Analogue of the Gellerstedt problem for a loaded equation of the third order*

Umida Baltaeva
National University of Uzbekistan

and

Pedro J. Torres
Department of Applied Mathematics and
Research Unit "Modelling Nature",
Universidad de Granada

Abstract. In this paper we study an analogue of the Gellerstedt problem for the third-order loaded equation with boundary conditions on non parallel characteristics in the hyperbolic domain of the equation.

Keywords. Equations of mixed type, loaded equation, boundary-value problem, gluing condition, integral equation.

1 Introduction

One of the main sections of the modern theory of partial differential equations is the theory of mixed type equations. Historically, the study of equations of mixed type was initiated by the pioneering works of F. Tricomi and S. Gellerstedt. Later, the works of M.A.Lavrent'ev, A.V.Bitsadze, M.S.Salakhitdinov, F.I.Frankl, M.Protter and J.M.Rassias had a considerable impact in this theory. In applications, equations of mixed type appear in a natural way in fluid mechanics and gas dynamics. For a detailed account of the classical theory and applications, the interested reader can consult the survey papers and monographies [1] - [5],[23] and the references therein.

The analysis of the third and higher order mixed and mixed-composite type equations with elliptic-hyperbolic or parabolic-hyperbolic operators began in the early seventies by A.V.Bitsadze, M.S.Salakhitdinov, T.D.Djuraev and others. In

*Partially supported by Spanish MICINN Grant with FEDER funds MTM2017-82348-C2-1-P.

particular, The works of M.S.Salakhitdinov and T.D.Djuraev, in 1971, investigate boundary-value problems for the third order equation with parabolic-hyperbolic operator in the form

$$\frac{\partial}{\partial x}L_k u = 0,\ (k = 1, 2),$$

where

$$L_1 u \equiv u_{xx} - \frac{1 - sgny}{2}u_{yy} - \frac{1 + sgny}{2}u_y,$$

$$L_2 u \equiv u_{xx} - \frac{1 + sgny + (1 - sgny)y}{2}u_{yy} - \frac{1 - sgny}{2}u_{yy} - \frac{1 + sgny}{2}u_y.$$

Also, the monographs of M.S.Salakhitdinov [6] and T.D.Djuraev, A.Sopuyev and M.Mamazhanov [7] investigated a number of well-posed local and nonlocal boundary-value problems for this model equation and for other equations with lower terms of the parabolic-hyperbolic and elliptic-hyperbolic type of the third order.

In the cited literature, the typical method to study boundary-value problems for the third-order equation is the representation of a general solution in the form of a sum of functions. This approach is efficient when equations are composed by products of permutating differential operators. In the present paper we use a different idea, i.e. reduction to second-order equations with unknown right-hand sides, which make it possible to solve boundary-value problems for generalized equations composed by a product of non-commuting differential operators [8], which are of actual interest for solving inverse problems of mechanics and physics. This observation is the basis for the study presented in this work.

The recent interest showed by the mathematical community in the study of fractional differential equations is due to their wide application in problems of physics, mechanics, control theory and other applied sciences [10]- [16]. A classical family of differential equations of fractional order is formed by the loaded equations [28]. The field of partial differential equations of fractional order is experiencing a sustained growing interest in the recent years, see for instance [17]- [22]. From the available bibliography, it is observed that the boundary-value problems for a third-order loaded differential equations containing mixed parabolic-hyperbolic and elliptic-parabolic operators in the principal part have been scarcely investigated [9]. On the other hand, it is well known that in connection with the important problems of partial differential equations of mixed type and gas dynamics of transonic currents, the construction of mathematical models describing various processes developed in media with a fractal structure leads to differential equations of fractional order.

Gellerstedt problems in the classical statement were studied in the monograph [23, p. 186]. The recent works [24]- [27] investigated the Gellerstedt problem for a mixed type equations. As far as we know, the Gellerstedt problem for a loaded mixed type equation of the third order has not been investigated before. The present paper is devoted to the formulation and investigation of an analogue problem to that of Gellerstedt, for a third-order equation with a loaded integro-differential operator in the form

$$0=\frac{\partial}{\partial x}\begin{cases} u_{xx}-u_y-\lambda_1 u-\mu_1\sum\limits_{i=1}^{n} D_{rx}^{\alpha_i}u(t,0), \ if \ \ y\geq 0,\\ u_{xx}-u_{yy}-\lambda_k u-\mu_k\sum\limits_{i=1}^{n} D_{r\xi}^{\beta_i}u(t,0), \ if \ y<0,\end{cases} \tag{1}$$

where $\xi=x+y$, λ_1,μ_1, λ_k,μ_k are given real parameters, and $\lambda_1>0$. The domain and boundary conditions will be explicited in Section 2. $D_{rx}^{\gamma_i}$ stands for the Riemann-Liouville fractional integral operator of order γ_i given by

$$D_{rx}^{\gamma_i}f(t)=\frac{sgn(x-r)}{\Gamma(-\gamma_i)}\int\limits_r^x\frac{f(t)dt}{(x-t)^{1+\gamma_i}}, \ \ \gamma_i<0,$$

where Γ is the Euler's Gamma function. This operator is often denoted by $I_{r+}^{-\gamma_i}$, if $x>r$ and $I_{r-}^{-\gamma_i}$, if $x<r$, called left-side and right-side Riemann-Liouville operator respectively [10]. From the basic theory of fractional derivatives [10], if $f\in L_1[a,b]$ and $\gamma_i<0$, then the integral exists almost everywhere and the function $D_{rx}^{\gamma_i}f$ belongs to $C[a,b]$.

The rest of the paper is organized as follows. In Section 2, we formulate the Gellerstedt problem under consideration and state the main result. Section 3 is devoted to derive some relevant functional relations. Finally, the proof of the main result is developed in Section 4.

