Existence of periodic and solitary waves for a Nonlinear Schrödinger Equation with nonlocal integral term of convolution type

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The model

$$iu_t + u_{xx} + V(x)u + u(x)\int_{-\infty}^{\infty} K(x,s)|u(s)|^2 ds = 0 \qquad (1)$$

where the kernel $K(x,s)$ is assumed to be of the form

$$K(x,s) = \gamma(x)W(x-s),$$

being W a function (or distribution) with non-negative values such that

$$\|W\|_1 = \int_{-\infty}^{\infty} W(s)ds < +\infty. \qquad (2)$$

The linear term $V(x)u$ is relevant in Bose-Einstein condensates as a model of a possible external magnetic trap.

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Possible choices for $K(x,s)$:

- Local interactions: $\gamma(x)\delta(x-s)$
- Step-like function: $\gamma(x)\theta(a-|x-s|)$
- Gaussian function: $\gamma(x)\exp\left(-(x-s)^2\right)$,
- super-Gaussian :$\gamma(x)\exp\left(-(x-s)^4\right)$

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Possible choices for $K(x,s)$:

- Local interactions:
  $K(x,s) = \gamma(x)\delta(x-s) \implies \gamma(x)|u(x)|^2 u(x)$
- Step-like function:
  $K(x,s) = \gamma(x)\theta(a-|x-s|) \implies u(x)\int_{x-a}^{x+a} K(x,s)|u(s)|^2 ds$
- Gaussian function: $K(x,s) = \gamma(x)\exp\left(-(x-s)^2\right)$
- super-Gaussian :$K(x,s) = \gamma(x)\exp\left(-(x-s)^4\right)$

Separation of variables

By setting $u(x,t) = e^{i\delta t}u(x)$, the above partial differential equation can be directly reduced to the second order integro-differential equation

$$-u''(x) + a(x)u(x) = \gamma(x)u(x)\int_{-\infty}^{+\infty} W(x-s)|u(s)|^2 ds \quad (3)$$

where $a(x) = \delta + V(x)$. We look for an analytical proof of the existence of two types of solutions:
(i) Periodic waves:

$$u(x) = u(x+T), \quad \text{for all } x$$

(ii) Solitary waves:

$$u(-\infty) = 0 = u(+\infty).$$

Main results

Theorem 1

Assume that $V(x), \gamma(x)$ are T-periodic functions. If γ takes non-negative values, $\delta > \|V\|_\infty$ and W verifies condition (2), then eq. (3) has at least one positive T-periodic solution $u \in W^{2,\infty}(0, T)$.

Theorem 2

If $\gamma(x)$ is a non-negative function with non-empty compact support, $\delta > \|V\|_\infty$ and W verifies condition (2), then eq. (3) has at least one non-negative solution (not identically zero) such that $u(-\infty) = 0 = u(+\infty)$. Besides, it has finite energy in the sense that $u \in H^1(\mathbb{R})$.

How to attack the problem

- Fixed point problem
- Krasnoselskii fixed point theorem for compact operators in cones of a Banach space
- Compactness criterion

Krasnoselskii fixed point Theorem

Let $\mathcal{B}$ be a Banach space.

Definition

A set $\mathcal{P} \subset \mathcal{B}$ is a cone if it is closed, nonempty, $\mathcal{P} \neq \{0\}$ and given $x, y \in \mathcal{P}$, $\lambda, \mu \in \mathbb{R}_+$ then $\lambda x + \mu y \in \mathcal{P}$.

A map $H : K \to K$ is completely continuous (or compact) if it is continuous and the image of a bounded set is relatively compact. Thereafter, we state a version of the well known Krasnoselskii fixed point Theorem for compact maps.

