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Modulated amplitude waves with non-trivial phase of multi-component Bose-Einstein condensates in optical lattices

QIHUAI LIU
School of Mathematics and Computing Sciences, Guangxi Colleges and Universities Key Laboratory of Data Analysis and Computation, Guilin University of Electronic Technology, Guilin 541002, China
qhuailiu@gmail.com

PEDRO J. TORRES
Departamento de Matemática Aplicada and Excellence Research Unit "Modeling Nature" (MNat), Universidad de Granada, Granada 18071, Spain
ptorres@ugr.es

AND

MINGYAN XING
School of Mathematics and Computing Sciences, Guilin University of Electronic Technology, Guilin 541002, China

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We study the existence of modulated amplitude waves with non-trivial phase of a quasi-1D multi-component Bose-Einstein condensate (BEC) in the presence of a external periodic potential. Mathematically, such coherent structures are doubly periodic solutions, in space and time, of a coupled system of Gross-Pitaevskii equations. For a binary BEC, the weak interaction regime is tackled by means of the averaging method and regular perturbation theory. The case of strong particle interaction is covered by a simple rescalling argument. One of the components is stationary, while the second component has non-trivial phase, meaning that there is circulation of matter only of the second component. The case of three components is briefly discussed as well.

Keywords: Modulated amplitude wave; Bose-Einstein condensate; Non-trivial phase; rotation number; averaging method

2000 Math Subject Classification: 35Q55; 34C25

1. Introduction

A Bose-Einstein condensate (BEC) arises when a dilute boson gas is cooled down to temperatures near the absolute zero. The theoretical prediction of this genuine quantum effect was done by Bose and Einstein in 1924, although the first experimental realizations had to wait until 1995 independently by two different teams, see Anderson *et al.* (1995); Davis *et al.* (1995). From then on, the scientific community has demonstrated a sustained interest on this field, in particular in the study of coherent states. The physical relevance of nonlinear waves in Bose-Einstein condensates and the related mathematical techniques have been reviewed comprehensively by Pitaevskii & Stringari (2003); Carretero-González *et al.* (2008).

Nowadays, it is possible to perform Bose-Einstein condensation of mixtures of different atomic species, which has recently been the subject of an intensive experimental and theoretical research. Experimental results have been reported for mixtures of different spin states of ^{87}Rb (Myatt *et al.* (1997); Hall *et al.* (1998)) and different atomic species, such as ^{41}K-^{87}Rb (Modugno *et al.* (2001)), ^{7}Li-^{133}Cs (Mudrich *et al.* (2002)). Motivated by the experimental observations, this phenomenon has been extensively studied also from the theoretical point of view. Without any intention of being exhaustive, we can cite Chui *et al.* (2000); Deconinck *et al.* (2003); Esry *et al.* (1997); Liu (2011); Ho & Shenoy (1996); Porter *et al.* (2004); Pu & Bigelow, (1998a,b); Riboli & Modugno (2002); Terracini & Verzini, (2009); Zhang *et al.* (2007) and many others.

The governing equations for the interaction of $N \geqslant 2$ condensates are given by a coupled system of nonlinear Schrödinger equations

$$i\hbar\frac{\partial \psi_j}{\partial t} = -\frac{\hbar^2}{2m_j}\nabla^2\psi_j + V_j(x)\psi_j + \sum_{l=1}^{N} u_{jl}|\psi_l|^2\psi_j,\ j = 1,2,\cdots,N, \tag{1.1}$$

see for instance Tsurumi *et al.* (2000); Deconinck *et al.* (2003). Here, the coupling constants u_{jl} are expressed in terms of the scattering length a_{jl} between the atomic species j and l by

$$u_{jl} = \frac{2\pi\hbar^2 a_{jl}(m_j + m_l)}{m_j m_l},$$

and m_j is the atom mass of the j-th atom species. The sign of the scattering length a_{jl} determines the nature of the interaction between the different atomic species: a positive value gives rise to a repulsive interatomic interaction, whereas a negative value causes an attractive interaction. $V_j(x)$ is the external potential experienced by the j-th condensate. Because of the different properties of constituting atoms of the different condensates, it is possible for different condensates in the same physical trap to experience different external potentials.

Mainly motivated by Deconinck *et al.* (2003); Porter *et al.* (2004), we consider a quasi-one dimensional cylindrical ("cigar-shaped") BEC mounted on an optical lattice (OL). Mathematically, this means that only one spatial variable is considered and the potentials $V_j(x)$ are periodic functions. Concretely, we take

$$V_j(x) = V_{j0}\cos(2\sqrt{k}x).$$

Spatially periodic potentials have been employed in experimental (see Anderson & Kasevich (1998); Hagley *et al.* (1999)) and theoretical studies of BECs (see for example Alfimov *et al.* (2002); Bronski *et al.* (2001a,b,c)). Then, our main aim is to identify doubly-periodic solutions of system (1.1), also called *modulated amplitude waves*.

We consider uniformly propagating coherent structures with the form

$$\psi_j(x,t) = R_j(x)\exp[i(\theta_j(x) - \mu_j t)],\ j = 1,2,\cdots,N, \tag{1.2}$$

where $R_j(x)$ is the amplitude of the wave function $\psi_j(x,t)$, $\theta_j(x)$ gives the phase dynamics, and μ_j is the BEC's chemical potential. When such a (temporally periodic) coherent structure (1.2) is also spatially periodic, it is called a modulated amplitude wave (MAW). Notice that (1.1) has a phase shift-invariance. This means that if $\Psi(x,t)$ is a solution of (1.1), then

$$\Psi(x,t) = \big(\psi_1(x,t)\exp(i\phi_1), \cdots, \psi_N(x,t)\exp(i\phi_N)\big), \quad \forall \phi_1, \cdots, \phi_N \in \mathbb{R}$$

also satisfies equation (1.1). Thus, if the phase variables $\theta_j(x)$ of a MAW are constant, we call it a modulated amplitude wave with trivial phase. In particular, if in addition $R_j(x)$ are also constant, it is called a standing wave. The dynamical behavior of MAWs with trivial phase for single BECs in lattice and superlattice potentials has been studied extensively by using different approaches, see Porter & Cvitanović (2004b); Porter & Kevrekidis (2005); Porter *et al.* (2007); Van Noort *et al.* (2007). In contrast, up to our knowledge the only two references for the existence of MAWs with trivial phase in two-component BECs are Deconinck *et al.* (2003); Porter *et al.* (2004).

Physically speaking, a modulated amplitude wave with non-trivial phase implies nonzero circulation of matter along the space, given by the scalar $c_j = R_j(x)^2\phi_j'(x)$. MAWs with non-trivial phase have been explicitly constructed in a single BEC by a particular choice of the trapping potential as the sine Jacobi elliptic function, see Bronski *et al.* (2001b). Moreover, an analytical study of the existence of MAWs with non-trivial phase in BECs of single particle has been established by the method of averaging, see Liu & Qian (2012a,b); Jia *et al.* (2014). On the other hand, using the theory of local continuation of solutions, the existence of MAWs with non-trivial phases in quasi-1 D inhomogeneous BECs was proved by Torres (2014).

In this paper, we are concerned with the existence of MAWs with non-trivial phase in the multi-component BEC described by (1.1), which includes nonlinear couplings and OL potentials. Up to our knowledge, this problem has not been treated with anteriority in the related literature. Form pure mathematical point of view, considering MAWs with non-trivial phase leads to a system that includes a strong singularity, which is a serious obstacle for finding exact solutions and also represents an additional difficulty for the qualitative study.

In order to illustrate our approach, we first consider the model of a two-component Bose-Einstein condensate in Section 2. To investigate MAWs, the evolution equations of amplitudes and phases are derived in Section 2.1. Then, in Section 2.2, we prove the existence of periodic solutions by using some results from averaging theory (see Theorem A.1 in the Appendix) for amplitude equations, a system of multiple coupled second order differential equations with singularities (for example, see (2.8) below), which is regarded as a perturbation of a quasi-periodic system with irrationally related frequencies by introducing a small parameter. After an analysis of the asymptotic profile of the wave functions given in Section 2.3 by means of the regular perturbation theory, we establish the existence of infinitely many MAWs with non-trivial phase in Section 2.4. The main particularity of the obtained solutions is that one of the components is stationary, while the second component has a non-trivial phase, meaning that there is circulation of matter only of the second component. Up to our knowledge, the observation of this type of solutions is new in the literature.

The method used in Section 2 can be applied to a multi-component ($N > 2$) BEC as well. We investigate the dynamics of MAWs for a three-component BEC in Section 3, but the method can be extended to a BEC system with arbitrary components. Finally, Section 4 contains a simple observation that permit the consideration of strong particle interactions by a simple rescalling. Finally, Section 5 summarizes the main conclusions and highlight some remarks.

2. Modulated amplitude waves in binary Bose-Einstein condensates

Based on a mean-field limit, the state of the BEC can be described by the condensate wave function, that in the case of a binary BEC is a function with two components. When the temperature is much lower than the critical temperature, the dynamics of the wave function of a quasi-1D binary BEC irradiated by an external electromagnetic field is well described by a system of two nonlinear Schrödinger equations, known as coupled Gross-Pitaevskii equations,

$$\begin{cases} i\hbar\dfrac{\partial\psi_1}{\partial t} = -\dfrac{\hbar^2}{2m_1}\dfrac{\partial^2\psi_1}{\partial x^2} + V_1(x)\psi_1 + u_{11}|\psi_1|^2\psi_1 + u_{12}|\psi_2|^2\psi_1, \\ i\hbar\dfrac{\partial\psi_2}{\partial t} = -\dfrac{\hbar^2}{2m_2}\dfrac{\partial^2\psi_2}{\partial x^2} + V_2(x)\psi_2 + u_{21}|\psi_1|^2\psi_2 + u_{22}|\psi_2|^2\psi_2, \end{cases} \tag{2.1}$$

where V_j is the magnetic trapping potential of the j-bosons for state $j\,(j=1,2)$, and the coupling constants u_{ij} are given in terms of the scattering length a_{ij} for binary collisions of distinguishable bosons given by Esry *et al.* (1997)

$$u_{11} = \frac{4\pi\hbar^2 a_{11}}{m_1}, \quad u_{22} = \frac{4\pi\hbar^2 a_{22}}{m_2}$$
$$u_{12} = 2\pi\hbar^2 a_{12}\frac{m_1+m_2}{m_1 m_2} = u_{21}.$$

