}\varepsilon(x, \tfrac{1}{2}u(x)^2)u(x)$$

Guidance conditions:

$$u \in H^1(\mathbb{R})$$

Main problem

To study the existence of solutions in $H^1(\mathbb{R})$ of the equation

$$-\ddot{u} + a(x)u = b(x)f(u) \tag{1}$$

with $a, b \in L^\infty(\mathbb{R})$.

Related work

N.N. Akhmediev, *Sov. Phys. JEPT*, **56** (1982), 299–303.

$$-\ddot{u} + k^2 u = \chi_A(x)u(x) + (1 - \chi_A(x))\, u(x)^3$$

where $A = [-d, d]$ is a closed interval and χ_A is the characteristic function. It corresponds to the propagation of a guided wave through an optical medium with dielectric function:

$$\varepsilon(x, s) = \begin{cases} \frac{\omega^2}{c^2} & \text{if } |x| < d, \\ \frac{\omega^2}{c^2}(1 + s) & \text{if } |x| > d, \end{cases}$$

Related work

In general, $\varepsilon(x,s) = A(x) + B(x)s^n$ with A, B piecewise-constant functions (Kerr nonlinearities).

- H.J. Ruppen [1997], *Ann. Mat. Pura. Appl.*
- A. Ambrosetti, D. Arcoya, J.L. Gámez [1998], *Rend. Sem. Mat. Univ. Padova*
- D. Arcoya, S. Cingolani, J.L. Gámez [1999], *SIAM J. Math. Anal.*
- K. Kurata, M. Shibata, T. Watanabe [2005], *Proc. Roy. Soc. Edin.*
- T. Watanabe [2005], *Nonlinear Anal.*

How to attack the problem

- Fixed point problem
- Krasnoselskii fixed point theorem for compact operators in cones os a Banach space
- Compactness criterion

Fixed point problem. Green's function

Let $a \in L^\infty(\mathbb{R})$ be such that $a_* > 0$ and $b \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$. Then, $u \in H^1(\mathbb{R})$ is solution of

$$-\ddot{u} + a(x)u = b(x)f(u)$$

iff it is a fixed point of the operator

$$T : BC(\mathbb{R}) \to H^1(\mathbb{R})$$

$$Tu(x) := \int_{\mathbb{R}} G(x,s)b(s)f(u(s))ds,$$

where $G(x,s)$ is the Green's function of the homogeneous problem

$$\begin{cases} -\ddot{u} + a(x)u = 0, \\ u(-\infty) = 0, u(+\infty) = 0. \end{cases}$$

Definition y properties of $G(x,s)$.

$$G(x,s) = \begin{cases} u_1(x)u_2(s), & \alpha < x \leq s < +\infty \\ u_1(s)u_2(x), & \alpha < s \leq x < +\infty \end{cases}$$

where u_1, u_2 are solutions of the homogeneous problem such that $u_1(-\infty) = 0, u_2(+\infty) = 0$. Moreover, u_1, u_2 are positive fucntions, u_1 increasing and u_2 decreasing.

Definition y properties of $G(x,s)$.

u_1, u_2 intersect in a unique point x_0. Let us define

$$p(x) = \begin{cases} \dfrac{1}{u_2(x)}, & x \le x_0, \\ \dfrac{1}{u_1(x)}, & x > x_0. \end{cases}$$

Properties

(P1) $G(x,s) > 0$ for all $(x,s) \in \mathbb{R}^2$.

(P2) $G(x,s) \le G(s,s)$ for all $(x,s) \in \mathbb{R}^2$.

(P3) Given a compact $P \subset \mathbb{R}$, we define

$$m_1(P) = \text{mín}\{u_1(\inf P), u_2(\sup P)\}.$$

Then,

$$G(x,s) \ge m_1(P)p(s)G(s,s) \text{ for all } (x,s) \in P \times \mathbb{R}.$$

(P4) $G(s,s)p(s) \ge G(x,s)p(x)$ for all $(x,s) \in \mathbb{R}^2$.

Fixed point theorem.

Let $\mathcal{B}$ be a Banach space.

Definition

A set $\mathcal{P} \subset \mathcal{B}$ is a cone if it is closed, nonempty, $\mathcal{P} \neq \{0\}$ and given $x, y \in \mathcal{P}$, $\lambda, \mu \in \mathbb{R}_+$ then $\lambda x + \mu y \in \mathcal{P}$.

