# GLOBAL BIFURCATION OF SOLUTIONS OF THE MEAN CURVATURE SPACELIKE EQUATION IN CERTAIN STANDARD STATIC SPACETIMES

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Abstract. We study the existence/nonexistence and multiplicity of spacelike graphs for the following mean curvature equation in a standard static spacetime

$$\operatorname{div}\left(\frac{a\nabla u}{\sqrt{1-a^2|\nabla u|^2}}\right)+\frac{g(\nabla u,\nabla a)}{\sqrt{1-a^2|\nabla u|^2}}=\lambda NH$$

with 0-Dirichlet boundary condition on the unit ball. According to the behavior of $H$ near 0, we obtain the global structure of one-sign radial spacelike graphs for this problem. Moreover, we also obtain the existence and multiplicity of entire spacelike graphs.

**Keywords**: Bifurcation; Mean curvature operator; One-sign solution; Entire spacelike graph

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1. **Introduction.** In a spacetime $M$, a unit timelike and future pointing vector fiel $U$ is called an observer field [22, p. 358] (or reference frame in the terminology of [26, Def. 2.3.1]). Each integral curve of $U$ models a physical observer parametrized by its proper time. The spacetime $M$ is said to be static respect to $U$ if it is irrotational (i.e., the distribution $U^\perp$ is integrable) and there exists a smooth function $a > 0$ on $M$ such that the vector field $aU$ is Killing. Thus, given any (local) flow $\{\varphi_t\}$ of this Killing vector field and a leaf $S$ of $U^\perp$ through $p \in M$, such that $\{\varphi_t\}$ is defined on $S$, we have $\varphi_t(S)$ is a leaf of $U^\perp$ through $\varphi_t(p) \in M$. Physically this means that the spatial universe looks the same, at least locally, for each observer in $U$.

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Given a static spacetime $M$, relative to $U$, for each $p \in M$ there is an open neighbourhood of $p$ which is isometric to the product manifold $I \times S$, endowed with the Lorentzian metric $-a^2dt^2 + g$, where $S$ is a leave of $U^\perp$ through $p$, $I$ an open interval of $\mathbb{R}$, $g$ the Riemannian metric on $S$ obtained by restriction of the Lorentzian metric of $M$ and $a > 0$ an smooth function on $S$ [22, Prop. 12.38]. Moreover, $U$ is represented in this decomposition just as the coordinate vector field $\partial/\partial t$. When this decomposition is global [1], then $M$ is called a standard static spacetime [22, Def. 12.36]. In other words, a standard static spacetime is a product manifold $I \times S$, endowed the Lorentzian metric

$$\overline{g} = -a^2dt^2 + g, \tag{1.1}$$

where $I$ is an open interval of the real line, $g$ is a Riemannian metric on the manifold $S$ and $a >$ is a smooth function on $S$. In the terminology of [22, Def. 7.33], the manifold with the Lorentzian metric (1.1) is a warped product with base $(S, g)$, fiber $(I, -dt^2)$ and warping function $a$.

Let us consider here the case $S = \Omega$ an open subdomain of $\mathbb{R}^N$, $N \geq 1$, with its canonical metric $g$ and $a$ invariant by certain isometry of $\overline{\Omega}$. Taking into account that an isometry $\phi$ of the base such that $a \circ \phi = a$ clearly induces an isometry of the standard static spacetime, this mathematical assumption may be interpreted as a natural symmetry of the spacetime. This is the case when $\Omega = I \times \mathbb{S}^{N-1}$, $I \subset \mathbb{R}_+$ an open interval ($\Omega$ may be seen as an open domain in $\mathbb{R}^N$),

$$g = E^2(r)dr^2 + r^2 d\sigma^2, \tag{1.2}$$

where $E > 0$ is defined on $I$, $d\sigma^2$ denotes the usual Riemannian metric of the unit sphere $\mathbb{S}^{N-1}$ and $a = a(r)$. In particular, for $I = (2m, \infty)$, $m > 0$ constant, $N = 4$, $E(r) = \left(1 - \dfrac{2m}{r}\right)^{-1/2}$ and $a(r) = \left(1 - \dfrac{2m}{r}\right)^{1/2}$, the corresponding standard static spacetime is the Schwarzschild (exterior) spacetime of mass $m$, [22, Def. 13.2].

For any $u \in C^2(\Omega)$, consider its graph

$$\Sigma_u = \{\, (u(x), x) \ : \ x \in \Omega \,\}$$

in the standard static spacetime $M := (I \times \Omega, \overline{g})$ which is spacelike, i.e., the induced metric on $\Sigma_u$ from (1.1) is Riemannian, if and only if $a|\nabla u| < 1$ holds on all $\Omega$, where $\nabla u$ is the gradient of the function $u$ and $|\nabla u|$ its $g$-length. In this case,

$$\xi = \frac{1}{\sqrt{1 - a^2|\nabla u|^2}} \left( \frac{1}{a}, \, a \nabla u \right) \tag{1.3}$$

is the unit timelike normal vector field on $\Sigma_u$ in $M$ in the same time-orientation as $\partial/\partial_t$.