Here, m_j denotes the mass of a gas particle of the j-th component, and $|\psi|_j^2$ is interpreted as the particle density of the j-th component ($j=1,2$). If $a_{jj} < 0\,(>0)$, the self-interaction is repulsive (attractive); analogously, if $a_{ij} < 0\,(>0), i \neq j$, the interspecies interaction is repulsive (attractive). For example, for two different spin states of ^{87}Rb, the scattering lengths are known at the 1% level to be in the proportion $a_{11} : a_{12} : a_{22} = 1.03 : 1 : 0.97$ Hall *et al.* (1998). In the ideal gas regime, $a_{ij} \approx 0$ $(j=1,2)$ and also the weak magnetic trapping potential is considered (for example, see Porter *et al.* (2004); Porter & Kevrekidis (2005); Porter *et al.* (2007)). Therefore, it is reasonable to introduce a small parameter $u_{ij} \equiv \varepsilon\tilde{u}_{ij}, V_j \equiv \varepsilon\tilde{V}_j, i,j=1,2$. For notational convenience, we drop the tildes from $\tilde{u}_{ij}$ and $\tilde{V}_j$, so that we obtain the system

$$\begin{cases} i\hbar\dfrac{\partial\psi_1}{\partial t} = -\dfrac{\hbar^2}{2m_1}\dfrac{\partial^2\psi_1}{\partial x^2} + \varepsilon V_1(x)\psi_1 + \varepsilon u_{11}|\psi_1|^2\psi_1 + \varepsilon u_{12}|\psi_2|^2\psi_1, \\ i\hbar\dfrac{\partial\psi_2}{\partial t} = -\dfrac{\hbar^2}{2m_2}\dfrac{\partial^2\psi_2}{\partial x^2} + \varepsilon V_2(x)\psi_2 + \varepsilon u_{21}|\psi_1|^2\psi_2 + \varepsilon u_{22}|\psi_2|^2\psi_2. \end{cases} \tag{2.2}$$

2.1 *Evolution equations of amplitudes and phases*

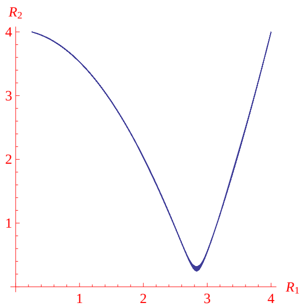
Substituting the ansatz (1.2) into the GP equations (2.2)and equating real and imaginary parts of the resulting equations, we obtain

$$R_1'' - R_1(\theta_1')^2 + k_1R_1 + \varepsilon g_{11}R_1^3 + \varepsilon\bar{V}_1(x)R_1 + \varepsilon g_{12}R_2^2R_1 = 0, \tag{2.3}$$
$$R_2'' - R_2(\theta_2')^2 + k_2R_2 + \varepsilon g_{22}R_2^3 + \varepsilon\bar{V}_2(x)R_2 + \varepsilon g_{21}R_1^2R_2 = 0, \tag{2.4}$$
$$2R_1'\theta_1' + R_1\theta_1'' = 0, \quad 2R_2'\theta_2' + R_2\theta_2'' = 0, \tag{2.5}$$

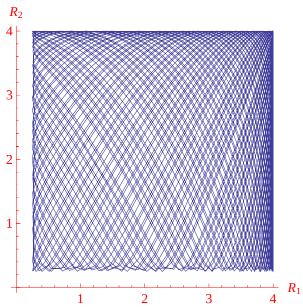
where the prime stands for $\mathrm{d}/\mathrm{d}x$, and

$$k_i = \frac{2m_i\mu_i}{\hbar}, \quad g_{ij} = -\frac{2m_iu_{ij}}{\hbar^2}, \quad \bar{V}_i(x) = -\frac{2m_iV_i(x)}{\hbar^2}, i,j=1,2.$$

Along this paper, we assume $k_i > 0$, corresponding to a positive chemical potential.



(a) Periodic case.



(b) Quasi-periodic case.

FIG. 1. In the projection plane R_1OR_2 of 4 dimensional phase space, we plot the projection curves of solutions for the unperturbed system of (2.8) with $c_1 \neq 0$ and $c_2 \neq 0$. *(a)* The projection curve of one periodic solution in case of $\sqrt{k_1}/\sqrt{k_2} \in \mathbb{Q}$. *(b)* The projection curve of one quasi-periodic solution, in case of $\sqrt{k_1}/\sqrt{k_2} \in \mathbb{R}/\mathbb{Q}$, is dense on the two-dimensional torus.

In view of (2.5) and noticing that

$$(R_i^2\theta_i')' = (2R_i'\theta_i' + R_i\theta_i'')R_i = 0,$$

we find that

$$\theta_i'(x) = \frac{c_i}{R_i^2(x)}, i = 1,2 \tag{2.6}$$

with arbitrary constants c_1 and c_2. Thus, phases are written in terms of amplitudes as

$$\theta_1(x) = c_1 \int \frac{\mathrm{d}x'}{R_1^2(x')}, \quad \theta_2(x) = c_2 \int \frac{\mathrm{d}x'}{R_2^2(x')}. \tag{2.7}$$

Substituting (2.6) into equations (2.3) and (2.4), we obtain the amplitude equations

$$\begin{cases} R_1'' + k_1R_1 - \dfrac{c_1^2}{R_1^3} + \varepsilon g_{11}R_1^3 + \varepsilon\bar{V}_1(x)R_1 + \varepsilon g_{12}R_2^2R_1 = 0, \\ R_2'' + k_2R_2 - \dfrac{c_2^2}{R_2^3} + \varepsilon g_{22}R_2^3 + \varepsilon\bar{V}_2(x)R_2 + \varepsilon g_{21}R_1^2R_2 = 0. \end{cases} \tag{2.8}$$

Without loss of generality, along this paper we assume that c_1, c_2 are selected as arbitrarily nonnegative constants.

In the case $\sqrt{k_1}/\sqrt{k_2} = p/q \in \mathbb{Q}$ with coprime positive integers q, p, the unperturbed system ($\varepsilon = 0$) of (2.8) is an isochronous system with minimal period $T = 2p\pi/\sqrt{k_1} = 2q\pi/\sqrt{k_2}$ for $c_1 = 0$ or $c_2 = 0$, and with period $T = p\pi/\sqrt{k_1} = q\pi/\sqrt{k_2}$ for $c_1 \neq 0$ and $c_2 \neq 0$, see Fig. 1(a). The continuation of periodic solutions of an unperturbed isochronous system has been studied before in different mathematical contexts, see for instance Liu *et al.* (2015). In case of the rational ratio, the persistence of periodic solutions (MAWs with the trivial phase) has been studied by Porter *et al.* (2004) by taking $c_1 = 0$ and $c_2 = 0$. The existence of periodic solutions is due to the high order resonance. We also refer to Porter

& Kevrekidis (2005); Porter & Cvitanović (2004a) for a method based on multiple scale perturbation theory, and Liu & Qian (2012a); Jia *et al.* (2014) for the method of averaging.

When $\sqrt{k_1}/\sqrt{k_2} \in \mathbb{R}/\mathbb{Q}$ is an irrational number, every solution of the unperturbed system ($\varepsilon = 0$) of (2.8) is quasi-periodic with two irrational related frequencies, see Fig. 1(b). Comparing with the periodic case, the classical theory of averaging for the existence of periodic solutions loses efficacy since we usually obtain a system of standard form with quasi-periodic coefficients. To overcome this difficulty, we perform a local averaging procedure due to Buica *et al.* (2007).

Along this paper we always assume $\sqrt{k_1}/\sqrt{k_2}$ is an irrational number. Roughly speaking, the assumption with respect to the ratio shall be used to eliminate the linear resonances and to ensure invertibility of the linear operator. To the best of our knowledge, the qualitative description of periodic MAWs with non-trivial phase for multi-component BECs form this point of view is completely new.

2.2 *Periodic solutions of amplitude equations*

In this subsection, we give a pure mathematical result upon the existence of periodic solutions. To seek periodic solutions with non-trivial phase, in view of (2.7), we take constants of integration $c_1 = 0$ and $c_2 \neq 0$, which means that the phase for the first component is trivial while the second component has a nontrivial phase. The case that $c_1 \neq 0$ and $c_2 = 0$ is analogous, and the case of both $c_1 = c_2 = 0$ has been studied by Porter *et al.* (2004).

Let usewrite equation (2.8) in the equivalent form

$$\begin{cases} R_1^{'} = S_1, \\ S_1^{'} = -k_1R_1 + \varepsilon F_1(x,R_1,R_2), \\ R_2^{'} = S_2, \\ S_2^{'} = -k_2R_2 + \dfrac{c_2^2}{R_2^3} + \varepsilon F_2(x,R_1,R_2), \end{cases} \tag{2.9}$$

where

$$F_1(x,R_1,R_2) = -(g_{11}R_1^3 + \bar{V}_1(x)R_1 + g_{12}R_2^2R_1),$$

$$F_2(x,R_1,R_2) = -(g_{22}R_2^3 + \bar{V}_2(x)R_2 + g_{21}R_1^2R_2).$$

The next theorem is an existence result of periodic solutions for system (2.9).

THEOREM 2.1 Assume that $\bar{V}_1(x) = \bar{V}_{10}\cos(2\sqrt{k_1}x)$ and $\bar{V}_2(x)$ is an arbitrary C^2 periodic function with the period $T = 2\pi/\sqrt{k_1}$. Fix $c_2 > 0$ and $g_{ij} \in \mathbb{R}$, $i,j = 1,2$, and take $k_2 > 0$ such that $\sqrt{k_1}/\sqrt{k_2}$ is an irrational number. Then, for $\varepsilon \neq 0$ sufficiently small, the following statements hold

(a) if $g_{11}(2g_{12}\beta_2^2 + \bar{V}_{10}) < 0$ and $\bar{V}_{10} \neq 0$, where $\beta_2 = \sqrt[4]{c_2^2/k_2}$, system (2.9) has a branch of T-periodic solutions $(R_1(x,\varepsilon), S_1(x,\varepsilon), R_2(x,\varepsilon), S_2(x,\varepsilon))$ such that

$$(R_1(0,\varepsilon), S_1(0,\varepsilon), R_2(0,\varepsilon), S_2(0,\varepsilon)) \to \left(\sqrt{\frac{-2}{3g_{11}}\left(2g_{12}\beta_2^2 + \bar{V}_{10}\right)}, 0, \beta_2, 0\right)$$

as $\varepsilon \to 0$.

(b) if $g_{11}(\bar{V}_{10}-2g_{12}\beta_2^2)>0$ and $\bar{V}_{10}\neq 0$, where $\beta_2=\sqrt[4]{c_2^2/k_2}$, system (2.9) has a branch of T-periodic solutions $(R_1(x,\varepsilon),S_1(x,\varepsilon),R_2(x,\varepsilon),S_2(x,\varepsilon))$ such that

$$(R_1(0,\varepsilon),S_1(0,\varepsilon),R_2(0,\varepsilon),S_2(0,\varepsilon))\to\left(0,\sqrt{\frac{2}{3g_{11}}\left(k_1\bar{V}_{10}-2g_{12}k_1\beta_2^2\right)},\beta_2,0\right)$$

as $\varepsilon\to 0$.