Krasnoselskii fixed point Theorem

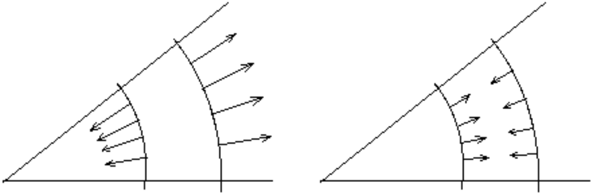
Theorem

Let X be a Banach space, and $K \subset X$ be a cone in X. Assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$ and let $H : K \bigcap (\bar{\Omega}_2 \backslash \Omega_1) \to K$ be a completely continuous operator such that one of the following conditions holds:

1. $\|Hu\| \leq \|u\|$, *if* $u \in K \bigcap \partial\Omega_1$, *and* $\|Hu\| \geq \|u\|$, *if* $u \in K \bigcap \partial\Omega_2$.
2. $\|Hu\| \geq \|u\|$, *if* $u \in K \bigcap \partial\Omega_1$, *and* $\|Hu\| \leq \|u\|$, *if* $u \in K \bigcap \partial\Omega_2$.

Then, H has at least one fixed point in $K \bigcap (\bar{\Omega}_2 \backslash \Omega_1)$.

Krasnoselskii fixed point Theorem



Periodic waves: Formulation of the fixed point problem

Denote by X_T the Banach space of bounded and T-periodic solutions endowed with the uniform norm $\|u\|_\infty$.
Consider the equation

$$-u''(x) + a(x)u(x) = w(x) \qquad (4)$$

with periodic boundary conditions. Given $w \in X_T$, eq. (4) admits a unique T-periodic solution by Fredholm's alternative, and it can be expressed as

$$u(x) = \int_0^T G(x,y)w(y)dy \qquad (5)$$

where $G(x,y)$ is the associated Green's function.
Recall that $a(x) = \delta + V(x)$. When $V(x) \equiv 0$, the Green's function has an explicit expression. In the more general case under consideration, such explicit expression is not available anymore, but the condition $\delta > \|V\|_\infty$ implies that $G(x,y) > 0$ for all $(x,y) \in [0,T] \times [0,T]$).

Periodic waves: Formulation of the fixed point problem

Now, we can define the operator $H : X_T \to W^{2,\infty}(0,T) \subset X_T$ by

$$Hu(x) = \int_0^T G(x,y) \left[\gamma(y)u(y) \int_{-\infty}^{+\infty} W(y-s)u(s)^2 ds \right] dy. \tag{6}$$

A fixed point of H is a periodic solution of eq. (3). The compactness of H is a direct consequence of Ascoli-Arzela Theorem.

Periodic waves:Application of KFPT

Let us define

$$m = \min_{x,y} G(x,y), \qquad M = \max_{x,y} G(x,y).$$

Our cone will be

$$K = \{u \in X_T \ : \min_x u \geq \frac{m}{M} \|u\|_\infty\}.$$

Lemma

$H(K) \subset K.$

Periodic waves:Application of KFPT

Proof: Take $u \in X_T$ and fix $u(x_0) = \text{mín}_{x\in[0,T]} Hu(x)$, then,

$$\begin{aligned}
Hu(x_0) &= \int_0^T G(x_0,y)\left[\gamma(y)u(y)\int_{-\infty}^{+\infty} W(y-s)u(s)^2 ds\right] dy \\
&\geq m\int_0^T \frac{\text{máx}_x\, G(x,y)}{M}\left[\gamma(y)u(y)\int_{-\infty}^{+\infty} W(y-s)u(s)^2 ds\right] \\
&= \frac{m}{M}\int_0^T \underset{x}{\text{máx}}\, G(x,y)\left[\gamma(y)u(y)\int_{-\infty}^{+\infty} W(y-s)u(s)^2 ds\right] d \\
&= \tfrac{m}{M}\left\|Hu\right\|_\infty ,
\end{aligned}$$

therefore the cone is invariant by H. □

Periodic waves:Application of KFPT

Proof of Theorem 1. Define $\Omega_1 = \{u \in X_T \ : \ \|u\|_\infty \leq r\}$. Given $u \in K \bigcap \partial\Omega_1$, it is evident that $\|u\|_\infty = r$. Then,

$$\begin{aligned} Hu(x) &= \int_0^T G(x,y)\left[\gamma(y)u(y)\int_{-\infty}^{+\infty} W(y-s)u(s)^2 ds\right] dy \leq \\ &\leq M\|\gamma\|_\infty r^3 \int_0^T \int_{-\infty}^{+\infty} W(y-s)dsdy = \\ &= M\|\gamma\|_\infty T\|W\|_1 r^3 < r, \end{aligned}$$

if r is small enough. Therefore $\|Hu\|_\infty < \|u\|_\infty$ for any $u \in K \bigcap \partial\Omega_1$.