If $H$ is the mean curvature of the spacelike graph $\Sigma_u$ with respect to the unit timelike normal vector field given in (1.3), the function $u$ may be seen as a solution of the mean curvature spacelike hypersuface equation (derived in Appendix, for the sake of completeness)

$$\begin{cases} \operatorname{div}\left(\dfrac{a\nabla u}{\sqrt{1 - a^2|\nabla u|^2}}\right) + \dfrac{g(\nabla u, \nabla a)}{\sqrt{1 - a^2|\nabla u|^2}} = NH, \\[2ex] \hfill |\nabla u| < \dfrac{1}{a}. \end{cases} \tag{1.4}$$

When $a \equiv 1$, $H \equiv 0$ and $\Omega = \mathbb{R}^N$, equation (1.4) reduces to the well-known maximal hypersurface equation in $(N+1)$-dimensional Lorentz-Minkowski spacetime. In this case, Calabi [6] proved that the only entire (i.e., defined on all $\mathbb{R}^N$) solutions are the affine functions defining spacelike hyperplanes for $N \leq 4$. Further, Cheng and Yau [8] extended this result for all $N$. This uniqueness result contracts with the behaviour of the entire solutions of the classical minimal hypersurface equation whose entire solutions are affine functions only for $N \leq 7$ and counter-examples exist for each $N > 7$, [23].

When $a \equiv 1$ and $H$ is a non-zero constant, some celebrated results for equation (1.4) were obtained by Treibergs [28]. If $a \equiv 1$, $\Omega$ is a bounded domain and $H$ is a bounded function defined on $\Omega \times \mathbb{R}$, Bartnik and Simon [2] proved that the equation (1.4) with Dirichlet boundary condition has a strictly spacelike solution. By topological degree or critical point theory, the authors of [5, 9] studied the nonexistence, existence and multiplicity of positive solutions for it in the case of $a \equiv 1$ and $\Omega$ being a bounded domain. When $a \equiv 1$ and $\Omega = B_R := B_R(0) = \{x \in \mathbb{R}^N : |x| < R\}$ with $R > 0$, Bereanu, Jebelean and Torres [3, 4] obtained some existence results for positive radial solutions of equation (1.4) with $u = 0$ on $\partial\Omega$. Recently, when $a \equiv 1$, the first author [10] studied the nonexistence, existence and multiplicity of positive radial solutions of equation (1.4) with $u = 0$ on $\partial\Omega$ and $NH = -\lambda f(x, s)$ on the unit ball via bifurcation method, which were extended to the general domain in [13, 14].

The main objective of this paper is to investigate the existence/nonexistence of one-sign radially symmetric spacelike solutions for equation (1.4) on the unit ball $B$ mainly by bifurcation method in the same philosophy as in [15]. The solution is understood in the classical sense.

For this aim to be achieved, we consider the following 0-Dirichlet boundary value problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{a\nabla u}{\sqrt{1-a^2|\nabla u|^2}}\right) - \dfrac{g(\nabla u, \nabla a)}{\sqrt{1-a^2|\nabla u|^2}} = -\lambda NH & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1.5}$$

where, here and that follows, the constrain $|\nabla u| < 1/a$ is understood, $\lambda$ is a nonnegative parameter which can represent in some sense the strength of mean curvature function, $a : \overline{B} \to \mathbb{R}$ is a smooth positive radially symmetric function, $H : \overline{B} \times [-\delta, \delta] \to \mathbb{R}$ is a continuous function and is radially symmetric with respect to $x$ with some positive constant $\delta$ determined later.

Taking $g$ as in (1.2) and following [19], we have that the problem (1.5) can be reduced to the following boundary value problem

$$\begin{cases} -\dfrac{1}{r^{N-1}E}\left(\dfrac{r^{N-1}av'}{E\sqrt{1-a^2v'^2}}\right)' - \dfrac{a'v'}{E^2\sqrt{1-a^2v'^2}} = -\lambda NH(r, v), \quad r \in (0, 1), \\ v'(0) = v(1) = 0, \end{cases} \tag{1.6}$$

where $r = |x|$, $v(r) = u(|x|)$ and $a(r) = a(|x|)$. Letting $\delta = \max_{\overline{B}}(E(x)/a(x))$, since the graph associated to $v$ is spacelike, we deduce that $\|v\|_\infty < \delta$. This is the reason we only require that $H$ is defined on $\overline{B} \times [-\delta, \delta]$.