Proof. The proof follows from Theorem A.1 in the Appendix, which is based on the Lyapunov-Schmidt reduction and the implicit function theorem. To apply Theorem A.1, we reset some notations. If we define

$$R=\begin{pmatrix}R_1\\S_1\\R_2\\S_2\end{pmatrix},G_0(x,R)=\begin{pmatrix}S_1\\-k_1R_1\\S_2\\-k_2R_2+\dfrac{c_2^2}{R_2^3}\end{pmatrix},G_1(x,R)=\begin{pmatrix}0\\F_1(x,R_1,R_2)\\0\\F_2(x,R_1,R_2)\end{pmatrix},$$

then (2.9) becomes

$$R'(x)=G_0(x,R)+\varepsilon G_1(x,R). \tag{2.10}$$

Let $r_1>0$ be arbitrarily small and $r_2>0$ be arbitrarily large and define the open and bounded subset V on the projection plane R_1OS_1 given by

$$V=\left\{\alpha\in\mathbb{R}^2:\ \alpha=(R_{10},S_{10})\text{ and }r_1<\sqrt{R_{10}^2+S_{10}^2}<r_2\right\}.$$

Let $\beta_2=\sqrt[4]{c_2^2/k_2}$, and the function $\beta:\overline{V}\to\mathbb{R}^2$ (see Theorem A.1 in the Appendix) is taken by $\beta(\alpha)\equiv(\beta_2,0)$. Consequently, we define the set $\mathscr{Z}$ by

$$\begin{aligned}\mathscr{Z}&=\left\{z_\alpha=(\alpha,\beta(\alpha)):\ \alpha\in\overline{V}\right\}\\&=\left\{(R_{10},S_{10},\beta_2,0)\in\mathbb{R}^4:\ r_1\leqslant\sqrt{R_{10}^2+S_{10}^2}\leqslant r_2\right\}.\end{aligned}$$

Obviously, for each $z_\alpha\in\mathscr{Z}$ and $\varepsilon=0$, the solution $R_\alpha(x;z_\alpha)$ of (2.10) with the initial value $R_\alpha(0;z_\alpha)=z_\alpha$ is T-periodic with respect to x with the least period $T=2\pi/\sqrt{k_1}$. In fact, every solution of (2.10) with $\varepsilon=0$ starting from $\mathscr{Z}$ can be written by coordinates as $R_\alpha(x;z_\alpha)=(R_{\alpha1}(x;z_\alpha),S_{\alpha1}(x;z_\alpha),R_{\alpha2}(x;z_\alpha),S_{\alpha2}(x;z_\alpha))$, where

$$\begin{aligned}&R_{\alpha1}(x)=R_{10}\cos(\sqrt{k_1}x)+\frac{1}{\sqrt{k_1}}S_{10}\sin(\sqrt{k_1}x),\quad R_{\alpha2}(x)=\beta_2,\\&S_{\alpha1}(x)=S_{10}\cos(\sqrt{k_1}x)-\sqrt{k_1}R_{10}\sin(\sqrt{k_1}x),\quad S_{\alpha2}(x)=0.\end{aligned}$$

The variational equation of the unperturbed system along the periodic solution $R_\alpha(x;z_\alpha)$ is given by

$$Y'=D_RG_0(t,R_\alpha(x;z_\alpha))Y=\begin{pmatrix}0&1&0&0\\-k_1&0&0&0\\0&0&0&1\\0&0&-4k_2&0\end{pmatrix}Y, \tag{2.11}$$

where Y is a 4×4 matrix. The fundamental matrix $M_{z_\alpha}(x)$ of the differential system (2.11) such that $M_{z_\alpha}(0)$ is the identity matrix of $\mathbb{R}^4$ adopts the simple form

$$\begin{pmatrix} \cos(\sqrt{k_1}x) & \frac{1}{\sqrt{k_1}}\sin(\sqrt{k_1}x) & 0 & 0 \\ -\sqrt{k_1}\sin(\sqrt{k_1}x) & \cos(\sqrt{k_1}x) & 0 & 0 \\ 0 & 0 & \cos(2\sqrt{k_2}x) & \frac{1}{2\sqrt{k_2}}\sin(2\sqrt{k_2}x) \\ 0 & 0 & -2\sqrt{k_2}\sin(2\sqrt{k_2}x) & \cos(2\sqrt{k_2}x) \end{pmatrix}.$$

Note that the fundamental matrix $M_{z_\alpha}(x)$ does not depend on the initial value z_α of the periodic solution $R(x,z_\alpha)$, so for simplicity we can drop the subscript z_α and reset $M(x) = M_{z_\alpha}(x)$. By an easy computation, we obtain that the matrix $M^{-1}(0) - M^{-1}(T)$ is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\cos\left(\dfrac{4\pi\sqrt{k_2}}{\sqrt{k_1}}\right) & \dfrac{1}{2\sqrt{k_1}}\sin\left(\dfrac{4\pi\sqrt{k_2}}{\sqrt{k_1}}\right) \\ 0 & 0 & -2\sqrt{k_1}\sin\left(\dfrac{4\pi\sqrt{k_2}}{\sqrt{k_1}}\right) & 1-\cos\left(\dfrac{4\pi\sqrt{k_2}}{\sqrt{k_1}}\right) \end{pmatrix}.$$

We observe that the matrix $M^{-1}(0) - M^{-1}(T)$ has in the upper right corner the 2×2 zero matrix, while in the lower right corner a 2×2 matrix Δ_α satisfies

$$\begin{aligned} \det(\Delta_\alpha) &= \begin{vmatrix} 1-\cos\left(\dfrac{4\pi\sqrt{k_2}}{\sqrt{k_1}}\right) & \dfrac{1}{2\sqrt{k_1}}\sin\left(\dfrac{4\pi\sqrt{k_2}}{\sqrt{k_1}}\right) \\ -2\sqrt{k_1}\sin\left(\dfrac{4\pi\sqrt{k_2}}{\sqrt{k_1}}\right) & 1-\cos\left(\dfrac{4\pi\sqrt{k_2}}{\sqrt{k_1}}\right) \end{vmatrix} \\ &= 4\sin^2\left(\dfrac{2\pi\sqrt{k_2}}{\sqrt{k_1}}\right) \neq 0, \end{aligned}$$

since $\sqrt{k_1}/\sqrt{k_2}$ is an irrational number. Consequently, the assumptions (i) and (ii) of Theorem A.1 are satisfied.

Following the notation of Theorem A.1, $\mathrm{P}(R_1,S_1,R_2,S_2) = (R_1,S_1)$. A straightforward computation leads to

$$\begin{aligned} \mathscr{G}(\alpha) = \mathscr{G}(R_{10},S_{10}) &= \mathrm{P}\left(\int_0^T M_{Z_\alpha}^{-1}(x) G_1(x,R_\alpha(x;Z_\alpha))\mathrm{d}x\right) \\ &= \begin{pmatrix} -\displaystyle\int_0^T \frac{\sin(\sqrt{k_1}x)}{\sqrt{k_1}} F_1(x,R_{\alpha 1},S_{\alpha 1},\beta_2,0)\mathrm{d}x \\ \displaystyle\int_0^T \cos(\sqrt{k_1}x) F_1(x,R_{\alpha 1},S_{\alpha 1},\beta_2,0)\mathrm{d}x \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{\pi S_{10}}{4k_1^{5/2}}\left[3g_{11}\left(k_1R_{10}^2 + S_{10}^2\right) - 2k_1\left(\bar{V}_{10} - 2g_{12}\beta_2^2\right)\right] \\ -\dfrac{\pi R_{10}}{4k_1^{3/2}}\left[3g_{11}\left(k_1R_{10}^2 + S_{10}^2\right) + 2k_1\left(\bar{V}_{10} + 2g_{12}\beta_2^2\right)\right] \end{pmatrix}. \end{aligned}$$

Now, we shall compute the simple zeroes of the function $\mathscr{G}(\alpha)$ in the open and bounded subset $V \subset \mathbb{R}^2$. The roots of the algebraic equation $\mathscr{G}(\alpha) = 0$ in V can be found as

$$\alpha_1 = (0,0), \quad \alpha_{2,3} = \left(\pm\sqrt{\frac{-2}{3g_{11}}\left(2g_{12}\beta_2^2+\bar{V}_{10}\right)},0\right), \tag{2.12}$$
$$\alpha_{4,5} = \left(0,\pm\sqrt{\frac{2k_1}{3g_{11}}\left(\bar{V}_{10}-2g_{12}\beta_2^2\right)}\right),$$

whenever the radical expressions of (2.12) are well defined. Notice that $\alpha_{2,3}$ are well defined provided $g_{11}(2g_{12}\beta_2^2+\bar{V}_{10}) < 0$; while $\alpha_{4,5}$ are well defined provided $g_{11}(\bar{V}_{10}-2g_{12}\beta_2^2) > 0$. The Jacobian determinant are given by

$$\det(\mathscr{G}'_\alpha(\alpha_{2,3})) = \frac{\pi^2\bar{V}_{10}\left(\bar{V}_{10}+2g_{12}\beta_2^2\right)}{k_1^2},$$
$$\det(\mathscr{G}'_\alpha(\alpha_{4,5})) = \frac{\pi^2\bar{V}_{10}\left(\bar{V}_{10}-2g_{12}\beta_2^2\right)}{k_1^2}.$$

By assumption, this means that $\det(\mathscr{G}'_\alpha(\alpha_i)) \neq 0\,(i = 2,\cdots,5)$. Therefore, α_i, $(i = 2,\cdots,5)$ are simple zeroes of the function $\mathscr{G}(\alpha)$. Now the proof is done by applying Theorem A.1 to α_2,α_4. □

REMARK 2.1 Note that, starting from α_3,α_5, we can construct two new branches in the same way, but such branches do not provide additional information since they can be obtained directly from the existing ones by means of the symmetry transformation $(R_1,S_1,R_2,S_2) \to (-R_1,-S_1,R_2,S_2)$. In the same way, the branch starting from α_1 correspond to a "semitrivial" solution $(0,0,R_2(x,\varepsilon),S_2(x,\varepsilon))$. In the original system, this means that $\psi_1 \equiv 0$ so it does not corresponds to a genuine two-component BEC.

REMARK 2.2 The method used to prove Theorem 2.1 is not suitable for the remaining case that both c_1 and c_2 are not zero, since the solutions of the unperturbed system are not concise, leading to a complex computation. We have even tried a symbolic computation approach, but failed. This problem is left for the further study.

2.3 *Asymptotic profiles of amplitude variables*

Theorem 2.1 has shown that there exist at least two $2\pi/\sqrt{k_1}$-periodic solutions $z_i(x,\varepsilon)$, $i = 1,2$ of system (2.9) with the initial values

$$z_1(0,\varepsilon) = \left(\sqrt{\frac{-2}{3g_{11}}\left(2g_{12}\beta_2^2+\bar{V}_{10}\right)}+O(\varepsilon),O(\varepsilon),\beta_2+O(\varepsilon),O(\varepsilon)\right)$$

and

$$z_2(0,\varepsilon) = \left(O(\varepsilon),\sqrt{\frac{2}{3g_{11}}\left(k_1\bar{V}_{10}-2g_{12}k_1\beta_2^2\right)}+O(\varepsilon),\beta_2+O(\varepsilon),O(\varepsilon)\right).$$

as $\varepsilon \to 0$, where we have used the notation $z_i(x,\varepsilon) = (R_i(x,\varepsilon),S_i(x,\varepsilon),\,R_i(x,\varepsilon),S_i(x,\varepsilon))$.

Consider a system of one order $z'(x) = f(x,z,\varepsilon)$, where $f : \mathbb{R}\times\mathbb{R}^n\times(-\varepsilon_0,\varepsilon_0) \to \mathbb{R}^n$ is a continuous function. The continuous dependence theorem of solutions with respect to parameters (see for instance

Theorem 5 of Sect. 1 in You (1981) or Sect. 8 in Arnold (1973)) states that, if the function f has a continuous partial derivative of m-th order with respect to both the variable z and the parameter ε on a domain G_ε of $\mathbb{R}^n$, then the solution $z(x,\varepsilon)$ with $x \in I$ has continuous partial derivatives up to m order with respect to the parameter ε. Here, the interval I denotes the maximal interval where the solution $z(x,\varepsilon)$ remain in G_ε.