2 Formulating the Gellerstedt problem.

Let us fix the points $A=(0,0), B=(1,0), A_0=(0,h), B_0=(1,h)$ in the plane of independent variables x and y. Let Ω be a simply connected domain bounded by

1) the segments BB_0, B_0A_0 and A_0A of the straight lines $x=1$, $y=h>0$ and $x=0$ respectively,

2) two segments of the characteristics of eq. (1) given by

$$x-y=r, \, x+y=r,$$

where $0\leq r\leq 1$ is fixed, starting at point $E=(r,0)$ to the points $C_1=(\frac{r}{2},-\frac{r}{2}), C_2=(\frac{r+1}{2},-\frac{r+1}{2})$ respectively, and

3) the segments AC_1, C_2B, belonging to the two characteristics $x+y=0, x-y=1$ respectively.

See Figure 1.

Let us introduce the following notation:

$$\Omega_1=\Omega\cap(y>0), \ \ \Omega_2=\Omega\cap\{(x,y):0<x<r, y<0\},$$

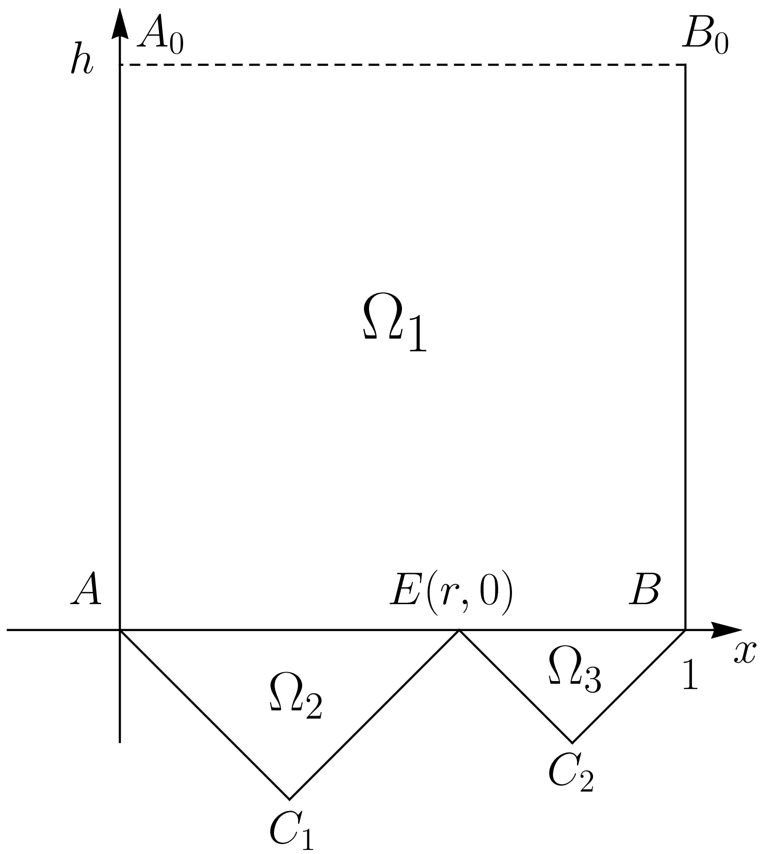

Figure 1: The domain Ω

$$\Omega_3 = \Omega \cap \{(x,y) : r < x < 1,\, y < 0\},$$
$$\gamma_1 = \{(x,y) : 0 < x < r,\, y = 0\}, \qquad \gamma_2 = \{(x,y) : r < x < 1,\, y = 0\},$$
$$\gamma_1' = \{(x,y) : x = 0,\ 0 < y < h\}, \qquad \gamma_2' = \{(x,y) : x = 1,\ 0 < y < h\},$$
$$\gamma_{11} = \left\{(x,y) : y = x - r,\ \frac{r}{2} < x < r\right\},$$
$$\gamma_{12} = \left\{(x,y) : y = r - x,\ r < x < \frac{r+1}{2}\right\},$$
$$\gamma_{13} = \left\{(x,y) : y = x - 1,\ \frac{r+1}{2} < x < 1\right\},$$

We consider an analog of the Gellerstedt problem for the loaded differential equation (1).

Problem $\mathbf{G}_1$. Find a function $u(x,y)$ possessing the following properties:

1. $u(x,y) \in C\left(\overline{\Omega}\right)$;
2. u_x (resp. u_y) is continuous up to $\gamma_1' \cup \gamma_j \cup \gamma_{1j}$ (resp. $\gamma_j \cup \gamma_{1j}$);
3. $u(x,y)$ is a regular solution of equation (1) in the domains $\Omega_1 \backslash \{x = r\}$, Ω_2 and Ω_3;
4. the gluing conditions
$$u_y(x,-0) = u_y(x,+0) \tag{2}$$
are satisfied on γ_j;
5. $u(x,y)$ satisfies the boundary-value conditions
$$u(x,y)|_{\bar{\gamma}_1'} = \varphi_1(y), \tag{3}$$
$$u_x(x,y)|_{\bar{\gamma}_1'} = \varphi_2(y), \tag{4}$$
$$u(x,y)|_{\bar{\gamma}_2'} = \varphi_3(y) \tag{5}$$
$$u(x,y)|_{\bar{\gamma}_{1j}} = \psi_j(x), \tag{6_j}$$
$$\left.\frac{\partial u(x,y)}{\partial n}\right|_{\bar{\gamma}_{1j}} = \psi_{2+j}(x), \tag{7_j}$$
where $\varphi_j(y), \varphi_3(y), \psi_j(x)$ and $\psi_{2+j}(x)$ are given real-valued functions such that $\psi_1(r) = \psi_2(r)$ and
$$\varphi_1(y) \in C^1\left(\bar{\gamma}_1'\right), \quad \varphi_2(y) = C\left(\bar{\gamma}_1'\right) \cap C^1\left(\gamma_1'\right), \quad \varphi_3(y) \in C^1\left(\bar{\gamma}_2'\right) \tag{8}$$
$$\psi_j(x) \in C^1\left(\bar{\gamma}_{1j}\right) \cap C^3\left(\gamma_{1j}\right),\ \psi_j(r) = 0, \tag{9_j}$$
$$\psi_{2+j}(x) \in C\left(\bar{\gamma}_{1j}\right) \cap C^2\left(\gamma_{1j}\right),\ (j = 1, 2). \tag{10_j}$$

Now, we are in disposition to state our main result.