Let $\lambda_1$ be the first eigenvalue of

$$\begin{cases} -\left(r^{N-1}\dfrac{a^2(r)}{E(r)}u'\right)' = \lambda r^{N-1}a(r)E(r)u, \quad r \in (0, 1), \\ u'(0) = u(1) = 0. \end{cases} \tag{1.7}$$

It is well known that $\lambda_1$ is simple, isolated and the associated eigenfunctions have one sign in $[0, 1)$ (see for instance [29, p. 284]). Let

$$X = \{u \in C^1[0, 1] \; : \; u'(0) = u(1) = 0\}$$

with the norm $\|u\| := \|(a/E)u'\|_\infty$. Clearly, one has $\|v\|_\infty \leq \|v'\|_\infty$ and $\|v\|/\rho \leq \|v'\|_\infty \leq \|v\|\,\delta$, where $\rho = \max_{[0,1]}(a(r)/E(r))$.

It follows that the norm $\|v\|$ is equivalent to the usual norm $\|v\|_\infty + \|v'\|_\infty$. Let $P^+ = \{v \in X : v > 0 \text{ on } [0,1)\}$ and $P^- = -P^+$. From now on, following [25], we add the point $\infty$ to our space $\mathbb{R} \times X$.

The main result of this paper is the following theorem.

**Theorem 1.1.** *Assume that $H(r,t)t < 0$ for any $r \in [0,1]$, $t \in (-\delta,\delta) \setminus \{0\}$ and there exists $H_0 \in [0,+\infty]$ such that*

$$\lim_{t\to 0}\frac{NH(r,t)}{t} = -H_0$$

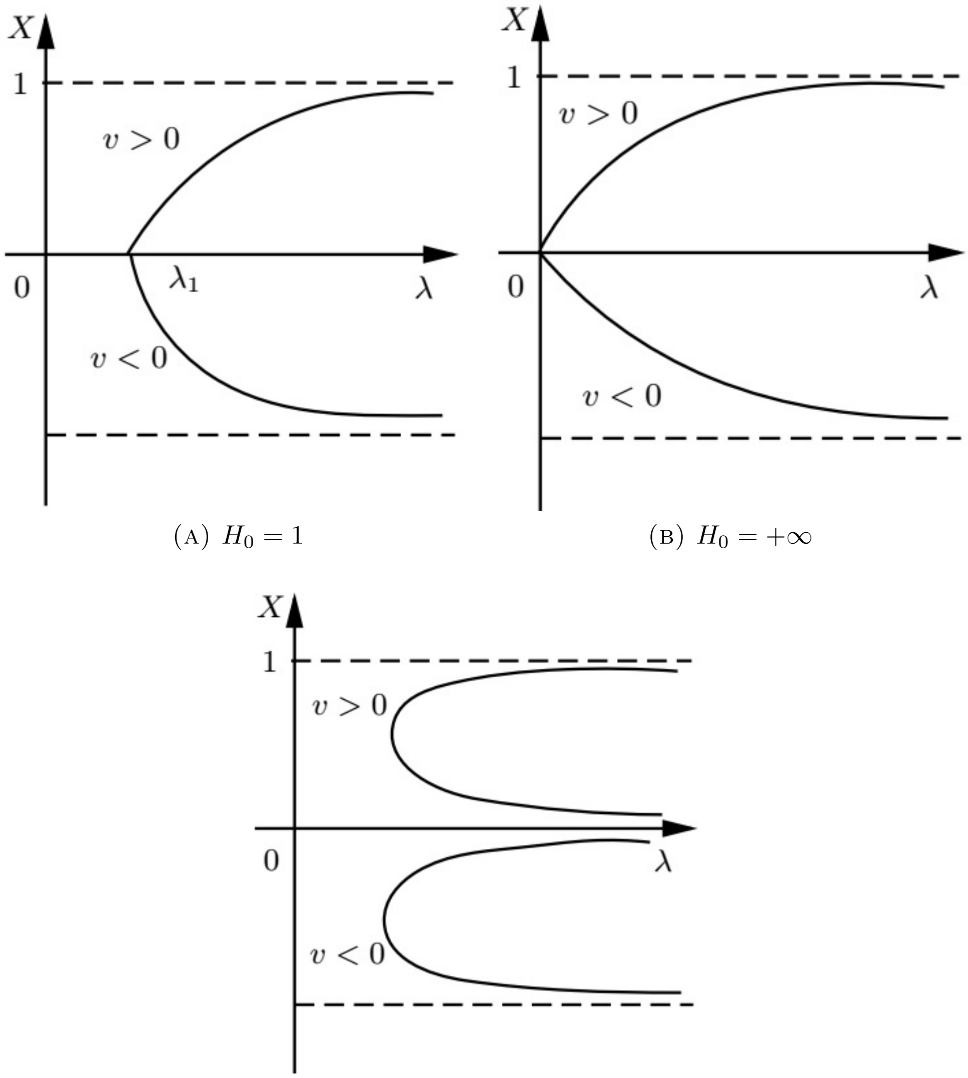
*uniformly for $r \in (0,1)$. Then,*

(a) *if $H_0 = 1$, there are two unbounded components, $\mathscr{C}^+$ and $\mathscr{C}^-$, of the set of nontrivial solutions of problem* (1.6) *bifurcating from $(\lambda_1, 0)$ such that $\mathscr{C}^\nu \subseteq (\mathbb{R} \times P^\nu) \cup \{(\lambda_1,0)\}$, $(\lambda_1,+\infty) \subseteq pr_{\mathbb{R}}(\mathscr{C}^\nu)$, $\|v_\lambda\| < 1$ and $\lim_{\lambda\to+\infty}\|v_\lambda\| = 1$ for $(\lambda, v_\lambda) \in (\mathscr{C}^+ \cup \mathscr{C}^-) \setminus \{(\lambda_1,0)\}$, where $\nu \in \{+,-\}$ and $pr_{\mathbb{R}}(\mathscr{C}^\nu)$ denotes the projection of $\mathscr{C}^\nu$ on $\mathbb{R}$,*