By compactness, we can find a domain G_ε of $(\mathbb{R}^+ \times \mathbb{R})^2 \times (-\varepsilon_0, \varepsilon_0)$ containing the periodic solutions $z_i(x,\varepsilon)$, $i=1,2$ of system (2.9) for all $x \in \mathbb{R}$. Moreover, the nonlinearity of system (2.9) is of class C^∞ with respect to both the variables R_i, S_i and the parameter ε on G_ε. By the continuous dependence theorem of solutions, we know that the periodic solutions $z_i(x,\varepsilon)$, $i=1,2$ have derivatives of any order with respect to the parameter ε. Now we can rewrite the periodic solutions $z_i(x,\varepsilon)$, $i=1,2$ of system (2.9) into Taylor expansions $z_i(x,\varepsilon) = z_i^{(0)}(x) + z_i^{(1)}(x)\varepsilon + O(\varepsilon^2)$ as $\varepsilon \to 0$.

In the following, we shall present an analysis of the asymptotic behavior of the amplitude variables for small ε by a regular perturbation theory. This method was introduced to find a finite order asymptotic approximation of a solution of a differential equation, see Holmes (1999).

According to Theorem 2.1, we distinguish two cases.

2.3.1 *Case I:* $g_{11}(2g_{12}\beta_2^2 + \bar{V}_{10}) < 0$ *and* $\bar{V}_{10} \neq 0$ By Theorem 2.1, if $g_{11}(2g_{12}\beta_2^2 + \bar{V}_{10}) < 0$ and $\bar{V}_{10} \neq 0$, there exists a T-periodic solution $(R_1(x,\varepsilon), S_1(x,\varepsilon), R_2(x,\varepsilon), S_2(x,\varepsilon))$ of system (2.9) with period $T = 2\pi/\sqrt{k_1}$ such that

$$(R_1(0,\varepsilon), S_1(0,\varepsilon), R_2(0,\varepsilon), S_2(0,\varepsilon)) \to \left(\sqrt{\frac{-2\left(\bar{V}_{10} + 2g_{12}\beta_2^2\right)}{3g_{11}}}, 0, \beta_2, 0 \right)$$

as $\varepsilon \to 0$.

In the following, we will use the regular perturbation theory to identify the first order approximation of the amplitude components $R_i(x,\varepsilon)$, $i=1,2$. Regular perturbation theory assumes an expansion of the form

$$R_i(x,\varepsilon) = R_i^{(0)}(x) + \varepsilon R_i^{(1)}(x) + \varepsilon^2 R_i^{(2)}(x) + \cdots, \tag{2.13}$$

where $R_i^{(j)}(x)$, $j=1,2,\ldots$ are all unknown T-periodic functions. Inserting (2.13) with $i=2$ into (2.9) and equating powers of ε, we have

$$O(1): \quad R_2^{(0)\prime\prime}(x) + k_2 R_2^{(0)}(x) - \frac{c_2^2}{\left(R_2^{(0)}(x)\right)^3} = 0, \; R_2^{(0)}(0) = \beta_2, \; R_2^{(0)\prime}(0) = 0, \tag{2.14}$$

$$O(\varepsilon): \quad R_2^{(1)\prime\prime}(x) + \left(k_2 + \frac{3c_2^2}{\left(R_2^{(0)}(x)\right)^4} \right) R_2^{(1)}(x) + g_{22}\left(R_2^{(0)}(x)\right)^3 + V_2(x)\left(R_2^{(0)}(x)\right) + g_{21}\left(R_1^{(0)}(x)\right)^2\left(R_2^{(0)}(x)\right) = 0. \tag{2.15}$$

Here, $R_1^{(0)}(x)$ is the solution of the zero-order equation for $R_1(x,\varepsilon)$

$$R_1^{(0)\prime\prime}(x) + k_1 R_1^{(0)}(x) = 0, \; R_1^{(0)}(0) = \sqrt{\frac{-2\left(\bar{V}_{10} + 2g_{12}\beta_2^2\right)}{3g_{11}}}, \; R_1^{(0)\prime}(0) = 0.$$

Obviously,

$$R_1^{(0)}(x) = \sqrt{\frac{-2\left(\bar{V}_{10} + 2g_{12}\beta_2^2\right)}{3g_{11}}}\cos(\sqrt{k_1}x).$$

Inserting this expression and the solution $R_2^{(0)}(x) = \beta_2$ of (2.14) into (2.15), it comes down to

$$R_2^{(1)\prime\prime}(x) + 4k_2R_2^{(1)}(x) + g_{22}\beta_2^3 - \frac{2\cos^2\left(\sqrt{k_1}x\right)g_{21}\beta_2\left(V_{10} + 2g_{12}\beta_2^2\right)}{3g_{11}} + \beta_2\bar{V}_2(x) = 0. \tag{2.16}$$

Taking $\bar{V}_2(x) = \bar{V}_{20}\cos(\sqrt{k_1}x)$, we know that (2.16) has a unique T-periodic solution with explicit expression

$$R_2^{(1)}(x) = \frac{-3g_{11}g_{22}\beta_2^3 + g_{21}\beta_2\left(\bar{V}_{10} + 2g_{12}\beta_2^2\right)}{12g_{11}k_2} - \frac{\cos\left(2\sqrt{k_1}x\right)\beta_2\left(-3g_{11}\bar{V}_{20} + g_{21}\left(\bar{V}_{10} + 2g_{12}\beta_2^2\right)\right)}{12g_{11}\left(k_1 - k_2\right)}.$$

Finally, by direct inspection the solution of (2.14) is simply $R_2^{(0)}(x) = \beta_2$.

In conclusion, we have obtained the asymptotic expansion of $R_1(x,\varepsilon)$ and $R_2(x,\varepsilon)$ as

$$R_1(x,\varepsilon) = \pm\sqrt{\frac{-2\left(\bar{V}_{10} + 2g_{12}\beta_2^2\right)}{3g_{11}}}\cos(\sqrt{k_1}x) + O(\varepsilon), \tag{2.17}$$

$$R_2(x,\varepsilon) = \beta_2 + \varepsilon\left(\frac{-3g_{11}g_{22}\beta_2^3 + g_{21}\beta_2\left(\bar{V}_{10} + 2g_{12}\beta_2^2\right)}{12g_{11}k_2} - \frac{\cos\left(2\sqrt{k_1}x\right)\beta_2\left(-3g_{11}\bar{V}_{20} + g_{21}\left(\bar{V}_{10} + 2g_{12}\beta_2^2\right)\right)}{12g_{11}\left(k_1 - k_2\right)}\right) + O(\varepsilon^2). \tag{2.18}$$

REMARK 2.3 Since equation (2.16) is $2\pi/\sqrt{k_1}$-periodic, and the period is irrational related to $2\pi/\sqrt{4k_2}$, we know that there is a unique $2\pi/\sqrt{4k_2}$-periodic solution for equation (2.16). However, while computing $R_1^{(1)}(x)$, the uniqueness of periodic solutions has been lost, since the considered equation has the same period $2\pi/\sqrt{4k_2}$. Therefore, $R_1^{(1)}(x)$ is unascertained. Fortunately, in our subsequent proof of our main results (see Theorem 2.2 to Theorem 2.5), we only need the first order expansion of $R_2(x)$.

2.3.2 *Case II:* $g_{11}\left(\bar{V}_{10} - 2g_{12}\beta_2^2\right) > 0$ *and* $\bar{V}_{10} \neq 0$ In this case, Theorem 2.1 provides a T-periodic solution $(R_1(x,\varepsilon), S_1(x,\varepsilon), R_2(x,\varepsilon), S_2(x,\varepsilon))$ with period $T = 2\pi/\sqrt{k_1}$ such that

$$(R_1(0,\varepsilon), S_1(0,\varepsilon), R_2(0,\varepsilon), S_2(0,\varepsilon)) \to \left(0, \sqrt{\frac{2k_1}{3g_{11}}\left(\bar{V}_{10} - 2g_{12}\beta_2^2\right)}, \beta_2, 0\right)$$

as $\varepsilon \to 0$.

Using again the regular perturbation theory, we obtain the asymptotic expansion of $R_1(x,\varepsilon)$ and $R_2(x,\varepsilon)$ as

$$R_1(x,\varepsilon) = \pm\sqrt{\frac{2\left(\bar V_{10} - 2g_{12}\beta_2^2\right)}{3g_{11}}}\sin(\sqrt{k_1}x) + O(\varepsilon), \tag{2.19}$$

$$\begin{aligned} R_2(x,\varepsilon) = \beta_2 + \varepsilon\Bigg(&\frac{(2g_{12}g_{21} - 3g_{11}g_{22})\beta_2^3 - g_{21}\beta_2\bar V_{10}}{12g_{11}k_2} \\ &+ \frac{\cos\left(2\sqrt{k_1}x\right)\beta_2\left(g_{21}\left(2g_{12}\beta_2^2 - \bar V_{10}\right) + 3g_{11}\bar V_{20}\right)}{12g_{11}\left(k_1 - k_2\right)}\Bigg) + O(\varepsilon^2). \end{aligned} \tag{2.20}$$

Since the computations are similar, we omit further details for the sake of brevity.

2.4 *Modulated amplitude waves with non-trivial phase*

In this subsection, we are going to present the main theorems for the binary BEC by taking advantage of the study performed in the last subsection. The first result is as follows.

THEOREM 2.2 Assume that $V_1(x) = V_{10}\cos(2\sqrt{k_1}x)$, $V_2(x) = V_{20}\cos(2\sqrt{k_1}x)$. Let k_2 be a positive constant such that $\sqrt{k_1/k_2}$ is an irrational number. Fix $\mu_1 = k_1\hbar/(2m_1)$ and $\mu_2 = k_2\hbar/(2m_2)$. If $u_{11}V_{10} <$ 0, then for each $c_2 \in (0, c_*)$ with

$$c_* := \frac{|V_{10}|\sqrt{k_2}}{2|u_{12}|},$$

there exists $\varepsilon(c_2) > 0$ such that system (2.2) has a modulated amplitude wave (ψ_1,ψ_2) of the form

$$\psi_j(x,t,\varepsilon) = R_j(x,\varepsilon)\exp[i(\theta_j(x,\varepsilon) - \mu_j t)],\ j = 1,2$$

for almost every for $|\varepsilon| \in (0,\varepsilon(c_2))$, where $\psi_j(x,t,\varepsilon)$ is periodic with respect to x, and such that

$$\begin{aligned} R_1(x,\varepsilon) &= \sqrt{-\frac{4\ c_2u_{12} + 2V_{10}\sqrt{k_2}}{3u_{11}\sqrt{k_2}}}\cos\left(\sqrt{k_1}x\right) + O(\varepsilon), \\ \theta_1(x,\varepsilon) &\equiv \theta_{10} = 0, \\ R_2(x,\varepsilon) &= \left(\frac{c_2^2}{k_2}\right)^{1/4} + \varepsilon\Bigg(-\frac{\sqrt{c_2}m_2}{6\hbar^2u_{11}\left(k_2\right)^{7/4}}\left[c_2\left(2u_{12}u_{21} - 3u_{11}u_{22}\right) + u_{21}V_{10}\sqrt{k_2}\right] \\ &\quad - \frac{\sqrt{c_2}m_2\left(u_{21}\left(2c_2\ u_{12} + V_{10}\sqrt{k_2}\right) - 3u_{11}V_{20}\sqrt{k_2}\right)}{6\hbar^2\left(k_1 - k_2\right)u_{11}\left(k_2\right)^{3/4}}\cos\left(2\sqrt{k_1}x\right)\Bigg) \\ &\quad + O(\varepsilon^2), \\ \theta_2(x,\varepsilon) &= \frac{\varepsilon\sin\left(2\sqrt{k_1}x\right)m_2\left(u_{21}\left(2c_2\ u_{12} + \sqrt{k_2}V_{10}\right) - 3\sqrt{k_2}u_{11}V_{20}\right)}{6\hbar^2\sqrt{k_1}\left(-k_1 + k_2\right)u_{11}} \\ &\qquad + \mathrm{rot}(R_2)(c_2,\varepsilon)x + O(\varepsilon^2) \end{aligned}$$

with

$$\mathrm{rot}(R_2)(c_2,\varepsilon) = \sqrt{k_2} + \frac{\varepsilon m_2\left(c_2\left(-2u_{12}u_{21} + 3u_{11}u_{22}\right) - u_{21}V_{10}\sqrt{k_2}\right)}{3\hbar^2k_2u_{11}} + O(\varepsilon)^2.$$