Theorem

Let $\mathcal{P}$ be a cone in the Banach space $\mathcal{B}$. Assume Ω^1, Ω^2 are open bounded subsets of $\mathcal{P}$ (in the relative topology of $\mathcal{P}$) with $0 \in \Omega^1$ and $\overline{\Omega^1} \subset \Omega^2$. If $T : \overline{\Omega^2} \to \mathcal{P}$ is a completely continuous map satisfying:

(H1) $Tu \neq \lambda u$ for all $u \in \partial_{\mathcal{P}}\Omega^1$ and $\lambda > 1$.

(H2) There exists $e \in \mathcal{P} \setminus \{0\}$ such that $u \neq Tu + \lambda e$ for all $u \in \partial_{\mathcal{P}}\Omega^2$ and all $\lambda > 0$,

Then, T has a fixed point in $\overline{\Omega^2} \setminus \Omega^1$.

Compactness criterion.

Proposition

Sea $\Omega \subset BC(\mathbb{R})$ whose functions are equicontinuous in each compact interval of $\mathbb{R}$. Let us assume that there exists $q \in BC(\mathbb{R})$ such that $\lim_{|x|\to+\infty} q(x) = 0$ and

$$|u(x)| \leq q(x) \qquad \forall x \in \mathbb{R}, u \in \Omega.$$

Then, Ω is relatively compact.

Main result.

Theorem 1

Let $[\beta,\gamma]$ be such that $x_0 \in (\beta,\gamma)$, $[\beta,\gamma] \cap Supp(b) \neq \emptyset$. Let $a \in L^\infty(\mathbb{R})$, $b \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ be such that $a_* > 0$, $b_* \geq 0$. Besides, $f(s) \geq 0$ para todo $s \geq 0$. In addition suppose that

(i) There exists $r > 0$ such that for every $u \in [0,r]$

$$f(u) \sup_{x\in\mathbb{R}} \int_{-\infty}^{\infty} G(x,s)b(s)ds \leq r.$$

(ii) There exists $R > r > 0$ such that for every $u \in [R, \frac{1}{m_1 p_0} R]$

$$f(u) \min_{x\in[\beta,\gamma]} \int_{\beta}^{\gamma} G(x,s)b(s)ds \geq R,$$

where $p_0 = \min_{x\in[\beta,\gamma]} p(x)$.

Then, there exists a positive solution $u \in H^1(\mathbb{R})$ of (1) such that $r \leq \|u\| \leq \frac{1}{m_1 p_0} R$.

Proof.

• Step 1: Definition of the cone.

$$\mathcal{P} = \left\{ u \in BC(\mathbb{R}) \,:\, u(x) \geq 0,\ \min_{y \in [\beta,\gamma]} u(y) \geq m_1 p(x) u(x) \text{ for all } x \right\}$$

• Step 2: $T(\mathcal{P}) \subset \mathcal{P}$.

Proof. For all $\tau \in \mathbb{R}$,

$$\begin{aligned} \min_{x \in [\beta,\gamma]} Tu(x) &= \min_{x \in [\beta,\gamma]} \int_{-\infty}^{+\infty} G(x,s) b(s) f(u(s)) ds \\ &\geq m_1 \int_{-\infty}^{+\infty} p(s) G(s,s) b(s) f(u(s)) ds \\ &\geq m_1 \int_{-\infty}^{+\infty} p(\tau) G(\tau,s) b(s) f(u(s)) ds \\ &= m_1 p(\tau) Tu(\tau), \end{aligned}$$

where $(P3)$ and $(P4)$ are used.

Proof.

• Step 3: T is completely continuous.

Proof. Given a bounded $\Omega \subset \mathcal{P}$, let us prove that $T(\Omega)$ is relatively compact. There is $M > 0$ such that $\|u\| \leq M$ for all $u \in \Omega$.

$$|Tu(x)| = \left| \int_{\mathbb{R}} G(x,s)b(s)f(u(s))ds \right| \leq M^* \int_{\mathbb{R}} G(x,s)b(s)ds =: q(x)$$

where $M^* = \text{máx}_{s\in[0,M]} f(s)$. Since $b \in L^1$, then $q \in H^1$. The compactness criterion concludes the proof.