Theorem 2.1. *If conditions* (8), (9_1), (9_2), (10_1) *and* (10_2) *hold, then there exists a unique solution to problem* G_1.

3 The main functional relations

Bearing in mind [9], after integration of the equation (1), with respect to x, we have the following form

$$u_{xx} - u_y - \lambda_1 u - \mu_1 \sum_{i=1}^{n} D_{rx}^{\alpha_i} u(t,0) = w_1(y) \quad \texttt{in} \quad \Omega_1 \tag{11}$$

$$u_{xx} - u_{yy} - \lambda_j u - \mu_k \sum_{i=1}^{n} D_{r\xi}^{\beta_i} u(t,0) = w_j(y) \quad \texttt{in} \quad \Omega_j \; (j = 2, 3), \tag{12_j}$$

where $w_k(y)$ are arbitrary continuous functions, and $\lambda_k > 0$ $(k = \overline{1,3})$.

The solution of the Cauchy problem for the equation (12_j), in Ω_j, with the conditions

$$u(x, -0) = \tau(x), \quad (x,0) \in \overline{\gamma_1 \cup \gamma_2}, \tag{13}$$

$$u_y(x, -0) = \nu(x), \quad (x,0) \in \gamma_1 \cup \gamma_2, \tag{14}$$

can be represented in the form

$$u(x,y) = \frac{1}{2}\left[\tau(x+y) + \tau(x-y)\right] + \frac{\lambda_j y}{2} \int\limits_{x+y}^{x-y} \tau(\xi)\, \bar{I}_1 \left[\sqrt{\lambda_j (\xi - x - y)(x - y - \xi)}\right] d\xi -$$

$$-\frac{1}{2} \int\limits_{x+y}^{x-y} \nu(\xi) I_0 \left[\sqrt{\lambda_j (\xi - x - y)(x - y - \xi)}\right] d\xi - \tag{15_j}$$

$$-\frac{1}{4} \int\limits_{x+y}^{x-y} d\xi \int\limits_{\xi}^{x-y} I_0 \left[\sqrt{\lambda_j (\xi - x - y)(\eta - x + y)}\right] \left(w_j\left(\frac{\xi - \eta}{2}\right) + \mu_j \sum_{i=1}^{n} D_{r\xi}^{\beta_i} u(t,0) \right) d\eta,$$

$(j = 2, 3)$, where $I_0(z)$, $I_1(z)$ are the modified Bessel function (Bessel function of the first kind with imaginary argument), $\bar{I}_1(z) = I_1(z)/z$. This result relies on the properties of Riemann functions [29, Chapter II, §5].

From (6_1) with respect to (15_2), we get

$$\tau(2x - r) - \lambda_2 (x - r) \int_{2x-r}^{r} \tau(\xi)\, \bar{I}_1 \left[\sqrt{\lambda_2 (\xi - 2x + r)(r - \xi)}\right] d\xi -$$

$$- \int_{2x-r}^{r} \nu(\xi)\, \bar{I}_0 \left[\sqrt{\lambda_2 (\xi - 2x + r)(r - \xi)}\right] d\xi = 2\psi_1(x) - \psi_1(r) +$$

$$+\frac{1}{2}\int_{2x-r}^{r}d\xi\int_{\xi}^{r}I_0\left[\sqrt{\lambda_2\left(\xi-2x+r\right)\left(\eta-r\right)}\right]\left(w_2\left(\frac{\xi-\eta}{2}\right)+\mu_2\sum_{i=1}^{n}D_{r\xi}^{\beta_i}\tau(t)\right)d\eta.$$

By replacing $2x-r=z$ and changing z for x in the next formula, we find the main functional relation between the functions $\tau(x)$ and $\nu(x)$ on γ_1 in the domain Ω_2, that is,

$$\tau(x)-\int_{x}^{r}K(x,t)\tau(t)dt-\int_{x}^{r}I_0\left[\sqrt{\lambda_2\left(t-x\right)\left(r-t\right)}\right]\nu(t)dt= \tag{16}$$

$$=2\psi_1\left(\frac{x+r}{2}\right)+\int_{\frac{x-r}{2}}^{0}w_2\left(t\right)dt\int_{x-t}^{t+r}I_0\left[\sqrt{\lambda_2\left(\xi+t-x\right)\left(\xi-t-r\right)}\right]d\xi-\psi_1(r),$$

where

$$K(x,t)=\frac{\lambda_2(x-r)}{2}\bar{I}_1\left[\sqrt{\lambda_2(t-x)(r-t)}\right]+$$

$$+\frac{\mu_2}{2}\sum_{i=1}^{n}\frac{1}{\Gamma(-\beta_i)}\int_{x}^{t}\frac{d\xi}{(\xi-t)^{1+\beta_i}}\int_{\xi}^{r}I_0\left[\sqrt{\lambda_2\left(\xi+r\right)\left(\eta-x\right)}\right]d\eta. \tag{17}$$