Proof. Note that

$$g_{11}\left(\bar{V}_{10}+2g_{12}\beta_2^2\right)=\frac{4m_1^2u_{11}}{\hbar^4\sqrt{k_2}}\left(2c_2u_{12}+V_{10}\sqrt{k_2}\right).$$

If $u_{11}V_{10}<0$, we select c_2 from the interval $(0,c_*)$ with

$$c_*:=\frac{|V_{10}|\sqrt{k_2}}{2|u_{12}|},$$

so that $g_{11}\left(\bar{V}_{10}+2g_{12}\beta_2^2\right)<0$. Moreover, it is obvious that $\bar{V}_{10}\neq 0$. Therefore, by applying Theorem 2.1 (a), for every $c_2\in(0,c_*)$, there exists $\varepsilon(c_2)>0$ such that for $|\varepsilon|<\varepsilon(c_2)$, system (2.9) has at least one T-periodic solution $(R_1(x,\varepsilon),S_1(x,\varepsilon),R_2(x,\varepsilon),S_2(x,\varepsilon))$, with $T=\frac{2\pi}{\sqrt{k_1}}$. Moreover, the regular perturbation analysis done in Subsection 2.3.1 gives an approximation of $R_1(x,\varepsilon)$ and $R_2(x,\varepsilon)$ as

$$\begin{aligned}R_1(x,\varepsilon)&=\sqrt{-\frac{4\ c_2u_{12}+2V_{10}\sqrt{k_2}}{3u_{11}\sqrt{k_2}}}\cos\left(\sqrt{k_1}x\right)+O(\varepsilon),\\ R_2(x,\varepsilon)&=\left(\frac{c_2^2}{k_2}\right)^{1/4}+\varepsilon\Bigg(-\frac{\sqrt{c_2}m_2\left(c_2\ (2u_{12}u_{21}-3u_{11}u_{22})+u_{21}V_{10}\sqrt{k_2}\right)}{6\hbar^2u_{11}\,(k_2)^{7/4}}\\ &\quad-\frac{\cos\left(2\sqrt{k_1}x\right)\ \sqrt{c_2}m_2\left(u_{21}\left(2c_2\ u_{12}+V_{10}\sqrt{k_2}\right)-3u_{11}V_{20}\sqrt{k_2}\right)}{6\hbar^2\,(k_1-k_2)\,u_{11}\,(k_2)^{3/4}}\Bigg)\\ &\quad+O(\varepsilon^2).\end{aligned}$$

Then, by a straightforward computation we have

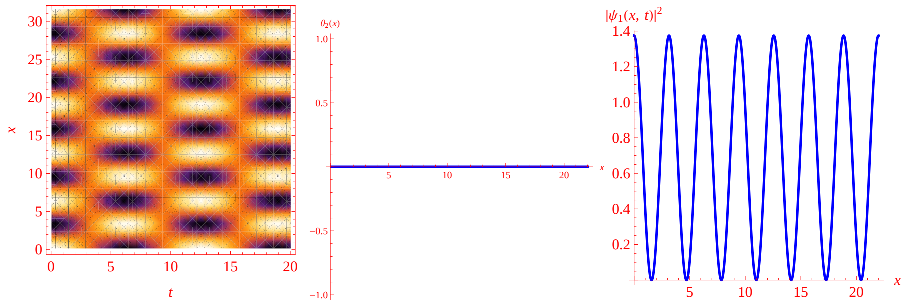
$$\begin{aligned}\operatorname{rot}(R_2)(c_2,\varepsilon)&=\frac{c_2}{\beta_2^2}-\frac{\left(c_2\left(-3g_{11}g_{22}\beta_2^2+g_{21}\left(\bar{V}_{10}+2g_{12}\beta_2^2\right)\right)\right)\varepsilon}{6\left(g_{11}k_2\beta_2^2\right)}+O(\varepsilon)^2\\ &=\sqrt{k_2}+\frac{\varepsilon m_2\left(c_2\left(-2u_{12}u_{21}+3u_{11}u_{22}\right)-u_{21}V_{10}\sqrt{k_2}\right)}{3\hbar^2k_2u_{11}}+O(\varepsilon)^2.\end{aligned}\tag{2.21}$$

Let

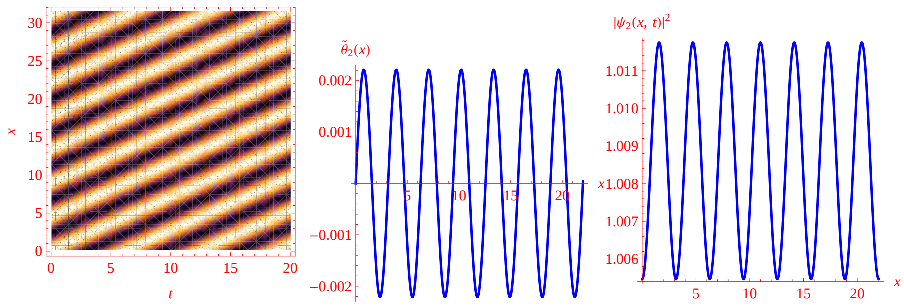
$$\begin{aligned}\tilde{\theta}_2(x,\varepsilon)&=\int_0^x\left(\frac{c_2}{R^2(x,\varepsilon)}-\operatorname{rot}(R_2)\right)\mathrm{d}x\\ &=\frac{\sin\left(2\sqrt{k_1}x\right)c_2\left(-3g_{11}\bar{V}_{20}+g_{21}\left(\bar{V}_{10}+2g_{12}\beta_2^2\right)\right)\varepsilon}{12g_{11}\sqrt{k_1}\,(k_1-k_2)\,\beta_2^2}+O(\varepsilon)^2\\ &=\frac{\varepsilon\sin\left(2\sqrt{k_1}x\right)m_2\left(u_{21}\left(2c_2\ u_{12}+\sqrt{k_2}V_{10}\right)-3\sqrt{k_2}u_{11}V_{20}\right)}{6\hbar^2\sqrt{k_1}\,(-k_1+k_2)\,u_{11}}+O(\varepsilon)^2,\end{aligned}$$

then we can verify that $\tilde{\theta}_2(x,\varepsilon)$ is a T-periodic function with respect to x. The coherent structure given by (1.2) can be written as

$$\begin{aligned}\psi_1(x,t)&=R_1(x)\exp[-i\mu_2t)],\\ \psi_2(x,t)&=R_2(x)\exp[i(\tilde{\theta}_2(x)+\operatorname{rot}(R_2)x-\mu_2t)].\end{aligned}$$



(a)



(b)

FIG. 2. An example of evolution of a MAW for (2.2) in a two-component BEC, where the parameters are $k_1 = 1, k_2 = 2, c_2 = \sqrt{2}, u_{11} = 0.97, u_{12} = u_{21} = 1, u_{22} = 1.03, V_{10} = -4, V_{20} = -1, \varepsilon = 0.01, \hbar = 1, m_1 = m_2 = 1$. (Recall that the sign of ε can be chosen arbitrarily.) (a) The left subplot shows the spatio-temporal evolution of $\mathrm{Re}(\psi_1)$ by mens of contour plots. The middle subplot and the right subplot displays the evolution of phase and amplitude, respectively. (b) The same as in (a) for ψ_2. We restate that the phase of the first component ψ_1 is trivial while the phase of the second component ψ_2 is non-trivial.

If $\mathrm{rot}(R_2)(c_2,\varepsilon) = m\sqrt{k_1}/n$, $\varepsilon \in (0, \varepsilon(c_2))$ and $c_2 \in (0, c_*)$ for some $m, n \in \mathbb{N}$ such that $(m,n) = 1$, that is,

$$\frac{2\pi}{\mathrm{rot}(R_2)} = \frac{n}{m}T,$$

then $\psi_2(x,t)$ is mT-periodic with respect to x. Therefore, $\Psi(x,t) = (\psi_1(x,t),\ \psi_2(x,t))$ is a MAW with the period mT. By the density of real numbers, for each $c_2 \in (0, c_*)$, the existence of MAWs for system (2.2) holds almost everywhere for $\varepsilon \in (0, \varepsilon(c_2))$. □

As an example, the evolution of a MAW for (2.2) is constructed with the parameters: $k_1 = 1, k_2 = 2, c_2 = \sqrt{2}, u_{11} = 0.97, u_{12} = u_{21} = 1, u_{22} = 1.03, V_{10} = -4, V_{20} = -1, \varepsilon = 0.01, \hbar = 1, m_1 = m_2 = 1$, see FIG. 2.

THEOREM 2.3 Assume that $V_1(x) = \bar{V}_{10}\cos(2\sqrt{k_1}x)$, $V_2(x) = \bar{V}_{20}\cos(2\sqrt{k_1}x)$. Let k_2 be a positive constant such that $\sqrt{k_1/k_2}$ is an irrational number, and fix $\mu_1 = k_1\hbar/(2m_1)$ and $\mu_2 = k_2\hbar/(2m_2)$. If $u_{11}u_{12} < 0$ and $V_{10} \neq 0$, then for each $c_2 \in (c_*, +\infty)$ with

$$c_* := \frac{|V_{10}|\sqrt{k_2}}{2|u_{12}|},$$

the conclusion of Theorem 2.2 holds.

Proof. Since

$$g_{11}\left(\bar{V}_{10}+2g_{12}\beta_2^2\right)=\frac{4m_1^2u_{11}}{\hbar^4\sqrt{k_2}}\left(2c_2u_{12}+V_{10}\sqrt{k_2}\right),$$

and $u_{11}u_{12}<0$, it is easy to check that $c_2>c_*$ implies the inequality $g_{11}(\bar{V}_{10}+2g_{12}\beta_2^2)<0$. Therefore, Theorem 2.1 (a) can be applied again, and the rest of the proof continues as in Theorem 2.2. □

In the latter results, we have used the first branch of solutions identified in Theorem 2.1 (a). By using the second branch found in Theorem 2.1 (b), we have two parallel results.