(b) *if $H_0 = +\infty$, there are two unbounded components, $\mathscr{C}^+$ and $\mathscr{C}^-$, of the set of nontrivial solutions of problem* (1.6) *emanating from $(0,0)$ such that $\mathscr{C}^\nu \subseteq (\mathbb{R} \times P^\nu) \cup \{(0,0)\}$, joins to $(+\infty,1)$ and $\|v_\lambda\| < 1$ for $(\lambda, v_\lambda) \in (\mathscr{C}^+ \cup \mathscr{C}^-) \setminus \{(0,0)\}$,*

(c) *if $H_0 = 0$, there are two unbounded components, $\mathscr{C}^+$ and $\mathscr{C}^-$, of the set of nontrivial solutions of problem* (1.6) *such that $\mathscr{C}^\nu \subseteq \mathbb{R} \times P^\nu$, joins $(+\infty,1)$ to $(+\infty,0)$ and $\|v_\lambda\| < 1$ for any $(\lambda, v_\lambda) \in \mathscr{C}^\nu$ with $\lambda < +\infty$.*

Figure 1 illustrates the global bifurcation branches of Theorem 1.1. The existence and multiplicity of one-sign solutions of problem (1.6) can be easily seen from these diagrams.




(A) $H_0 = 1$ (B) $H_0 = +\infty$

(C) $H_0 = 0$

FIGURE 1. Bifurcation diagrams of Theorem 1.1.

The following result is concerning the nonexistence.

**Theorem 1.2.** *Assume that $a(r)$ is nondecreasing and there exists a positive constant $\varrho$ such that*

$$\frac{-H(r,s)}{s} \leq \varrho$$

*for any $s \neq 0$ and $r \in (0,1)$. Then there exists $\varrho_* > 0$ such that problem (1.6) has not any one-sign solution for $\lambda \in (0, \varrho_*)$.*

By arguments similar to those of Theorems 1.1–1.2, we can also show that the conclusions of Theorems 1.1–1.2 are valid for equation (1.4) on any annular domain with the Robin boundary condition. Concretely, let $R_1$, $R_2 \in \mathbb{R}$ with $0 < R_1 < R_2$ and

$\mathcal{A} := \{x \in \mathbb{R}^N : R_1 \le |x| \le R_2\}$. Consider the following problem with the Robin boundary condition

$$\begin{cases} -\operatorname{div}\left(\dfrac{a\nabla u}{\sqrt{1-a^2|\nabla u|^2}}\right) + \dfrac{g(\nabla u, \nabla a)}{\sqrt{1-a^2|\nabla u|^2}} = -\lambda NH \text{ in } \mathcal{A}, \\ \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial B_{R_1},\ u = 0 \text{ on } \partial B_{R_2}, \end{cases} \tag{1.8}$$

where $H : \overline{\mathcal{A}} \times \big(-(R_2-R_1)\,\delta, (R_2-R_1)\,\delta\big) \longrightarrow \mathbb{R}$ is a continuous function and is radially symmetric with respect to $x$, $\partial v/\partial \nu$ is the outward normal derivative of $v$ and $a : \overline{\mathcal{A}} \to \mathbb{R}$ is a smooth positive radially symmetric function. As that of (1.6), the problem (1.8) is reduced to the following one

$$\begin{cases} -\dfrac{1}{r^{N-1}E}\left(\dfrac{r^{N-1}av'}{E\sqrt{1-f^2v'^2}}\right)' - \dfrac{a'v'}{E^2\sqrt{1-f^2v'^2}} = -\lambda NH\,(r,v)\,,\ r \in (R_1,R_2)\,, \\ v'\,(R_1) = v\,(R_2) = 0. \end{cases} \tag{1.9}$$

Let $\lambda_1$ be the first eigenvalue (see [20]) of

$$\begin{cases} -\left(r^{N-1}\dfrac{a^2(r)}{E(r)}u'\right)' = \lambda r^{N-1}a(r)E(r)u, \quad r \in (R_1,R_2)\,, \\ u'\,(R_1) = u\,(R_2) = 0 \end{cases}$$

and

$$X = \{u \in C^1\,[R_1,R_2]\ :\ u'\,(R_1) = u\,(R_2) = 0\}$$

with the norm $\|u\| := \|(a/E)u'\|_\infty$. Then, in particular, we have the following consequence.