Similarly,introducing (15_3) into (6_2), we get the following functional relation between $\tau(x)$ and $\nu(x)$, transferred from the domain Ω_3 to γ_2

$$\tau(x)+\int_{r}^{x}\bar{K}(x,t)\tau(t)dt-\int_{r}^{x}I_0\left[\sqrt{\lambda_3\left(t-r\right)\left(x-t\right)}\right]\nu(t)dt=$$

$$=2\psi_2\left(\frac{x+r}{2}\right)-\psi_2(r)+\int_{\frac{r-x}{2}}^{0}w_3\left(t\right)dt\int_{r}^{2t+x}I_0\left[\sqrt{\lambda_3\left(\xi-r\right)\left(\xi-2t-x\right)}\right]d\xi, \tag{18}$$

where

$$\bar{K}(x,t)=\frac{\lambda_3(x-r)}{2}\bar{I}_1\left[\sqrt{\lambda_3(t-r)(x-t)}\right]+$$

$$+\frac{\mu_3}{2}\sum_{i=1}^{n}\frac{1}{\Gamma(-\beta_i)}\int_{x}^{t}\frac{d\xi}{(\xi-t)^{1+\beta_i}}\int_{\xi}^{r}I_0\left[\sqrt{\lambda_3\left(\xi-r\right)\left(\eta-x\right)}\right]d\eta. \tag{19}$$

Further, from (11) and considering (2), (13) and (14), passing through the limit $y\to+0$ we obtain a second functional relation between $\tau(x)$ and $\nu(x)$, transferred from the domain Ω_1 to $\gamma_1\cup\gamma_2$ as

$$\tau''(x)-\nu(x)-\lambda_1\tau(x)-\mu_1\sum_{i=1}^{n}D_{rx}^{\alpha_i}\tau(t)=w_1(0), \tag{20}$$

where $w_1(0)$ is an unknown constant to be defined.

4 Proof of Theorem 2.1.

Substituting

$$\nu(x)=\tau''(x)-\lambda_1\tau(x)-\mu_1\sum_{i=1}^{n}\frac{1}{\Gamma(-\alpha_i)}\int_x^r\frac{\tau(t)dt}{(x-t)^{1+\alpha_i}}-w_1(0),$$

into the main functional relation (16), and considering the agreement conditions (9_j),

$$\tau(r)=\psi_1(r),\ \tau(x)=\psi_1(r)-\int_x^r\tau'(t)dt$$

and after some transformations, we have the following integro-differential relation

$$\tau'(x)-\int_x^r K_1(x,t)\tau'(t)dt=f_1(x)+\tau'(r-0)-w_1\left(0\right)\int_x^r I_0\left[\sqrt{\lambda_2\left(t-x\right)\left(r-t\right)}\right]dt+$$

$$+\int_{\frac{x-r}{2}}^{0}w_2\left(t\right)dt\int_{x-t}^{t+r}I_0\left[\sqrt{\lambda_2\left(\xi+t-x\right)\left(\xi-t-r\right)}\right]d\xi, \tag{21}$$

where

$$K_1(x,t)=1-\frac{\partial}{\partial t}I_0\left[\sqrt{\lambda_2(t-x)(r-t)}\right]-\int_x^t\left(K(x,\xi)+\mu_1k_1(x,\xi)+\right.$$

$$\left.+\lambda_1I_0\left[\sqrt{\lambda_2(\xi-x)(r-\xi)}\right]\right)d\xi, \tag{22}$$

$$k_1(x,t)=\sum_{i=1}^{n}\frac{1}{\Gamma(-\alpha_i)}\int_x^t\frac{I_0\left[\sqrt{\lambda_2(\xi-x)(r-\xi)}\right]}{(\xi-t)^{\alpha_i+1}}d\xi,$$

$$f_1(x)=2\psi_1\left(\frac{x+r}{2}\right), \tag{23}$$

$\tau'(r-0),w_1(0)$ are unknown parameters.

Thus, by considering (9_1) we conclude that $f_1(x)\in C^2(\gamma_1)$. By virtue of (9_1), (17), (22), we can conclude that $K_1(x,t)\in C\left(\overline{\gamma}_1\times\overline{\gamma}_1\right)$, as then

$$|K_1(x,t)|\leq const. \tag{24}$$

Thus, taking into account of the theory of Volterra integral equations [30, p. 10], equation (21) has a unique solution, which is representable in the form

$$\tau(x) = H_1(x)+\tau'(r-0)H_2(x)+w_1(0)H_3(x)-\int\limits_{\frac{x-r}{2}}^{0} H(x,t)w_2(t)dt, \quad 0\le x\le r, \tag{25}$$

where

$$H(x,t) = \int\limits_{x}^{2t+r}\left(H^*(s,t)-\int\limits_{s}^{2t+r} R_1(s,\xi)H^*(\xi,t)d\xi\right)ds, \tag{26}$$

$$H_1(x) = -\int\limits_{x}^{r} f_1(t)\left(1-\int\limits_{x}^{t} R_1(s,t)ds\right)dt, \tag{27_1}$$

$$H_2(x) = x-r+\int\limits_{x}^{r} dt\int\limits_{t}^{r} R_1(t,s)ds, \tag{27_2}$$

$$H_3(x) = \int\limits_{x}^{r} dt\int\limits_{t}^{r}\left(I_0\left[\sqrt{\lambda_2(s-t)(r-s)}\right]-\int\limits_{t}^{s} R_1(t,s)I_0\left[\sqrt{\lambda_2(s-\xi)(r-s)}\right]d\xi\right)ds, \tag{27_3}$$

$$H^*(x,t) = \int\limits_{x-t}^{t+r} I_0\left[\sqrt{\lambda_2\,(\xi+t-x)\,(\xi-t-r)}\right]d\xi,$$

where $R_1(x,t)$ is the resolvent of the kernel $K_1(x,t)$. Such solution belongs to the class of $\tau(x)\in C^1(\overline{\gamma}_1\backslash E)\cap C^3(\gamma_1)$.