Proof.

- Step 4: Krasnoselskii conditions.

(H1) Let be $\Omega_1 = \{u \in \mathcal{P} : \|u\| \leq r\}$.
Given $u \in \partial_{\mathcal{P}}\Omega_1$,

$$Tu(x) \leq M_r \sup_{x\in\mathbb{R}} \int_{-\infty}^{+\infty} G(x,s)b(s)ds \leq r,$$

where $M_r = \text{máx}_{s\in[0,r]} f(s)$. Therefore, $Tu \neq \lambda u$ para $\lambda > 1$, $u \in \partial_{\mathcal{P}}\Omega_1$.

Proof.

- Step 4: Krasnoselskii conditions.

(H2) Let be $\Omega_2 = \{u \in \mathcal{P} : \text{mín}_{y\in[\beta,\gamma]} u(y) < R\}$

$$\overline{\Omega}_1 \subset \Omega_2$$

If $u \in \partial_{\mathcal{P}}\Omega_2$,

$$R = \min_{y\in[\beta,\gamma]} u(y) \geq m_1 p(x)\|u\| \geq m_1 p_0 \|u\|,$$

therefore

$$R \leq u(x) \leq \frac{R}{m_1 p_0} \; \forall \, x \in [\beta,\gamma].$$

Take $e \in \mathcal{P}$ such that $e(x) = 1$, $x \in [\beta,\gamma]$. If there exists $\lambda \in (0,1)$ such that $u = Tu + \lambda e$, given $x \in [\beta,\gamma]$,

$$\begin{aligned} u(x) = & \int_{-\infty}^{+\infty} G(x,s)b(s)f(u(s))ds + \lambda e(x) \\ \geq & \; m_R \min_{x\in[\beta,\gamma]} \int_{\beta}^{\gamma} G(x,s)b(s)ds + \lambda \geq R + \lambda > R \end{aligned}$$

where $m_R = \text{mín}_{s\in[R,\frac{R}{m_1 p_0}]} f(s)$. This contradicts $R = \text{mín}_{y\in[\beta,\gamma]} u(y)$.

Odd homoclinics under symmetry conditions.

$$-\ddot{u} + a(x)u = b(x)f(u)$$

Theorem 2

Under the assumptions of Theorem 1, if:

- $a(x) = a(-x)$, $b(x) = b(-x) \quad \forall x$,
- $f(-s) = -f(s) \quad \forall s$,

there exists an odd solution $u \in H^1(\mathbb{R})$, positive in $\mathbb{R}^+$ such that $r \leq \|u\| \leq \frac{1}{m_1 p_0} R$.

Consequences for models in Nonlinear Optics.

$$-\ddot{u}(x) + k^2 u(x) = \frac{\omega^2}{c^2}\varepsilon(x, \tfrac{1}{2}u(x)^2)u(x)$$

Nonlinear contribution of the dielectric function is isolated with the decomposition

$$\varepsilon_L(x) = \varepsilon(x, 0), \qquad \varepsilon_{NL}(x, s) = \varepsilon(x, s) - \varepsilon_L(x).$$

For simplicity, it is assumed $\varepsilon_{NL}(x, s) = B(x)F(s)$.

Consequences for models in Nonlinear Optics.

$$-\ddot{u}(x) + (k^2 - \frac{\omega^2}{c^2}\varepsilon_L(x))u(x) = \frac{\omega^2}{c^2}B(x)F(\tfrac{1}{2}u(x)^2)u(x) \qquad (2)$$

Definition

A dielectric function $\varepsilon(x,s) = \varepsilon_L(x) + B(x)F(s)$ is called of Kerr type if F is increasing, $B_* \geq 0$, $Supp(B) \neq \emptyset$ and

$$F(0) = 0, \qquad \lim_{s \to +\infty} F(s) = +\infty.$$

Consequences for models in Nonlinear Optics.

Corollary

Let us assume a dielectric function of Kerr type with $B \in L^1 \cap L^\infty$. If $k^2 > \frac{\omega^2}{c^2} \|\varepsilon_L\|$, then there exists a positive solution $u \in H^1(\mathbb{R})$ of (2). Moreover, if $\varepsilon_L(x), B(x)$ are even functions, there exists an odd solution $\tilde{u} \in H^1(\mathbb{R})$ of (2).