**Corollary 1.1.** *Assume that $a'(r) \le 0$ for any $r \in (R_1,R_2)$ and $H(r,t)t < 0$ for any $r \in [R_1,R_2]$, $t \in \big(-(R_2-R_1)\,\delta, (R_2-R_1)\,\delta\big) \setminus \{0\}$ and there exists $H_0 \in [0,+\infty]$ such that*

$$\lim_{t\to 0}\frac{NH\,(r,t)}{t} = -H_0$$

*uniformly for $r \in (R_1,R_2)$. Then,*

(a) *if $H_0 = 1$, there are two unbounded components, $\mathscr{C}^+$ and $\mathscr{C}^-$, of the set of nontrivial solutions of problem (1.9) bifurcating from $(\lambda_1,0)$ such that $\mathscr{C}^\nu \subseteq (\mathbb{R}\times P^\nu)\cup\{(\lambda_1,0)\}$, $(\lambda_1,+\infty) \subseteq pr_{\mathbb{R}}(\mathscr{C}^\nu)$, $\|v_\lambda\| < 1$ and $\lim_{\lambda\to+\infty}\|v_\lambda\| = 1$ for $(\lambda,v_\lambda) \in (\mathscr{C}^+\cup\mathscr{C}^-)\setminus\{(\lambda_1,0)\}$,*

(b) *if $H_0 = +\infty$, there are two unbounded components, $\mathscr{C}^+$ and $\mathscr{C}^-$, of the set of nontrivial solutions of problem (1.9) emanating from $(0,0)$ such that $\mathscr{C}^\nu \subseteq (\mathbb{R}\times P^\nu)\cup\{(0,0)\}$, joins to $(+\infty,1)$ and $\|v_\lambda\| < 1$ for $(\lambda,v_\lambda) \in (\mathscr{C}^+\cup\mathscr{C}^-)\setminus\{(0,0)\}$,*

(c) *if $H_0 = 0$, there are two unbounded components, $\mathscr{C}^+$ and $\mathscr{C}^-$, of the set of nontrivial solutions of problem (1.9) such that $\mathscr{C}^\nu \subseteq \mathbb{R}\times P^\nu$, joins $(+\infty,1)$ to $(+\infty,0)$ and $\|v_\lambda\| < 1$ for any $(\lambda,v_\lambda) \in \mathscr{C}^\nu$ with $\lambda < +\infty$.*

The rest of this paper is arranged as follows. In Section 2, we introduce an approximation problem of problem (1.6) and investigate its global bifurcation phenomenon. Section 3 is devoted to the study of the convergence of solutions of our approximation problem as $\epsilon \to 0^+$. The proofs of Theorems 1.1 and 1.2 will be given in Section 4. In Section 5, on the basis of Theorem 1.1 (or Corollary 1.1) and the standard prolongability theorem of ordinary differential equations, we study the existence of entire radially spacelike graphs of equation (1.4) for $\Omega = \mathbb{R}^N$ (or exterior region). Two examples are also given in this section. Finally, the derivation of equation (1.4) is given in Appendix.

2. **Bifurcation for an approximation problem.** If $v$ is a solution of problem (1.6), then we have that

$$\begin{aligned} -\lambda N H(r,v) &= -\frac{1}{r^{N-1}E}\left(\frac{r^{N-1}av'}{E\sqrt{1-f^2v'^2}}\right)' - \frac{a'v'}{E^2\sqrt{1-f^2v'^2}} \\ &= -\frac{1}{r^{N-1}E}\left(r^{N-1}(a/E)v'\right)'\frac{1}{\sqrt{1-f^2v'^2}} \\ &\quad -\frac{av'}{E^2}\left(\frac{1}{\sqrt{1-f^2v'^2}}\right)' - \frac{a'v'}{E^2\sqrt{1-f^2v'^2}} \\ &= -\frac{a}{E^2}\frac{v''}{\left(1-f^2v'^2\right)^{3/2}} - \frac{N-1}{r}\frac{a}{E^2}\frac{v'}{\sqrt{1-f^2v'^2}} \\ &\quad -\frac{(a/E)'v'}{E\sqrt{1-f^2v'^2}} - \frac{av'}{E^2}\frac{f(r)f'(r)v'^2}{\left(1-f^2v'^2\right)^{3/2}} - \\ &\quad -\frac{a'v'}{E^2\sqrt{1-f^2v'^2}}, \end{aligned}$$

where $f(r) = a(r)/E(r)$. Furthermore, we obtain that

$$\begin{aligned} -v'' &= -\lambda N\frac{E^2}{a}H(r,v)\left(1-f^2v'^2\right)^{3/2} + \frac{N-1}{r}v'\left(1-f^2v'^2\right) \\ &\quad +\frac{f'v'}{f}\left(1-f^2v'^2\right) + ff'v'^3 + \frac{a'v'}{a}\left(1-f^2v'^2\right). \end{aligned}$$