Further, by using (7_2) from (15_2), and taking into account (16), (20), (25) we have the integral equation

$$w_2(y)-\int\limits_{y}^{0} H_1(2y+r,t)w_2(t)dt-\int\limits_{y/2}^{0} H_2(2y+r,t)w_2(t)dt=$$

$$=H_{11}(2y+r)+\tau'(r-0)H_{12}(2y+r)+w_1(0)H_{13}(2y+r), \tag{28}$$

where

$$H_1(y,t) = -H(y,t)\bar{H}(y,y)+\lambda_2\int\limits_{y}^{2t+r} H(\xi,t)\bar{H}'(y,\xi)d\xi+$$

$$+\lambda_2\int\limits_{y}^{2t+r}\left(H''(\xi,t)\bar{I}_1\left[\sqrt{\lambda_2(\xi-y)(r-\xi)}\right]+\bar{I}_1\left[\sqrt{\lambda_2(\xi-y)(\xi-2t-r)}\right]\right)d\xi+ \tag{29_1}$$

$$+\frac{\lambda_2}{2}\int\limits_y^{2t+r}\left(H''(\xi,t)I_2\left[\sqrt{\lambda_2(\xi-y)(r-\xi)}\right]+I_2\left[\sqrt{\lambda_2(\xi-y)(\xi-2t-r)}\right]\right)d\xi,$$

$$H_2(y,t)=\mu_2 H\left(\frac{y+r}{2},t\right), \tag{29$_2$}$$

$$H_{11}(y)=\lambda_2 H_1(y)\bar{H}(y,y)-\mu_2 H_1\left(\frac{y-r}{2}\right)-\lambda_2\int\limits_y^r\left[\bar{H}'_y(y,\xi)H_1(\xi)+\right.$$

$$\left.+\left(\bar{I}_1\left[\sqrt{\lambda_2(\xi-y)(r-\xi)}\right]+\frac{1}{2}I_2\left[\sqrt{\lambda_2(\xi-y)(r-\xi)}\right]\right)H''_1(\xi)\right]d\xi-$$

$$-\sqrt{2}\psi'_3\left(\frac{y+r}{2}\right), \tag{30$_1$}$$

$$H_{12}(y)=\lambda_2 H_2(y)\bar{H}(y,y)-\mu_2 H_2\left(\frac{y-r}{2}\right)-\lambda_2\int\limits_y^r\left[\bar{H}'_y(y,\xi)H_2(\xi)+\right.$$

$$\left.+\left(\bar{I}_1\left[\sqrt{\lambda_2(\xi-y)(r-\xi)}\right]+\frac{1}{2}I_2\left[\sqrt{\lambda_2(\xi-y)(r-\xi)}\right]\right)H''_2(\xi)\right]d\xi, \tag{30$_2$}$$

$$H_{13}(y)=\lambda_2 H_3(y)\bar{H}(y,y)-\mu_2 H_3\left(\frac{y-r}{2}\right)-\lambda_2\int\limits_y^r\left[\bar{H}'_y(y,\xi)H_3(\xi)+\right.$$

$$\left.+\left(\bar{I}_1\left[\sqrt{\lambda_2(\xi-y)(r-\xi)}\right]+\frac{1}{2}I_2\left[\sqrt{\lambda_2(\xi-y)(r-\xi)}\right]\right)H''_3(\xi)\right]d\xi+$$

$$+\lambda_2\int\limits_y^r\left(\bar{I}_1\left[\sqrt{\lambda_2(\xi-y)(r-\xi)}\right]+\frac{1}{2}I_2\left[\sqrt{\lambda_2(\xi-y)(r-\xi)}\right]\right)d\xi, \tag{30$_3$}$$

where $\bar{H}(y,t)$, $\bar{S}_1(y,t)$ are continuous function.

By virtue of (9_1), (10_1), (26), (27_1), (27_2), and (27_3), we conclude that the functions $H_1(2y+r,t)$, $H_{1j}(y)\,(j=1,3)$ are bounded,

$$|H_1(2y+r,t)|\le const,\quad |H_{1j}(y)|\le const\quad (j=1,3). \tag{31}$$

By setting

$$F(y)=H_{11}(2y+r)+\tau'(r-0)H_{12}(2y+r)+w_1(0)H_{13}(2y+r)+$$

$$+\int\limits_{y/2}^{0}H_2(2y+r,t)w_2(t)dt, \tag{32}$$

equation (28) is written as

$$w_2(y) + \int\limits_{y}^{0} H_1(2y+r,t) w_2(t) dt = F(y), \ \ -\frac{r}{2} \le y \le 0. \tag{33}$$

Assuming that the right-hand side is known, the kernel $H_1(2y+r,t)$ and the function $F(y)$ with regards 1)-3) conditions of the problem $\mathtt{G}_1$ belong to the class L_2, has one and essentially only one solution in the same class L_2 (respectively in C). This solution is given by the formula

$$w_2(y) = F(y) + \int\limits_{y}^{0} R_{11}(y,t) F(t) dt, \tag{34}$$

where the "resolvent kernel" $R_{11}(y,t)$ is given by the series of iterated kernels $H_1(2y+r,t)$ [30, p. 11].

Using (32) into (34), some transformations lead to

$$w_2(y) - \int\limits_{y/2}^{0} H_3(y,t) w_2(t) dt = \bar{F}(y), \tag{35}$$

where

$$H_3(y,t) = H_2(2y+r,t) + \int\limits_{y}^{2t} R_{11}(y,\xi) H_2(2\xi+r,t) d\xi, \tag{36}$$

$$\bar{F}(y) = H_{11}(2y+r) + \int\limits_{y}^{0} R_{11}(y,t) H_{11}(2t+r) dt+$$

$$+\tau'(r-0)(H_{12}(2y+r) + \int\limits_{y}^{0} R_{11}(y,t) H_{12}(2t+r) dt) + w_1(0)(H_{13}(2y+r)+$$

$$+ \int\limits_{y}^{0} R_{11}(y,t) H_{13}(2t+r) dt). \tag{37}$$

Note that by (31), (36), (37), and also (9_1), (10_1) , we have that that

$$|H_3(y,t)| \le const, \quad \left|\bar{F}(y)\right| \le const. \tag{38}$$

To solve equation (35), we use the method successive approximations. Let us construct the following sequence

$$\begin{aligned}
&w_{20}(y) = \bar{F}(y),\\
&w_{21}(y) = \bar{F}(y) + \int\limits_{y/2}^{0} H_3(y,t)\bar{F}(t)dt,\\
&w_{22}(y) = \bar{F}(y) + \int\limits_{y/2}^{0} H_3(y,t)w_{21}(t)dt, \ .....\\
&w_{2n}(y) = \bar{F}(y) + \int\limits_{y/2}^{0} H_3(y,t)w_{2n-1}(t)dt, \ ....
\end{aligned} \tag{39}$$