Consequences for models in Nonlinear Optics.

Proof.

$$a(x) = k^2 - \frac{\omega^2}{c^2}\varepsilon_L(x),\ b(x) = \frac{\omega^2}{c^2}B(x),\ f(s) = F(\tfrac{1}{2}s^2)s$$

[(*i*)] *There exists* $r > 0$ *such that*

$$f(u)\sup_{x\in\mathbb{R}}\int_{-\infty}^{\infty} G(x,s)b(s)ds \leq r \qquad \forall u \in [0,r]$$

This amounts to:

$$F(\tfrac{1}{2}r^2)\sup_{x\in\mathbb{R}}\int_{-\infty}^{\infty} G(x,s)b(s)ds \leq 1$$

[(*ii*)] *There exists* $R > r > 0$ *such that*

$$f(u)\min_{x\in[\beta,\gamma]}\int_{\beta}^{\gamma} G(x,s)b(s)ds \geq R \qquad \forall u \in [R, \frac{1}{m_1 p_0}R]$$

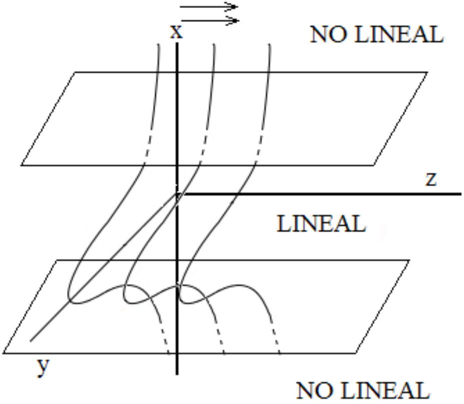
This amounts to:

$$F(\tfrac{1}{2}R^2)\min_{x\in[\beta,\gamma]}\int_{\beta}^{\gamma} G(x,s)b(s)ds \geq 1$$

Consequences for models in Nonlinear Optics.

An example (Akhmediev):

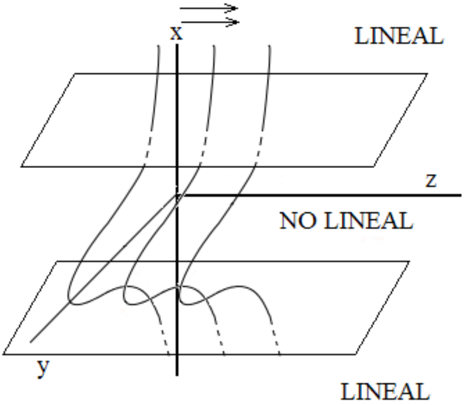
$$\varepsilon(x,s) = \begin{cases} q^2 + s & \text{si } |x| \geq d, \\ q^2 + p^2 & \text{si } |x| < d, \end{cases}$$



Consequences for models in Nonlinear Optics.

An example (Akhmediev):

$$\varepsilon(x,s) = \begin{cases} q^2 + s & \text{si } |x| \leq d, \\ q^2 + p^2 & \text{si } |x| > d, \end{cases}$$



Consequences for models in Nonlinear Optics.

$$\varepsilon(x,s) = \begin{cases} q^2 + b(x)s & \text{if } |x| \geq d, \\ q^2 + p^2 & \text{if } |x| < d, \end{cases}$$

with $b \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ positive.

Consequences for models in Nonlinear Optics.

$$\varepsilon(x,s) = \begin{cases} q^2 + b(x)s & \text{if } |x| \geq d, \\ q^2 + p^2 & \text{if } |x| < d, \end{cases}$$

with $b \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ positive.

The “limit case” $\varepsilon(x,s) = b(x)s$ is interesting in the context of Bose-Einstein condensates.

Optical media with saturation

Definition

Una dielectric function $\varepsilon(x,s) = \varepsilon_L(x) + B(x)F(s)$ is called saturable if F is increasing, $B_* \geq 0$, $Supp(B) \neq \emptyset$ and

$$F(0) = 0, \qquad \lim_{s \to +\infty} F(s) = F_\infty < +\infty.$$

$$\varepsilon(x,s) = \varepsilon_L(x) + B(x)\frac{s}{1+s},$$

$$\varepsilon(x,s) = \varepsilon_L(x) + B(x)(1 - e^{-s}).$$

[C.A. Stuart, Guidance properties of nonlinear planar waveguides, *Arch. Rational Mech. Anal.*, **125** (1993), 145–200.]