Note that the above equation is singular at $r = 0$. To overcome this singularity, we consider the following approximation equation

$$\begin{aligned} -v'' &= -\lambda N\frac{E^2}{a}H(r,v)\left(1-f^2v'^2\right)^{3/2} + \frac{N-1}{r+\epsilon}v'\left(1-f^2v'^2\right) \\ &\quad +\frac{f'v'}{f}\left(1-f^2v'^2\right) + ff'v'^3 + \frac{a'v'}{a}\left(1-f^2v'^2\right) \end{aligned}$$

for any $\epsilon \in (0,1]$. It follows that

$$\begin{aligned} -\frac{1}{(r+\epsilon)^{N-1}}\left((r+\epsilon)^{N-1}v'\right)' &= -\lambda N\frac{E^2}{a}H(r,v)\left(1-f^2v'^2\right)^{3/2} - \frac{N-1}{r+\epsilon}f^2v'^3 \\ &\quad +\frac{f'v'}{f}\left(1-f^2v'^2\right) + ff'v'^3 + \frac{a'v'}{a}\left(1-f^2v'^2\right) \\ &= -\lambda N\frac{E^2}{a}H(r,v)\left(1-f^2v'^2\right)^{3/2} - \frac{N-1}{r+\epsilon}f^2v'^3 \\ &\quad +\frac{2a'v'}{a} - \frac{E'v'}{E} - \frac{aa'v'^3}{E^2}. \end{aligned}$$

Consequently, we have that

$$\begin{aligned}
-\frac{1}{(r+\epsilon)^{N-1}}\left((r+\epsilon)^{N-1}\frac{a^2(r)}{E(r)}v'\right)' &= -\lambda NaEH(r,v)\left(1-f^2v'^2\right)^{3/2}-\frac{N-1}{r+\epsilon}af^3v'^3\\
&\quad+\frac{a^2}{E}\left(\frac{2a'v'}{a}-\frac{E'v'}{E}\right)-a'f^3v'^3-\left(\frac{a^2}{E}\right)'v'\\
&= -\lambda NaEH(r,v)\left(1-f^2v'^2\right)^{3/2}-\frac{N-1}{r+\epsilon}af^3v'^3\\
&\quad-a'f^3v'^3.
\end{aligned}$$

So, we have that

$$\begin{aligned}
-\left((r+\epsilon)^{N-1}\frac{a^2(r)}{E(r)}v'\right)' &= -\lambda N(r+\epsilon)^{N-1}aEH(r,v)\left(1-f^2v'^2\right)^{3/2}\\
&\quad-(N-1)(r+\epsilon)^{N-2}af^3v'^3-(r+\epsilon)^{N-1}a'f^3v'^3\\
&:= F(\lambda,r,v,v').
\end{aligned}$$

Now, we consider the following problem

$$\begin{cases}
-\left((r+\epsilon)^{N-1}\dfrac{a^2(r)}{E(r)}v'\right)'=F(\lambda,r,v,v'),\\
v'(0)=v(1)=0.
\end{cases}\tag{2.1}$$

Using the above expanding process, we can see that if $v$ is a solution of the following approximation problem

$$\begin{cases}
-\dfrac{1}{(r+\epsilon)^{N-1}E}\left(\dfrac{(r+\epsilon)^{N-1}av'}{E\sqrt{1-f^2v'^2}}\right)'-\dfrac{a'v'}{E^2\sqrt{1-f^2v'^2}}=-\lambda NH(r,v),\ r\in(0,1),\\
v'(0)=v(1)=0,
\end{cases}\tag{2.2}$$

then $v$ is a solution of problem (2.1).

Conversely, the following lemma implies that $v$ is a solution of problem (2.2) if $v$ is a solution of problem (2.1).

**Lemma 2.1.** *For any solution $v$ of problem* (2.1) *with $\epsilon \geq 0$, we have that $|v'| < E/a$ on* $[0,1]$.

**Proof.** Suppose, by contradiction, that $(a|v'|)/E$ can achieve 1 on $[0,1]$. Let $r^*$ be the first such kind of point. Since $v'(0)=0$, one has that $r^*\in(0,1]$. Note that $v$ satisfies

$$\left((r+\epsilon)^{N-1}\phi(fv')\right)'=\lambda N(r+\epsilon)^{N-1}EH(r,v)-\frac{(r+\epsilon)^{N-1}a'v'}{E\sqrt{1-f^2v'^2}},\ r\in(0,r^*),$$

where $\phi(s)=s/\sqrt{1-s^2}$. Since $a(r^*)|v'(r^*)|=E(r^*)$, there exists $r_*\in(0,r^*)$ such that $f|v'|>1/2$ for all $r\in(r_*,r^*)$. It follows that

$$\frac{\left((r+\epsilon)^{N-1}\phi(fv')\right)'}{(r+\epsilon)^{N-1}\phi(fv')}=\lambda N\frac{EH(r,v)}{\phi(fv')}-\frac{a'}{a}$$

for $r \in (r_*, r^*)$. Integrating this equality from $r_*$ to $r \in (r_*, r^*)$, we obtain that

$$\begin{aligned}\ln\left|(r+\epsilon)^{N-1}\phi\left(fv'\right)\right| &- \ln\left|(r_*+\epsilon)^{N-1}\phi\left(f\left(r^*\right)v'\left(r^*\right)\right)\right| \\ &= N\lambda\int_{r_*}^{r}\frac{EH\left(\tau,v\right)}{\phi\left(fv'\right)}\,d\tau - \int_{r_*}^{r}\frac{a'}{a}\,d\tau.\end{aligned}$$