Periodic waves:Application of KFPT

On the other hand, define $\Omega_2 = \{u \in X_T \,:\, \|u\|_\infty \le R\}$. Assume that $u \in K \bigcap \partial\Omega_2$, then by the own definition of the cone $\text{mín}_x\, u \ge \frac{m}{M} R$. Hence,

$$\begin{aligned} Hu(x) &= \int_0^T G(x,y)\left[\gamma(y)u(y)\int_{-\infty}^{+\infty} W(y-s)u(s)^2 ds\right] dy \ge \\ &\ge \left(\frac{m}{M}R\right)^3 \|W\|_1 \int_0^T G(x,y)\gamma(y)dy \ge \\ &\ge \left(\tfrac{m}{M}R\right)^3 \|W\|_1\, m \textstyle\int_0^T \gamma(y)dy. \end{aligned}$$

Note that γ is not identically zero, so $\int_0^T \gamma(y)dy > 0$ and the latter inequality holds for any x. In consequence, taking R big enough we get

$$\|Hu\|_\infty > R = \|u\|_\infty .$$

Therefore, the assumptions of KFPT are fulfilled, in consequence H has a fixed point in $K \bigcap (\bar{\Omega}_2 \backslash \Omega_1)$, which is equivalent to a positive T-periodic solution of eq. (3).

Solitary waves. Green's function

Let us denote by $BC(\mathbb{R})$ the Banach space of the bounded and continuous functions in $\mathbb{R}$ with the uniform norm. The following result is well-known.

Lemma

Assume that there exists a_* such thath $a(x) \geq a_* > 0$ for a.e. x. If $w \in L^\infty(\mathbb{R})$, then the linear equation

$$-u''(x) + a(x)u(x) = w(x)$$

admits a unique bounded solution $u \in W^{2,\infty}(\mathbb{R})$ and it can be expressed as

$$u(x) = \int_{-\infty}^{+\infty} G(x,y)w(y)dy.$$

Besides, if $w \in L^1(\mathbb{R})$, then $u \in H^1(\mathbb{R})$.

Solitary waves. Green's function

When $V(x) \equiv 0$ then $a(x) \equiv \delta$ and the Green's function has the simple expression

$$G(x, y) = \frac{1}{2\sqrt{\delta}} e^{-\sqrt{\delta}|x-y|}.$$

However, as remarked in the periodic case, the Green's function for the general case of a variable $a(x)$ does not have such an explicit formula and requires a more careful study of its properties.

Definition and properties of $G(x, s)$.

$$G(x,s) = \begin{cases} u_1(x)u_2(s), & \alpha < x \leq s < +\infty \\ u_1(s)u_2(x), & \alpha < s \leq x < +\infty \end{cases}$$

where u_1, u_2 are solutions of the homogeneous problem such that $u_1(-\infty) = 0, u_2(+\infty) = 0$. Moreover, u_1, u_2 are positive fucntions, u_1 increasing and u_2 decreasing.

Definition and properties of $G(x, s)$.

u_1, u_2 intersect in a unique point x_0. Let us define

$$p(x) = \begin{cases} \dfrac{1}{u_2(x)}, & x \leq x_0, \\ \dfrac{1}{u_1(x)}, & x > x_0. \end{cases}$$

Properties

(*P*1) $G(x, s) > 0$ for all $(x, s) \in \mathbb{R}^2$.

(*P*2) $G(x, s) \leq G(s, s)$ for all $(x, s) \in \mathbb{R}^2$.

(*P*3) Given a compact $P \subset \mathbb{R}$, we define

$$m_1(P) = \text{mín}\{u_1(\text{ínf}\, P), u_2(\sup P)\}.$$

Then,

$$G(x, s) \geq m_1(P)p(s)G(s, s) \text{ for all } (x, s) \in P \times \mathbb{R}.$$

(*P*4) $G(s, s)p(s) \geq G(x, s)p(x)$ for all $(x, s) \in \mathbb{R}^2$.

Solitary waves.