Optical media with saturation

Corollary

Let us assume a saturable dielectric function with $B \in L^1 \cap L^\infty$. If:

- $k^2 > \frac{\omega^2}{c^2}\|\varepsilon_L\|$,
- $\exists[\beta,\gamma]$ such that $x_0 \in (\beta,\gamma)$, $[\beta,\gamma] \cap Supp(B) \neq \emptyset$ and

$$\omega^2 F_\infty \min_{x\in[\beta,\gamma]} \int_\beta^\gamma G(x,s)B(s)ds > c^2,$$

then there exists a positive solution $u \in H^1(\mathbb{R})$ of (2). Moreover, if $\varepsilon_L(x), B(x)$ are even functions, there exists an odd solution $\tilde{u} \in H^1(\mathbb{R})$ of (2).

Optical media with saturation

As a consequence, it is proved the existence of guided waves

$$E(x, y, z, t) = u(x)e_2 \cos(kz - \omega t)$$

in saturable media for high values of ω, k.

Consequences for models in Nonlinear Optics.

Branches of solutions in systems controlled by parameters:

$$\varepsilon(x,s) = \varepsilon_L(x) + \lambda B(x) F(s),$$

Corollary

Let us assume a dielectric function of Kerr type. If $k^2 > \frac{\omega^2}{c^2} \|\varepsilon_L\|$, then for all $\lambda > 0$ there exists a positive solution $u_\lambda \in H^1(\mathbb{R})$ of (2). Besides,

$$\text{lím}_{\lambda \to 0^+} \|u_\lambda\| = +\infty, \qquad \text{lím}_{\lambda \to +\infty} \|u_\lambda\| = 0.$$

If $\varepsilon_L(x), B(x)$ are even, then there exists a second branch of odd solutions $\tilde{u}_\lambda \in H^1(\mathbb{R})$ with the same behaviour.

Consequences for models in Nonlinear Optics.

Proof. We need r_λ, R_λ such that

$$F(\tfrac{1}{2}r_\lambda^2) \leq \left(\tfrac{1}{c^2}\lambda\omega^2 \sup_{x\in\mathbb{R}} \int_{-\infty}^{\infty} G(x,s)B(s)ds\right)^{-1} \leq$$
$$\leq \left(\tfrac{1}{c^2}\lambda\omega^2 m_1 p_0 \operatorname{mín}_{x\in[\beta,\gamma]} \int_\beta^\gamma G(x,s)B(s)ds\right)^{-1} \leq F(\tfrac{1}{2}R_\lambda^2).$$

By using that the nonlinearity is of Kerr type, $r_\lambda < R_\lambda$ can be chosen such that

$$\operatorname{lím}_{\lambda\to 0^+} r_\lambda = +\infty, \qquad \operatorname{lím}_{\lambda\to+\infty} R_\lambda = 0.$$

We obtain a branch u_λ such that $r_\lambda \leq \|u_\lambda\| \leq \frac{R_\lambda}{m_1 p_0}$.

Consequences for models in Nonlinear Optics.

Corollary

Let us assume a dielectric function is saturable. If $k^2 > \frac{\omega^2}{c^2}\|\varepsilon_L\|$, then there exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ there exists a positive solution $u_\lambda \in H^1(\mathbb{R})$ of (2). Besides,

$$\lim_{\lambda\to+\infty} \|u_\lambda\| = 0.$$

If $\varepsilon_L(x), B(x)$ are even, then there exists a second branch of odd solutions $\tilde{u}_\lambda \in H^1(\mathbb{R})$ with the same behaviour.

Open directions of research.

- Extension to systems:

$$\begin{cases} -\ddot{u} + a_1(t)u & = & b_1(t)(u+v)u^2 \\ -\ddot{v} + a_2(t)v & = & b_2(t)(u+v)v^2 \end{cases}$$

"Manakov systems"

Open directions of research.

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  “Manakov systems”
- higher dimensions:

$$-\Delta u + a(t)u = b(t)u^3$$

Open directions of research.

- Extension to systems:

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  "Manakov systems"
- higher dimensions:

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- Dark solitons (heteroclinics).