To prove the absolute and uniform convergence of $\{w_{2n}(y)\}_{n=1}^{\infty}$ is equivalent to the convergence of the series

$$w_{20}(y) + \sum_{n=1}^{\infty} [w_{2n}(y) - w_{2n-1}(y)]. \tag{40}$$

By naming

$$\max |H_3(y,t)| = M, \qquad \max \left|\bar{F}(y)\right| = m_1, \tag{41}$$

we can estimate

$$\begin{aligned}
&|w_{20}(y)| \le m_1,\\
&|w_{21}(y) - w_{20}(y)| \le \left| \int\limits_{y/2}^{0} H_3(y,t)w_{20}(t)dt \right| \le m_1 M \frac{y}{2},\\
&|w_{22}(y) - w_{21}(y)| \le \left| \int\limits_{y/2}^{0} H_3(y,t)\left[w_{21}(t) - w_{20}(t)\right]dt \right| \le m_1 M^2 \frac{y^2}{2!\,2^3},\\
&\text{..}\\
&|w_{2n}(y) - w_{2n-1}(y)| \le \left| \int\limits_{y/2}^{0} H_3(y,t)\left[w_{2n-1}(t) - w_{2n-2}(t)\right]dt \right| \le \left| m_1 M^n \frac{(-1)^n y^n}{n! 2^{\frac{n^2+n}{2}}} \right|.
\end{aligned}$$

From the last estimate one concludes that (40) converge absolutely and uniformly. Hence, we conclude that (35) has a solution $w_2(y)$ in $[-r/2, 0]$. For proving uniqueness to the solution of (35), we proceed as in [9], that is, we prove that the corresponding homogeneous equation has only a the trivial solution.

Let $R_{12}(y,t)$ be the resolvent of the kernel $H_3(y,t)$ [30, p. 11]. Then, the solution of equation (35) is

$$w_2(y) = \tilde{H}_{11}(y) + \tau'(r-0)\tilde{H}_{12}(y) + w_1(0)\tilde{H}_{13}(y), \quad -\frac{r}{2} < y < 0, \tag{42}$$

where

$$\tilde{H}_{1j}(y) = H_{1j}(2y+r) + \int\limits_{y}^{0} R_{11}(y,t) H_{1j}(2t+r) dt +$$

$$+ \int\limits_{y/2}^{0} H_{1j}(2t+r) \left(R_{12}(y,t) + \int\limits_{y/2}^{t} R_{12}(y,s) \, R_{11}(s,t) \, ds \right) dt, \;\; j = \overline{1,3}.$$

Analogously as (21), from the main functional relations (18) and (20) with regard to the agreement condition $\psi_1(r) = \psi_2(r) = 0$ we arrive to the following Volterra integral equation

$$\tau'(x) - \int\limits_{x}^{r} \bar{K}_1(x,t) \tau'(t) dt = g_1(x) -$$

$$- \int\limits_{\frac{r-x}{2}}^{0} w_3(t) dt \int\limits_{r}^{2t+x} I_0 \left[\sqrt{\lambda_3 \left(\xi - r\right) \left(\xi - 2t - x\right)} \right] d\xi + \tau'(r+0) + w_1(0) \times$$

$$\times \int\limits_{r}^{x} I_0 \left[\sqrt{\lambda_3 \left(t - r\right) \left(x - t\right)} \right] dt, \tag{43}$$

where

$$\bar{K}_1(x,t) = 1 + \frac{\partial}{\partial t} I_0 \left[\sqrt{\lambda_3 (t-x)(x-t)} \right] + \int\limits_{t}^{x} \lambda_1 I_0 \left[\sqrt{\lambda_3 (\xi - r)(x - \xi)} \right] d\xi +$$

$$+ \int\limits_{t}^{x} \left(\bar{K}(x,\xi) + \mu_1 \sum_{i=1}^{n} \int\limits_{\xi}^{x} \frac{I_0 \left[\sqrt{\lambda_3 (s-r)(x-s)} \right]}{\Gamma(-\alpha_i)(s-\xi)^{\alpha_i + 1}} ds \right) d\xi, \tag{44}$$

$$g_1(x) = -2\psi_2 \left(\frac{x+r}{2} \right), \tag{45}$$

where $\tau'(r+0), w_1(0)$ are unknown constants.

We investigate the right side of the integral equation (43)

$$g_1^*(x) = g_1(x) - \int\limits_{\frac{r-x}{2}}^{0} w_3(t) dt \int\limits_{r}^{2t+x} I_0 \left[\sqrt{\lambda_3 (\xi - r)(\xi - 2t - x)} \right] d\xi +$$

$$+ \left(\tau'(r+0) + w_1(0) \int\limits_{r}^{x} I_0 \left[\sqrt{\lambda_3 (t-r)(x-t)} \right] dt \right). \tag{46}$$