To find a solitary wave of eq. (3) is equivalent to find a fixed point of the operator $H : BC(\mathbb{R}) \to WH^1(\mathbb{R}) \subset BC(\mathbb{R})$ defined by

$$Hu(x) = \int_{-\infty}^{+\infty} G(x,y) \left[\gamma(y)u(y) \int_{-\infty}^{+\infty} W(y-s)u(s)^2 ds \right] dy. \tag{7}$$

Compactness

The compactness of H is a consequence of the following lemma.

Lemma

Let $\Omega \subset BC(\mathbb{R})$. Let us assume that the functions $u \in \Omega$ are equicontinuous in each compact interval of $\mathbb{R}$ and that for all $u \in \Omega$ we have

$$|u(x)| \leq \xi(x), \qquad \forall x \in \mathbb{R} \tag{8}$$

where $\xi \in BC(\mathbb{R})$ satisfies

$$\lim_{|x| \to +\infty} \xi(x) = 0. \tag{9}$$

Then, Ω is relatively compact.

Main result.

Theorem 2

If $\gamma(x)$ is a non-negative function with non-empty compact support, $\delta > \|V\|_\infty$ and W verifies condition (2), then eq. (3) has at least one non-negative solution (not identically zero) such that $u(-\infty) = 0 = u(+\infty)$. Besides, it has finite energy in the sense that $u \in H^1(\mathbb{R})$.

Proof.

- Step 1: Definition of the cone.

$$K = \{u \in BC(\mathbb{R}) : \ u(x) \geq 0 \ \text{ for all } x, \quad \min_D u(x) \geq m_1 p_0 \|u\|_\infty\}.$$

Recall that D is the (compact) support of γ.

- Step 2: $H(K) \subset K$.

Proof.

- Step 3: Krasnoselskii conditions.

$(H1)$ Let be $\Omega_1 = \{u \in BC(\mathbb{R}) \,:\, \|u\|_\infty \leq r\}$. Given $u \in K \bigcap \partial\Omega_1$,

$$\begin{aligned} \|Hu\|_\infty &= \max_x \int_D G(x,y) \left[\gamma(y)u(y) \int_{-\infty}^{+\infty} W(y-s)u(s)^2 ds\right] \\ &\leq r^3 \max_x \int_D G(x,s)\gamma(s) \int_{-\infty}^{+\infty} W(y-s) ds dy = \\ &= \|W\|_1\, r^3 \max_x \textstyle\int_D G(x,y)\gamma(y)dy. \end{aligned}$$

Note that by definition, $\int_D G(x,y)\gamma(y)dy$ is the unique solution belonging to $BC_0(\mathbb{R})$ of the linear problem $-u'' + a(x)u = \gamma(x)$. Of course, such a solution is bounded and the maximum in the previous inequality makes sense. In conclusion, if r is small enough,

$$\|Hu\|_\infty \leq \|W\|_1\, r^3 \max_x \int_D G(x,y)\gamma(y)dy < r = \|u\|_\infty$$

for every $u \in K \bigcap \partial\Omega_1$.

Proof.

- Step 4: Krasnoselskii conditions.

(*H2*) Define $\Omega_2 = \{u \in BC(\mathbb{R}) \,:\, \|u\|_\infty \le R\}$. Assume that $u \in K \bigcap \partial\Omega_2$, then by definition of the cone $\min_{x\in D} u \ge m_1 p_0 R$. Hence,

$$\begin{aligned} \|Hu\|_\infty &= \max_x \int_D G(x,y)\left[\gamma(y)u(y)\int_{-\infty}^{+\infty} W(y-s)u(s)^2 ds\right] dy \ge \\ &\ge (m_1 p_0 R)^3 \int_D G(x,y)\left[\gamma(y)\int_{-\infty}^{+\infty} W(y-s)ds\right] dy \\ &\ge (m_1 p_0 R)^3 \|W\|_1 \max_x \textstyle\int_D G(x,y)\gamma(y)dy. \end{aligned}$$

Note that γ is not identically zero, so $\max_x \int_D G(x,y)\gamma(y)dy > 0$. In consequence, taking R big enough we get

$$\|Hu\|_\infty > R = \|u\|_\infty .$$