Letting $r \to r^*$, we see that the left side term tends to infinity while the right one is bounded, which is a contradiction. ■

Due to Lemma 2.1, problem (2.2) is equivalent to problem (2.1). Furthermore, we have the following monotonicity result.

**Lemma 2.2.** *Assume that $H(r,t)t < 0$ for any $r \in (0,1)$, $t \in (-\delta,\delta) \setminus \{0\}$. Then, for any nontrivial solution $v$ with $v \geq 0$ ($v \leq 0$) of problem (2.2) with $\epsilon \geq 0$, one has that $v > 0$ ($v < 0$) on $[0,1)$ and $v' < 0$ ($v' > 0$) on $(0,1]$.*

**Proof.** We only prove the case of $v \geq 0$ because the proof of $v \leq 0$ is similar. Note that

$$\left((r+\epsilon)^{N-1}\phi\left(fv'\right)\right)' + \frac{a'}{a}(r+\epsilon)^{N-1}\phi\left(fv'\right) = \lambda N(r+\epsilon)^{N-1}EH(r,v).$$

By the variation of constants formula, we have that

$$\begin{aligned}(r+\epsilon)^{N-1}\phi\left(fv'\right) &= \int_0^1 \lambda N(t+\epsilon)^{N-1}EH(t,v)e^{-\int_t^r \frac{a'}{a}\,ds}\,dt \\ &= \lambda N\int_0^1 (t+\epsilon)^{N-1}E(t)H(t,v)\frac{a(t)}{a(r)}\,dt.\end{aligned}$$

It follows that

$$fv' = \phi^{-1}\left(\frac{\lambda N}{(r+\epsilon)^{N-1}}\int_0^1 (t+\epsilon)^{N-1}E(t)H(t,v)\frac{a(t)}{a(r)}\,dt\right),$$

where $\phi^{-1}$ is the inverse function of $\phi$. It is easy to verify that $\phi^{-1}$ is an odd increasing diffeomorphism. It follows that $v' < 0$ on $(0,1]$. Since $v$ is nontrivial, we must have $v(0) > 0$. In view of $v(1) = 0$, we get that $v > 0$ on $[0,1)$. ■

Next, we give a Dancer's type unilateral global bifurcation result which will be used later. Let $E$ be a real Banach space with the norm $\|\cdot\|$ and $\mathcal{O}$ be an open subset of $\mathbb{R} \times E$. Consider the following operator equation

$$u = \lambda Lu + H(\lambda, u), \tag{2.3}$$

where $L : X \longrightarrow X$ is a linear compact operator and $H : \overline{\mathcal{O}} \longrightarrow \mathrm{pr}_E\left(\overline{\mathcal{O}}\right)$ is completely continuous with $H = o(\|u\|)$ at $u = 0$ uniformly on bounded $\lambda$ intervals in $\mathrm{pr}_{\mathbb{R}}\left(\overline{\mathcal{O}}\right)$. Let

$$\mathscr{S} := \overline{\{(\lambda,u) : (\lambda,u) \text{ satisfies equation (2.3) and } u \not\equiv 0\ \}}^{\mathcal{O}}.$$

Let $r(L)$ be the characteristic value set of $L$. By an arguments similar to that of [17, Theorem 2] (or [18, Theorem]) with obvious changes, we obtain the following result, which is the local version of Theorem 2 of [17].

**Lemma 2.3.** *If $\mu \in pr_{\mathbb{R}}(\mathcal{O}) \cap r(L)$ is geometric multiplicity 1 and odd algebraic multiplicity, then $\mathscr{S}$ possesses two maximal continua $\mathscr{C}_\mu^+$, $\mathscr{C}_\mu^- \subset \overline{\mathcal{O}}$ such that $(\mu,0) \in \mathscr{C}_\mu^\pm$ and one of the following three properties is satisfied by $\mathscr{C}_\mu^\pm$:*

(i) *$\mathscr{C}_\mu^+$ and $\mathscr{C}_\mu^-$ are both unbounded in $\overline{\mathcal{O}}$,*

(ii) $\mathscr{C}_\mu^+$ *and* $\mathscr{C}_\mu^-$ *both meets* $\partial\mathcal{O}$,
(iii) $\mathscr{C}_\mu^+ \cap \mathscr{C}_\mu^- \neq \{(\mu,0)\}$.