From (44) and considering (6_2), (19) we conclude that $\bar{K}_1(x,t) \in C\left(\overline{\gamma}_2 \times \overline{\gamma}_2\right)$ and consequently,

$$\left|\bar{K}_1(x,t)\right| \leq const. \tag{47}$$

Thus from (43) and with regards (9_2) we have

$$\tau(x) = Q_1(x) + \tau(r+0)Q_2(x) + w_1(0)Q_3(x) - \int\limits_{\frac{r-x}{2}}^{0} Q(x,t)w_3(t)dt, \quad r \leq x \leq 1, \tag{48}$$

where

$$Q(x,t) = \int\limits_{r-2t}^{x} \left(Q^*(t,s) - \int\limits_{r-2t}^{s} Q^*(t,s)\bar{R}_1(s,\xi)d\xi \right) ds, \tag{49}$$

$$Q_1(x) = \psi_2(r) + \int\limits_{r}^{x} g_1(t) \left(1 + \int\limits_{t}^{x} \bar{R}_1(s,t)ds \right) dt, \tag{49_1}$$

$$Q_2(x) = \int\limits_{r}^{x} \left(1 + \int\limits_{r}^{s} \bar{R}_1(s,t)dt \right) ds, \tag{49_2}$$

$$Q_3(x) = \int\limits_{r}^{x}\int\limits_{r}^{s} \left(I_0\left[\sqrt{\lambda_3\left(\xi - r\right)\left(s - \xi\right)}\right] + \int\limits_{\xi}^{s} \bar{R}_1\left(s,t\right) I_0\left[\sqrt{\lambda_3\left(\xi - r\right)\left(t - \xi\right)}\right] dt \right) d\xi ds, \tag{49_3}$$

$$Q^*(x,t) = \int\limits_{r}^{2x+t} I_0\left[\sqrt{\lambda_3\left(\xi - r\right)\left(\xi - 2x - t\right)}\right] d\xi,$$

and $\bar{R}_1(x,t)$ is the resolvent of the kernel $\bar{K}_1(x,t)$.

On the other hand, conducting an analogous reasoning to that of $w_2(y)$, from (15_3) and (7_2), with regards of (20), (48), we find the relation

$$w_3(y) = \tilde{Q}_{11}(y) + \tau'(r+0)\tilde{Q}_{12}(y) + w_1(0)\tilde{Q}_{13}(y), \quad \frac{r-1}{2} < y < 0, \tag{50}$$

where $\tilde{Q}_{1j}(y)$,$(j = 1,3)$ are continuous functions.

In conclusion, by using the relations (25), (48) and considering (42), (50) and the agreement conditions

$$\tau(0) = \varphi_1(0), \;\; \tau'(0) = \varphi_2(0), \;\; \tau(1) = \varphi_3(0), \tag{51}$$

we uniquely define the unknown constants $\tau'(r-0), \tau'(r+0)$ and $w_1(0)$.

Now, from the function $\tau(x)$ and using relation (20), we uniquely define the function $\nu(x)$.

Consequently, problem $\mathtt{G}_1$ is identically solvable by the equivalence for Volterra integral equations of the second kind with shift in Ω_j $(j = 2, 3)$.

Hence, by virtue of the (20), (25), (48), with considering (42), (50), (51), we uniquely define functions $\tau(x)$ and $\nu(x)$.

Thus, the solution $u(x, y)$ of the problem $\mathtt{G}_1$ in Ω_2 and Ω_3 dis efined by the formula (15_j) respectively $j = 2, 3$. For the determination of $u(x, y)$ in the domain Ω_1, we arrive to the non loaded equation

$$\frac{\partial}{\partial x}\left(u_{xx} - u_y - \lambda_1 u\right) = \widetilde{\Phi}(x, y), \tag{52}$$

with boundary conditions (3-5) and

$$u(x, +0) = \tau(x),$$

where $\widetilde{\Phi}(x, y) = \mu_1 \frac{\partial}{\partial x} \sum\limits_{i=1}^{n} D_{rx}^{\alpha_i} \tau(x)$ is a known function. The unique solution of this problem is restored as the solution of problem T_0 in [31].

In a similar way, we can investigate the following analogue of Gellerstedt problem, when the boundary-value conditions are given on parallel characteristics [26].

Problem $\mathbf{G}_2$. Find a function that satisfies all the conditions of the problem G_1, except (6_2) and (7_2) for which the following conditions must be satisfied: $u_x(u_y)$ is continuous up to γ_{13} and satisfies the boundary-value conditions

$$u(x, y)|_{\gamma_{13}} = \tilde{\psi}_2(x), \;\; (x, 0) \in \bar{\gamma}_{13}; \tag{53}$$

$$\left.\frac{\partial u(x, y)}{\partial n}\right|_{\gamma_{13}} = \tilde{\psi}_4(x), \;\; (x, 0) \in \gamma_{13}, \tag{54}$$

where $\tilde{\psi}_2(x)$ and $\tilde{\psi}_4(x)$ are given real-valued functions, moreover $\tilde{\psi}_2(r) = \varphi_3(0)$,

$$\tilde{\psi}_2(x) \in C^1\left(\bar{\gamma}_{13}\right) \cap C^3\left(\gamma_{13}\right), \, \tilde{\psi}_2(r) = 0, \tag{55}$$

$$\tilde{\psi}_4(x) \in C\left(\bar{\gamma}_{13}\right) \cap C^2\left(\gamma_{13}\right). \tag{56}.$$

Theorem 4.1. *If conditions (8),* (9_1), (10_1), (55) *and* (56) *are satisfied, then there exists a unique solution of problem G_2.*

Theorem 4.1 is proved in the same way as Theorem 2.1. The main functional relations (16) and (20), from the domains Ω_2 and Ω_1 is correctly. On the other side, using the solution of the Cauchy problem (15_3), taking into account (53),

$$\tau(1) = \widetilde{\psi}_2(1), \; \tau(x) = \widetilde{\psi}_2(1) - \int\limits_{x}^{1} \tau'(t)dt$$

and (20) we obtain the Volterra integral equation of the second kind with respect to $\tau'(x)$ from the domain Ω_3 to γ_3. Further, conducting an analogous reasoning to that of $w_3(y)$, from (15_3) and (54), with regards of (20) and $\tau(x)$ on $\frac{r+1}{2} < x < 1$, we can to find the $w_3(y)$ on $\frac{r-1}{2} < y < 0$. The subsequent investigations are performed by analogy with problem G_1.