THEOREM 2.4 Assume that $V_1(x)=V_{10}\cos(2\sqrt{k_1}x)$, $V_2(x)=V_{20}\cos(2\sqrt{k_1}x)$. Let k_2 be a positive constant such that $\sqrt{k_1/k_2}$ is an irrational number, and fix $\mu_1=k_1\hbar/(2m_1)$ and $\mu_2=k_2\hbar/(2m_2)$. If $u_{11}V_{10}>0$, then for each $c_2\in(0,c_*)$ with

$$c_*:=\frac{|V_{10}|\sqrt{k_2}}{2|u_{12}|},$$

there exists $\varepsilon(c_2)>0$ such that system (2.2) has a modulated amplitude wave (ψ_1,ψ_2) of the form

$$\psi_j(x,t,\varepsilon)=R_j(x,\varepsilon)\exp[i(\theta_j(x,\varepsilon)-\mu_jt)],\ i=1,2$$

for almost every $|\varepsilon|\in(0,\varepsilon(c_2))$, where $\psi_i(x,t,\varepsilon)$ is periodic with respect to x, and

$$\begin{aligned}
R_1(x,\varepsilon)&=\sqrt{\frac{2V_{10}\sqrt{k_2}-4c_2u_{12}}{3u_{11}\sqrt{k_2}}}\sin\left(\sqrt{k_1}x\right)+O(\varepsilon),\\
\theta_1(x,\varepsilon)&\equiv\theta_{10}=0,\\
R_2(x,\varepsilon)&=\left(\frac{c_2^2}{k_2}\right)^{1/4}+\varepsilon\left(\frac{\left(\frac{c_2^2}{k_2}\right)^{1/4}m_2\left(-2\sqrt{\frac{c_2^2}{k_2}}u_{12}u_{21}+3\sqrt{\frac{c_2^2}{k_2}}u_{11}u_{22}+u_{21}V_{10}\right)}{6\hbar^2k_2u_{11}}\right.\\
&\quad\left.+\frac{\cos\left(2\sqrt{k_1}x\right)\left(\frac{c_2^2}{k_2}\right)^{1/4}m_2\left(u_{21}\left(-2\sqrt{\frac{c_2^2}{k_2}}u_{12}+V_{10}\right)-3u_{11}V_{20}\right)}{6\hbar^2\left(k_1-k_2\right)u_{11}}\right)\\
&\quad+O(\varepsilon^2)\\
\theta_2(x,\varepsilon)&=\frac{\varepsilon\sin\left(2\sqrt{k_1}x\right)m_2\left(u_{21}\left(V_{10}\sqrt{k_2}-2c_2u_{12}\right)-3u_{11}V_{20}\sqrt{k_2}\right)}{6\hbar^2\sqrt{k_1}\ (-k_1+k_2)u_{11}}+\\
&\quad+\mathrm{rot}(R_2)(c_2,\varepsilon)x+O(\varepsilon^2)
\end{aligned}$$

with

$$\mathrm{rot}(R_2)(c_2,\varepsilon)=\sqrt{k_2}+\frac{m_2\left(c_2\left(2u_{12}u_{21}-3u_{11}u_{22}\right)-u_{21}V_{10}\sqrt{k_2}\right)\varepsilon}{3\hbar^2k_2u_{11}}+O(\varepsilon^2).$$

Proof. In this case,

$$g_{11}\left(\bar{V}_{10}-2g_{12}\beta_2^2\right)=\frac{4m_1^2u_{11}}{\hbar^4\sqrt{k_2}}\left(-2c_2\,u_{12}+V_{10}\sqrt{k_2}\right).$$

If $u_{11}V_{10}>0$, we select the constant c_2 of integration from the interval $(0,c_*)$ with

$$c_*:=\frac{|V_{10}|\sqrt{k_2}}{2|u_{12}|},$$

so that $g_{11}\left(\bar{V}_{10}-2g_{12}\beta_2^2\right)>0$. Therefore, by applying Theorem 2.1 (b), for every $c_2\in(0,c_*)$, there exists $\varepsilon(c_2)>0$ such that for $|\varepsilon|<\varepsilon(c_2)$, system (2.9) has a T-periodic solution with the assymptotic approximation shown in Subsection 2.3.2. The rest of proof is analogous to the proof of Theorem 2.2. □

THEOREM 2.5 Assume that $V_1(x)=V_{10}\cos(2\sqrt{k_1}x)$, $V_2(x)=V_{20}\cos(2\sqrt{k_1}x)$. Let k_2 be a positive constant such that $\sqrt{k_1/k_2}$ is an irrational number ,and fix $\mu_1=k_1\hbar/(2m_1)$ and $\mu_2=k_2\hbar/(2m_2)$. If $u_{11}u_{12}<0$ and $V_{10}\neq 0$, then for each $c_2\in(c_*,+\infty)$ with

$$c_*:=\frac{|V_{10}|\sqrt{k_2}}{2|u_{12}|},$$

the conclusion of Theorem 2.4 holds.

Proof. If $u_{11}u_{12}<0$ and $V_{10}\neq 0$, taking $c_2>c_*$, one has $g_{11}\left(\bar{V}_{10}-2g_{12}\beta_2^2\right)>0$ and the remaining proof is the same. □

3. Extension to three–component Bose-Einstein condensates

To evince the generality of the above approach, we briefly consider its extension to a BEC model of three hyperfine states coupled by two different microwave fields, which is also a physically relevant situation. The system under study is

$$\begin{cases} i\hbar\dfrac{\partial\psi_1}{\partial t}=-\dfrac{\hbar^2}{2m_1}\nabla^2\psi_1+\varepsilon V_1(x)\psi_1+\varepsilon u_{11}|\psi_1|^2\psi_1+\varepsilon u_{12}|\psi_2|^2\psi_1+\varepsilon u_{13}|\psi_3|^2\psi_1,\\ i\hbar\dfrac{\partial\psi_2}{\partial t}=-\dfrac{\hbar^2}{2m_2}\nabla^2\psi_2+\varepsilon V_2(x)\psi_2+\varepsilon u_{21}|\psi_2|^2\psi_2+\varepsilon u_{22}|\psi_1|^2\psi_2+\varepsilon u_{23}|\psi_3|^2\psi_2,\\ i\hbar\dfrac{\partial\psi_3}{\partial t}=-\dfrac{\hbar^2}{2m_3}\nabla^2\psi_3+\varepsilon V_3(x)\psi_1+\varepsilon u_{31}|\psi_1|^2\psi_3+\varepsilon u_{32}|\psi_2|^2\psi_3+\varepsilon u_{33}|\psi_3|^2\psi_3, \end{cases}\tag{3.1}$$

where $V_i(x)=\tilde{V}_{i0}\cos(2\sqrt{k_1}x)\,(i=1,2,3)$ are the sinusoidal OL potentials considered before.

Given a solution $\Psi=(\psi_1,\psi_2,\psi_3)$, if one of the components is identically zero then the remaining components are solutions of a binary BEC. Therefore, to avoid trivialities and cases yet studied in Section 2, we look for solutions with every component different from zero. As in the two-component case, we start with the general ansantz

$$\psi_j(x,t)=R_j(x)\exp[i(\theta_j(x)-\mu_jt)],\ j=1,2,3$$

for solutions of coupled GP equation (3.1), where $\mu_1=2k_1\hbar/(2m_1)$, μ_2 and μ_3 are arbitrary constants, and

$$\theta_i(x)=\int_0^x\frac{c_i}{R_i^2(x')}\mathrm{d}x',\ i=1,2,3$$

with arbitrary constants c_1, c_2 and c_3. Without loss of generality, we assume that $c_i \geqslant 0$ for $i = 1,2,3$. Then we arrive at the following equations

$$\begin{cases} R_1' = S_1, \\ S_1' = -k_1 R_1 + \dfrac{c_1^2}{R_1^3} + \varepsilon F_1(R_1,R_2,R_3), \\ R_2' = S_2, \\ S_2' = -k_2 R_2 + \dfrac{c_2^2}{R_2^3} + \varepsilon F_2(R_1,R_2,R_3), \\ R_3' = S_3, \\ S_3' = -k_3 R_3 + \dfrac{c_3^2}{R_3^3} + \varepsilon F_3(R_1,R_2,R_3), \end{cases} \tag{3.2}$$

where

$$F_i(x,R_1,R_2,R_3) = -(V_{i0}\cos(2\sqrt{k_1}x)R_i + \sum_{j=1}^{3} g_{ij}R_j^2 R_i);$$

$$g_{ij} = -\frac{2m_i u_{ij}}{\hbar^2}, \quad k_i = \frac{2m_i \mu_i}{\hbar}, \quad V_{i0} = -\frac{2m_i \tilde{V}_{i0}}{\hbar^2}, i,j = 1,2,3.$$

Along this section, we assume as standing hypothesis that $c_1 = 0$, $c_2, c_3 > 0$ and

$$\sqrt{k_1/k_2}, \sqrt{k_1/k_3} \in \mathbb{R} \setminus \mathbb{Q}.$$

Let $r_1 > 0$ be arbitrarily small and $r_2 > 0$ be arbitrarily large. Define an open and bounded subset V on the projection subspace R_1OS_1 by

$$V = \left\{\alpha \in \mathbb{R}^2 : \alpha = (R_{10}, S_{10}) \text{ and } r_1 < R_{10}^2 + S_{10}^2 < r_2\right\}.$$

Let $\beta_2 = \sqrt[4]{c_2^2/k_2}$ and $\beta_3 = \sqrt[4]{c_3^2/k_3}$. The function $\beta : \overline{V} \to \mathbb{R}^4$ is taken by $\beta(\alpha) \equiv (\beta_2, 0, \beta_3, 0), \alpha \in \overline{V}$. Consequently, we define the set $\mathscr{Z}$ by

$$\begin{aligned} \mathscr{Z} &= \left\{z_\alpha = (\alpha, \beta(\alpha)) : \alpha \in \overline{V}\right\} \\ &= \left\{(R_{10}, S_{10}, \beta_2, 0, \beta_3, 0) \in \mathbb{R}^6 : r_1 < R_{10}^2 + S_{10}^2 < r_2\right\}. \end{aligned}$$

Obviously, for each $z_\alpha \in \mathscr{Z}$ and $\varepsilon = 0$, the solution $R_\alpha(x; z_\alpha)$ of (3.2) with the initial value $R_\alpha(0; z_\alpha) = z_\alpha$ is T-periodic with respect to x with the least period $T = 2\pi/\sqrt{k_1}$. In fact, every solution of (3.2) with $\varepsilon = 0$ starting from $\mathscr{Z}$ can be obtained as an explicit formulation $R_\alpha(x; z_\alpha) = (R_{\alpha 1}(x; z_\alpha), S_{\alpha 1}(x; z_\alpha),$ $R_{\alpha 2}(x; z_\alpha), S_{\alpha 2}(x; z_\alpha), R_{\alpha 3}(x; z_\alpha), S_{\alpha 3}(x; z_\alpha))$, where

$$\begin{aligned} R_{\alpha 1}(x) &= R_{10}\cos(\sqrt{k_1}x) + \frac{1}{\sqrt{k_1}} S_{10}\sin(\sqrt{k_1}x), \\ S_{\alpha 1}(x) &= S_{10}\cos(\sqrt{k_1}x) - \sqrt{k_1}R_{10}\sin(\sqrt{k_1}x), \\ R_{\alpha 2}(x) &= \beta_2, \quad S_{\alpha 2}(x) = 0, \\ R_{\alpha 3}(x) &= \beta_3, \quad S_{\alpha 3}(x) = 0. \end{aligned}$$

The variational equation of the unperturbed system along the periodic solution $R_\alpha(x;z_\alpha)$ is given by

$$Y' = D_R G_0(t, R_\alpha(x;z_\alpha))Y = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -k_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -4k_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -4k_3 & 0 \end{pmatrix} Y, \tag{3.3}$$

where Y is a 6×6 matrix. The fundamental matrix $M_{Z_\alpha}(x)$ of the differential system (3.3) such that $M_{Z_\alpha}(0)$ is the identity matrix of $\mathbb{R}^6$ takes the simple form

$$M_{Z_\alpha}(x) = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{pmatrix},$$

where

$$A_1 = \begin{pmatrix} \cos(\sqrt{k_1}x) & \frac{1}{\sqrt{k_1}}\sin(\sqrt{k_1}x) \\ -\sqrt{k_1}\sin(\sqrt{k_1}x) & \cos(\sqrt{k_1}x) \end{pmatrix},$$

$$A_i = \begin{pmatrix} \cos(2\sqrt{k_i}x) & \frac{1}{2\sqrt{k_i}}\sin(2\sqrt{k_i}x) \\ -2\sqrt{k_i}\sin(2\sqrt{k_i}x) & \cos(2\sqrt{k_i}x) \end{pmatrix}, i = 2,3.$$

Note that the fundamental matrix $M_{z_\alpha}(x)$ does not depend on the initial value z_α of the periodic solution $R(x, z_\alpha)$. Therefore, we drop the subscript z_α and reset $M(x) = M_{Z_\alpha}(x)$. With an easy computation, we obtain that the matrix $M^{-1}(0) - M^{-1}(T)$ is given by

$$M^{-1}(0) - M^{-1}(T) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & B_1 & 0 \\ 0 & 0 & B_2 \end{pmatrix},$$

where

$$B_{i-1} = \begin{pmatrix} 1-\cos\left(\frac{4\pi\sqrt{k_i}}{\sqrt{k_1}}\right) & \frac{1}{2\sqrt{k_1}}\sin\left(\frac{4\pi\sqrt{k_i}}{\sqrt{k_1}}\right) \\ -2\sqrt{k_1}\sin\left(\frac{4\pi\sqrt{k_i}}{\sqrt{k_1}}\right) & 1-\cos\left(\frac{4\pi\sqrt{k_i}}{\sqrt{k_1}}\right) \end{pmatrix}, i = 2,3.$$

We observe that the matrix $M^{-1}(0) - M^{-1}(T)$ has in the upper right corner the 2×2 zero matrix, while in the lower right corner a 4×4 matrix Δ_α satisfies

$$\det(\Delta_\alpha) = 16\sin^2\left(\frac{2\pi\sqrt{k_2}}{\sqrt{k_1}}\right)\sin^2\left(\frac{2\pi\sqrt{k_3}}{\sqrt{k_1}}\right) \neq 0,$$

since $\sqrt{k_1}/\sqrt{k_2}, \sqrt{k_1}/\sqrt{k_3}$ are all irrational. Consequently, the assumptions (i) and (ii) of Theorem A.1 are satisfied.

In the notation of Theorem A.1, let us define $\mathrm{P}(R_1,S_1,R_2,S_2,R_3,S_3)=(R_1,S_1)$. By a straightforward computation, we obtain that

$$\begin{aligned}\mathscr{G}(\alpha)=\mathscr{G}(R_{10},S_{10})&=\mathrm{P}\left(\int_0^T M_{Z_\alpha}^{-1}(x)G_1(x,R_\alpha(x;Z_\alpha))\mathrm{d}x\right)\\ &=\begin{pmatrix}\dfrac{\pi S_{10}}{4k_1^{5/2}}\left[3S_{10}^2g_{11}+k_1\left(-2V_{10}+3R_{10}^2g_{11}+4\beta_2^2g_{12}+4\beta_3^2g_{13}\right)\right]\\ -\dfrac{\pi R_{10}}{4k_1^{3/2}}\left[3S_{10}^2g_{11}+k_1\left(2V_{10}+3R_{10}^2g_{11}+4\beta_2^2g_{12}+4\beta_3^2g_{13}\right)\right]\end{pmatrix}.\end{aligned}$$

It is easy to compute the simple zeroes of the function $\mathscr{G}(\alpha)$ in the open and bounded subset $V\subset\mathbb{R}^2$. Besides the trivial zero $\alpha_1=(0,0)$, we have

(i) $\alpha_{2,3}=\left(\pm\sqrt{2V_{10}+4\left(g_{12}\beta_2^2+g_{13}\beta_3^2\right)/(-3g_{11})},0\right)$, if $(V_{10}+2(g_{12}\beta_2^2+g_{13}\beta_3^2))g_{11}<0$. Moreover, if $V_{10}\neq 0$, the corresponding Jacobian determinant is

$$\mathrm{Det}(\mathscr{G}'(\alpha_{2,3}))=\frac{\pi^2V_{10}\left(V_{10}+2\left(g_{12}\beta_2^2+g_{13}\beta_3^2\right)\right)}{k_1^2}\neq 0.$$

(ii) (iii) $\alpha_{4,5}=\left(0,\pm\sqrt{2k_1\left(V_{10}-2\left(g_{12}\beta_2^2+g_{13}\beta_3^2\right)\right)/(3g_{11})}\right)$, if $(V_{10}-2(g_{12}\beta_2^2+g_{13}\beta_3^2))g_{11}>0$. Moreover, if $V_{10}\neq 0$, the corresponding Jacobian determinant is

$$\mathrm{Det}(\mathscr{G}'(\alpha_{4,5}))=\frac{\pi^2V_{10}\left(V_{10}-2\left(g_{12}\beta_2^2+g_{13}\beta_3^2\right)\right)}{k_1^2}\neq 0.$$

By application of Theorem A.1, we find a branch of periodic solutions starting from α_2, and by means of an asymptotic analysis like that of Subsection 2.3, we can formulate the following result. The proof is done by mimicking the arguments used in Section 2, we skip detailed computations for conciseness.

THEOREM 3.1 Assume that $V_i(x)=\tilde{V}_{i0}\cos(2\sqrt{k_1}x)$, $i=1,2,3$ and fix positive constants k_2,k_3 such that $\sqrt{k_1/k_2},\sqrt{k_1/k_3}\in\mathbb{R}\setminus\mathbb{Q}$. Let

$$\mu_i=\frac{k_i\hbar}{2m_i},\quad \beta_i=\sqrt[4]{\frac{c_i^2}{k_i}},\quad g_{ij}=-\frac{2m_iu_{ij}}{\hbar^2},\quad V_{i0}=-\frac{2m_i\tilde{V}_{i0}}{\hbar^2},\, i,j=1,2,3.$$

Assume that $\left(V_{10}+2\left(g_{12}\beta_2^2+g_{13}\beta_3^2\right)\right)g_{11}<0$ and $V_{10}\neq 0$. Then, for each $c_2,c_3\in(0,+\infty)$, there exists $\varepsilon(c_2,c_3)>0$ such that system (3.1) has one modulated amplitude wave

$$\psi_j(x,t,\varepsilon)=R_j(x,\varepsilon)\exp[i(\theta_j(x,\varepsilon)-\mu_jt)],\ j=1,2,3$$

for almost every $|\varepsilon| \in (0, \varepsilon(c_2, c_3))$, where $\psi_j(x,t,\varepsilon)$ is periodic with respect to x, and

$$
\begin{aligned}
R_1(x,\varepsilon) &= \sqrt{\frac{2V_{10}+4\left(g_{12}\beta_2^2+g_{13}\beta_3^2\right)}{-3g_{11}}}\cos\left(\sqrt{k_1}x\right)+O(\varepsilon)\\
\theta_1(x,\varepsilon) &\equiv \theta_0 = 0,\\
R_2(x,\varepsilon) &= \beta_2+\varepsilon\frac{\beta_2}{12}\left(\frac{g_{21}\left(V_{10}+2g_{12}\beta_2^2+2g_{13}\beta_3^2\right)-3g_{11}\left(g_{22}\beta_2^2+g_{23}\beta_3^2\right)}{g_{11}k_2}\right.\\
&\qquad\left.+\frac{3g_{11}V_{30}-g_{21}\left(V_{10}+2g_{12}\beta_2^2+2g_{13}\beta_3^2\right)}{g_{11}\left(k_1-k_2\right)}\cos\left(2\sqrt{k_1}x\right)\right)+O(\varepsilon^2),\\
\theta_2(x,\varepsilon) &= \frac{\sin\left(2\sqrt{k_1}x\right)c_2\left(-3g_{11}V_{30}+g_{21}\left(V_{10}+2g_{12}\beta_2^2+2g_{13}\beta_3^2\right)\right)\varepsilon}{12g_{11}\sqrt{k_1}\left(k_1-k_2\right)\beta_2^2}\\
&\qquad+\mathrm{rot}(R_2)(c_2,c_3,\varepsilon)x+O(\varepsilon^2),\\
R_3(x,\varepsilon) &= \beta_3+\varepsilon\frac{\beta_3}{12}\left(\frac{g_{31}\left(V_{10}+2g_{12}\beta_2^2+2g_{13}\beta_3^2\right)-3g_{11}\left(g_{32}\beta_2^2+g_{33}\beta_3^2\right)}{g_{11}k_3}\right.\\
&\qquad\left.+\frac{3g_{11}V_{30}-g_{31}\left(V_{10}+2g_{12}\beta_2^2+2g_{13}\beta_3^2\right)}{g_{11}\left(k_1-k_3\right)}\cos\left(2\sqrt{k_1}x\right)\right)+O(\varepsilon^2),\\
\theta_3(x,\varepsilon) &= \frac{\sin\left(2\sqrt{k_1}x\right)c_3\left(-3g_{11}V_{30}+g_{31}\left(V_{10}+2g_{12}\beta_2^2+2g_{13}\beta_3^2\right)\right)\varepsilon}{12g_{11}\sqrt{k_1}\left(k_1-k_3\right)\beta_3^2}\\
&\qquad+\mathrm{rot}(R_3)(c_2,c_3,\varepsilon)x+O(\varepsilon^2),
\end{aligned}
$$

with

$$
\begin{aligned}
\mathrm{rot}(R_2)(c_2,c_3,\varepsilon) &= \sqrt{k_2}-\frac{\left(c_2\left(g_{21}\left(V_{10}+2g_{12}\beta_2^2+2g_{13}\beta_3^2\right)-3g_{11}\left(g_{22}\beta_2^2+g_{23}\beta_3^2\right)\right)\right)\varepsilon}{6\left(g_{11}k_2\beta_2^2\right)}\\
&\qquad+O(\varepsilon^2),\\
\mathrm{rot}(R_3)(c_2,c_3,\varepsilon) &= \sqrt{k_3}-\frac{\left(c_3\left(g_{31}\left(V_{10}+2g_{12}\beta_2^2+2g_{13}\beta_3^2\right)-3g_{11}\left(g_{32}\beta_2^2+g_{33}\beta_3^2\right)\right)\right)\varepsilon}{6\left(g_{11}k_3\beta_3^2\right)}\\
&\qquad+O(\varepsilon^2).
\end{aligned}
$$

A second result is obtained by using now α_4.