References

[1] Soldatov AP. On the theory of mixed-type equations, Journal of Mathematical Sciences, August 2005, Volume 129, Issue 1:3670-3679.

[2] Rassias JM, Mixed type partial differential equations with initial and boundary values in fluid mechanics, International Journal of Applied Mathematics and Statistics, 2008. Vol. 13; No. J08:77-107.

[3] Chen SX. Mixed type equations in gas dynamics, Quart.Appl. Math. LVXIII (3) (2010):487-511.

[4] Marcus A. Khuri. Boundary value problems for mixed type equations and applications. June 2011, Nonlinear Analysis 74(17):1-15.

[5] Salakhitdinov MS, Islomov B. *Equations of a mixed type with two lines of degeneracy*, Mumtoz suz, Tashkent, 2009. - 264 p.

[6] Salakhitdinov MS. *Equation of a mixed composite type*, Fan, Tashkent, 1974, 156p.

[7] Djuraev TD, Sopuev A, Mamajonov M. *Boundary value problems for the problems for the parabolic-hyperbolic type equations*, Fan, Tashkent, 1986.

[8] Sabitov KB. Boundary Value Problem for a Third-Order Equation of Mixed Type in a Rectangular domain, Differential Equations, 2013, Vol. 49, No. 2:187-197.

[9] Islomov B, Baltaeva UI. Boundary value problems for a third-order loaded parabolic-hyperbolic equation with variable coefficients. Electronic Journal of Differential Equations, Vol. 2015; 221:1-10.

[10] Samko SG, Kilbas AA, Marichev OI. *Fractional Integral and Derivatives. Theory and Applications*, Gordon and Breach, Longhorne, PA, 1993.

[11] Miller KS and Ross B. *An Introduction to the Fractional Calculus and Fractional Differential Equations.* Wiley and Sons: New York, 1993.

[12] Mainardi F. Fractional calculus: some basic problem in continuum and statistical mechanics, in Fractal and Fractional Calculus in Continuum Mechanics, A. Carpinteri and F. Mainardi, Eds. Sprienger, Vienna, Austria, 1997:291-948.

[13] Hilfer R. *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[14] Yang XJ. Fractional derivatives of constant and variable orders applied to anomalous relaxation models in heat-transfer problems. *Thermal Science*. 2017, No 3(21):1161-1171.

[15] Podlubny I. Geometric and physical interpretation of fractional integration and fractional differentiation. Dedicated to the 60th anniversary of Prof. Francesco Mainardi. Fract. Calc. Appl. Anal. 5(2002), No. 4:367-386.

[16] Tenreiro Machado J, Virginia Kiryakova, Francesco Mainardi. Recent history of fractional calculus. Commun Nonlinear Sci Numer Simulat, 16 (2011) 1140-1153.

[17] Pskhu AV. *Partial Differential Equation of Fractional Order*, Nauka, Moscow, Russia, 2005.

[18] Kilbas AA, Srivastava HM, Trujillo JJ. *Theory and Applications of Fractional Differential Equations*. Elsevier, Amsterdam, 2006. -539 p.

[19] Yang Q, Liu F, Turner I. Numerical methods for fractional partial differential equations with Riesz space fractional derivatives. Applied Mathematical Modelling Volume 34, Issue 1, January 2010:200-218.

[20] Agarwal P, Karimov E, Mamchuev M and Ruzhansky M. On boundary value problems for a partial differential equation with Caputo and Bessel operators, in Novel Methods in Harmonic Analysis, Vol. 2:707-719, Applied and Numerical Harmonic Analysis. Birkhauser Basel, 2017. arXivmath/01624.

[21] Kishin B. Sadarangani and Obidjon Kh. Abdullaev. Non-local problems with integral gluing condition for loaded mixed type equations involving the Caputo fractional derivative, Electronic Journal of Differential Equations, Vol. 2016 (2016), ISSN: 1072-6691, No. 164: 1-10.

[22] Baltaeva UI. Boundary value problems The loaded parabolic-hyperbolic equation and its relation to non-local problems, Nanosystems physics, chemistry, mathematics, 2017. 8 (4):413-419.

[23] Smirnov MM. *Equations of the Mixed Type*, Moscow: Vysshaya Shkola, 1985.

[24] Mamadaliev NK. The Gellerstedt problem for a parabolic-hyperbolic equation of the second kind, International Journal of Dynamical Systems and Differential Equations (IJDSDE) 2007, Vol. 1, No. 2:102-108.

[25] Moiseev TE. Gellerstedt Problem with a Generalized Frankl Matching Condition on the Type Change Line with Data on External Characteristics, Differential Equations, 2016, Vol. 52, No. 2:240-û247.

[26] Moiseev EI, Moiseev TE and Kholomeeva AA. Solvability of the Gellerstedt problem with data on parallel characteristics. ISSN 0012-2661. Differential equations, 2017. Vol. 53, No. 10:1346-1351.

[27] Zarubin AN. Gellerstedt problem for a Differential-Difference Equation of Mixed Type with Advanced-Retarded Multiple Deviations of the Argument, ISSN 0012-2661. Differential Equations, 2013, Vol. 49, No. 10:1274-1281.

[28] Nakhushev AM. *Loaded Equations and Their Applications*, M. Nauka, 2012.

[29] Tikhonov AN, Samarskii AA. *Equations of mathematical physics*, M: Nauka, 1977, 736 p.

[30] Tricomi FG. *Integral Equations*, Dover Publications, Inc. New York. 1985. 245p.

[31] Baltaeva UI. Solvability of the analogs of the problem Tricomi for the mixed type loaded equations with parabolic-hyperbolic operators. Boundary Value Problems 2014:211, 2014:1-12

[32] Agarwal P, Baltaeva UI. Boundary-value problems for the third-order loaded equation with noncharacteristic type-change boundaries. Mathematical Methods in the Applied Sciences, Volume 2018:1-9.