THEOREM 3.2 Assume that $V_i(x) = \tilde{V}_{i0}\cos(2\sqrt{k_1}x)$, $i = 1,2,3$ and fix positive constants k_2, k_3 such that $\sqrt{k_1/k_2}, \sqrt{k_1/k_3} \in \mathbb{R}\setminus\mathbb{Q}$. Let

$$
\mu_i = \frac{k_i\hbar}{2m_i},\quad \beta_i = \sqrt[4]{\frac{c_i^2}{k_i}},\quad g_{ij} = -\frac{2m_iu_{ij}}{\hbar^2},\quad V_{i0} = -\frac{2m_i\tilde{V}_{i0}}{\hbar^2}, i,j = 1,2,3.
$$

Assume that $(V_{10}-2(g_{12}\beta_2^2+g_{13}\beta_3^2))g_{11} > 0$ and $V_{10} \neq 0$. Then, for each $c_2, c_3 \in (0,+\infty)$, there exists $\varepsilon(c_2,c_3) > 0$ such that system (3.1) has one modulated amplitude wave

$$
\psi_j(x,t,\varepsilon) = R_j(x,\varepsilon)\exp[i(\theta_j(x,\varepsilon)-\mu_jt)],\ j = 1,2,3
$$

for almost every $|\varepsilon| \in (0, \varepsilon(c_2, c_3))$, where $\psi_i(x,t,\varepsilon)$ is periodic with respect to x, and

$$
\begin{aligned}
R_1(x,\varepsilon) &= \sqrt{\frac{2\left(V_{10}-2\left(g_{12}\beta_2^2+g_{13}\beta_3^2\right)\right)}{3g_{11}}}\sin\left(\sqrt{k_1}x\right)+O(\varepsilon)\\
\theta_1(x,\varepsilon) &\equiv \theta_0 = 0,\\
R_2(x,\varepsilon) &= \beta_2+\varepsilon\frac{\beta_2}{12}\left(\frac{g_{21}\left(-V_{10}+2g_{12}\beta_2^2+2g_{13}\beta_3^2\right)-3g_{11}\left(g_{22}\beta_2^2+g_{23}\beta_3^2\right)}{g_{11}k_2}\right.\\
&\qquad\left.+\frac{3g_{11}V_{30}+g_{21}\left(-V_{10}+2g_{12}\beta_2^2+2g_{13}\beta_3^2\right)}{g_{11}\left(k_1-k_2\right)}\cos\left(2\sqrt{k_1}x\right)\right)+O(\varepsilon^2),\\
\theta_2(x,\varepsilon) &= \varepsilon\frac{c_2\left(-3g_{11}V_{30}+g_{21}\left(V_{10}-2\left(g_{12}\beta_2^2+g_{13}\beta_3^2\right)\right)\right)}{12g_{11}\sqrt{k_1}\left(k_1-k_2\right)\beta_2^2}\sin\left(2\sqrt{k_1}x\right)\\
&\qquad+\operatorname{rot}(R_2)(c_2,\varepsilon)x+O(\varepsilon^2),
\end{aligned}
$$

$$
\begin{aligned}
R_3(x,\varepsilon) &= \beta_3+\varepsilon\frac{\beta_3}{12}\left(\frac{g_{31}\left(-V_{10}+2g_{12}\beta_2^2+2g_{13}\beta_3^2\right)-3g_{11}\left(g_{32}\beta_2^2+g_{33}\beta_3^2\right)}{g_{11}k_3}\right.\\
&\qquad\left.+\frac{3g_{11}V_{30}+g_{31}\left(-V_{10}+2g_{12}\beta_2^2+2g_{13}\beta_3^2\right)}{g_{11}\left(k_1-k_3\right)}\cos\left(2\sqrt{k_1}x\right)\right)+O(\varepsilon^2),\\
\theta_3(x,\varepsilon) &= \frac{\sin\left(2\sqrt{k_1}x\right)c_3\left(-3g_{11}V_{30}+g_{31}\left(V_{10}-2\left(g_{12}\beta_2^2+g_{13}\beta_3^2\right)\right)\right)\varepsilon}{12g_{11}\sqrt{k_1}\left(k_1-k_3\right)\beta_3^2}\\
&\qquad+\operatorname{rot}(R_3)(c_3,\varepsilon)x+O(\varepsilon^2),
\end{aligned}
$$

with

$$
\begin{aligned}
\operatorname{rot}(R_2)(c_2,\varepsilon) &= \sqrt{k_2}+\frac{c_2\left(3g_{11}\left(g_{22}\beta_2^2+g_{23}\beta_3^2\right)+g_{21}\left(V_{10}-2\left(g_{12}\beta_2^2+g_{13}\beta_3^2\right)\right)\right)\varepsilon}{6g_{11}k_2\beta_2^2}\\
&\qquad+O(\varepsilon^2),\\
\operatorname{rot}(R_3)(c_3,\varepsilon) &= \sqrt{k_3}+\frac{c_3\left(3g_{11}\left(g_{32}\beta_2^2+g_{33}\beta_3^2\right)+g_{31}\left(V_{10}-2\left(g_{12}\beta_2^2+g_{13}\beta_3^2\right)\right)\right)\varepsilon}{6g_{11}k_3\beta_3^2}\\
&\qquad+O(\varepsilon^2).
\end{aligned}
$$

4. Strong interactions

Due to the presence of the small parameter on the cubic terms, system (2.2) models a binary BEC with weak atomic interactions. However, note that if (ψ_1, ψ_2) is a solution of (2.2), then

$$
(\tilde{\psi}_1, \tilde{\psi}_2) = \left(\psi_1 \exp[-i\frac{V_{01}}{\hbar}t]/\sqrt{\varepsilon}, \psi_2 \exp[-i\frac{V_{02}}{\hbar}t]/\sqrt{\varepsilon}\right)
$$

solves the system with strong interactions

$$\begin{cases} i\hbar\dfrac{\partial\tilde{\psi}_1}{\partial t} = -\dfrac{\hbar^2}{2m_1}\nabla^2\tilde{\psi}_1 + [V_{01}+\varepsilon V_1(x)]\tilde{\psi}_1 + u_{11}|\tilde{\psi}_1|^2\tilde{\psi}_1 + u_{12}|\tilde{\psi}_2|^2\tilde{\psi}_1, \\ i\hbar\dfrac{\partial\tilde{\psi}_2}{\partial t} = -\dfrac{\hbar^2}{2m_2}\nabla^2\tilde{\psi}_2 + [V_{02}+\varepsilon V_2(x)]\tilde{\psi}_2 + u_{21}|\tilde{\psi}_1|^2\tilde{\psi}_2 + u_{22}|\tilde{\psi}_2|^2\tilde{\psi}_2. \end{cases} \tag{4.1}$$

Therefore, the main results of Section 2 have a direct interpretation for BECs with strong particle interactions and general OL potentials. Obviously, the asymptotic profiles of the solutions remain valid just dividing the radial coordinates by $\sqrt{\varepsilon}$. The rescalling argument is valid for a general multi-component BEC.

5. Conclusions and further remarks

In this paper, we investigate the existence of modulated amplitude waves in quasi-one-dimensional condensate mixtures in the presence of a external periodic potential. Mathematically, such coherent structures are doubly periodic solutions, in space and time, of a coupled system of Gross-Pitaevskii equations. For a binary BEC, the weak interaction regime is tackled by means of the averaging method and regular perturbation theory. The case of strong particle interaction is covered by a simple rescalling argument. The key feature that distinguish our results from earlier ones is that one of the wave functions is stationary, while the second component has non-trivial phase, meaning that there is circulation of matter only of the second component. A second novelty with respect to similar works (see for instance Porter *et al.* (2004)) is that the chemical potentials μ_j of each component may be different. Analogous results for the three-component BEC are presented as well.

In future works, we would like to find a rigourous way to identify modulated amplitude waves with non-trivial phases in all the components. As it is pointed out in Remark 2.2, some technical issues must be solved. Also, it would be desirable to perform a stability analysis of the coherent structures obtained in our results, as it is done for instance in (Deconinck *et al.* , 2003, Section 4). Such study is difficult by the fact that an exact formulation of the solutions is not available, only asymptotic profiles, but specially by the presence of non-trivial phases.

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A. Some results from averaging theory

We present a result from the averaging theory which we shall need for proving our main results of this paper.

We consider the problem of the bifurcation of $T-$periodic solutions from differential systems of the form

$$\dot{X}(t) = G_0(t,X) + \varepsilon G_1(t,X) + \varepsilon^2 G_2(t,X,\varepsilon), \tag{A.1}$$

with $\varepsilon \neq 0$ sufficiently small, where $G_0, G_1 : \mathbb{R}\times\Omega \to \mathbb{R}^n$ and $G_2 : \mathbb{R}\times\Omega\times(-\varepsilon_0,\varepsilon_0) \to \mathbb{R}^n$ are C^2 functions with T-periodic dependence in the first variable, and Ω is an open subset of $\mathbb{R}^n$.The main assumption is that the unperturbed system with $\varepsilon = 0$

$$\dot{X}(t) = G_0(t,X) \tag{A.2}$$

has a submanifold of periodic solutions.

We denote the linearization of the unperturbed system (A.2) by

$$\dot{y}(t) = U(t,z)y, \tag{A.3}$$

where

$$U(t,z) = D_X G_0(t, X(t,z)), \tag{A.4}$$

and $X(t,z)$ is the solution of the system (A.2) such that $X(0,z) = z$. In what follows, we denote by $M_z(t)$ the fundamental matrix of the linear differential system (A.3), and denote by $P : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ the projection of $\mathbb{R}^n$ onto its first k coordinates; i.e., $P(x_1,\cdots,x_n) = (x_1,\cdots,x_k)$.

We assume that there exists a $k-$dimensional submanifold $\mathscr{Z}$ of Ω filled with T -periodic solutions of (A.2). Then an answer to the problem of bifurcation of T -periodic solutions from the periodic solutions contained in $\mathscr{Z}$ for system (A.1) is given in the following result.

THEOREM A.1 Let $\beta : \overline{V} \to \mathbb{R}^{n-k}$ be a C^2 function, where V be an open and bounded subset of $\mathbb{R}^k$. We assume that

(i) $\mathscr{Z} = \{z_\alpha = (\alpha, \beta(\alpha)), \alpha \in \overline{V}\} \subset \Omega$ and that for each $z_\alpha \in \mathscr{Z}$ the solution $X(t, z_\alpha)$ of (A.2) is T-periodic;

(ii) for each $z_\alpha \in \mathscr{Z}$, there is a fundamental matrix $M_{z_\alpha}(t)$ of $(A.3)$ such that the matrix $M_{z_\alpha}^{-1}(0) - M_{z_\alpha}^{-1}(T)$ has in the upper right corner the $k \times (n-k)$ zero matrix, while in the lower right corner a $(n-k) \times (n-k)$ matrix Δ_α with $\det(\Delta_\alpha) \neq 0$.

We consider the function $\mathscr{G} : \overline{V} \to \mathbb{R}^k$

$$\mathscr{G}(\alpha) = P\left(\int_0^T M_{z_\alpha}^{-1}(t) G_1(t, X(t, z_\alpha))\mathrm{d}t\right). \tag{A.5}$$

If there exists $a \in V$ with $\mathscr{G}(a) = 0$ and $\det(\mathrm{d}\mathscr{G}(a)/\mathrm{d}\alpha) \neq 0$, then there is a T-periodic solution $X(t,\varepsilon)$ of system (A.1) such that $X(0,\varepsilon) \to z_a$ as $\varepsilon \to 0$.

For a proof of Theorem A.1 one can consult Malkin (1959) and Roseau (1966), or Buica *et al.* (2007) for a more recent and shorter proof.

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