To investigate the bifurcation phenomenon of problem (2.1), we consider the following eigenvalue problem

$$\begin{cases} -\left((r+\epsilon)^{N-1}\dfrac{a^2(r)}{E(r)}u'\right)' = \lambda(r+\epsilon)^{N-1}a(r)E(r)u, \ r\in(0,1), \\ u'(0)=u(1)=0. \end{cases} \tag{2.4}$$

From [29, p. 269], we know that problem (2.4) has a principal eigenvalue $\lambda_1(\epsilon)$, which is simple and isolated. Then, we have the following unilateral global bifurcation result for problem (2.1).

**Theorem 2.1.** *Under the assumptions of Theorem* 1.1*,*

(a) *if* $H_0=1$*, there are two unbounded components,* $\mathscr{C}_\epsilon^+$ *and* $\mathscr{C}_\epsilon^-$*, of the set of nontrivial solutions of problem* (2.1) *bifurcating from* $(\lambda_1(\epsilon),0)$ *such that* $\mathscr{C}_\epsilon^\nu\subseteq(\mathbb{R}\times P^\nu)\cup\{(\lambda_1(\epsilon),0)\}$*,* $(\lambda_1(\epsilon),+\infty)\subseteq pr_{\mathbb{R}}(\mathscr{C}_\epsilon^\nu)$*,* $\|v_\lambda^\epsilon\|<1$ *and* $\lim_{\lambda\to+\infty}\|v_\lambda^\epsilon\|=1$ *for* $(\lambda,v_\lambda^\epsilon)\in(\mathscr{C}_\epsilon^+\cup\mathscr{C}_\epsilon^-)\setminus\{(\lambda_1(\epsilon),0)\}$,

(b) *if* $H_0=+\infty$*, there are two unbounded components,* $\mathscr{C}_\epsilon^+$ *and* $\mathscr{C}_\epsilon^-$*, of the set of nontrivial solutions of problem* (2.1) *emanating from* $(0,0)$ *such that* $\mathscr{C}_\epsilon^\nu\subseteq(\mathbb{R}\times P^\nu)\cup\{(0,0)\}$*, joins to* $(+\infty,1)$ *and* $\|v_\lambda^\epsilon\|<1$ *for* $(\lambda,v_\lambda^\epsilon)\in(\mathscr{C}_\epsilon^+\cup\mathscr{C}_\epsilon^-)\setminus\{(0,0)\}$,

(c) *if* $H_0=0$*, there are two unbounded components,* $\mathscr{C}_\epsilon^+$ *and* $\mathscr{C}_\epsilon^-$*, of the set of nontrivial solutions of problem* (2.1) *such that* $\mathscr{C}_\epsilon^\nu\subseteq\mathbb{R}\times P^\nu$*, joins* $(+\infty,1)$ *to* $(+\infty,0)$ *and* $\|v_\lambda^\epsilon\|<1$ *for any* $(\lambda,v_\lambda^\epsilon)\in\mathscr{C}^\nu$ *with* $\lambda<+\infty$.

**Proof.** (a) Define $\xi(r,t)=NH(r,t)+t$, then we have that

$$\lim_{t\to0}\frac{\xi(r,t)}{t}=0.$$

Consider

$$\begin{cases} \mathscr{L}v=\lambda(r+\epsilon)^{N-1}\left(1-f^2v'^2\right)^{3/2}aE\left(v-\xi(r,v)\right)-K\left(\lambda,r,v,v'\right), \\ v'(0)=v(1)=0, \end{cases} \tag{2.5}$$

where $\mathscr{L}v=-\left((r+\epsilon)^{N-1}\dfrac{a^2(r)}{E(r)}v'\right)'$ and

$$K\left(\lambda,r,v,v'\right)=(r+\epsilon)^{N-1}a'f^3v'^3+(N-1)(r+\epsilon)^{N-2}af^3(r)v'^3.$$

Let $G(r,s)$ be the Green's function associated to the operator $\mathscr{L}v$ with the same boundary condition as problem (2.5). Then problem (2.5) can be equivalently written as

$$v=\lambda Lv+H(\lambda,v):=\Psi(\lambda,v),$$

where

$$Lv=\int_0^1 G(r,s)(s+\epsilon)^{N-1}E(s)a(s)v(s)\,ds$$

and

$$\begin{aligned} H(\lambda, v) &= \lambda \int_0^1 G(r,s)(s+\epsilon)^{N-1} Eav\left[\left(1-f^2v'^2\right)^{3/2}-1\right] ds \\ &\quad -\lambda \int_0^1 G(r,s)(s+\epsilon)^{N-1}\left(1-f^2v'^2\right)^{3/2} Ea\xi(s,v)\, ds \\ &\quad -(N-1)\int_0^1 G(r,s)(s+\epsilon)^{N-2} af^3(s)v'^3\, ds - \int_0^1 G(r,s)(s+\epsilon)^{N-1}a'f^3v'^3\, ds. \end{aligned}$$

It is well known that $L : X \longrightarrow X$ is linear compact (see [24]). It follows from $f\,|v'| < 1$ on $[0,1]$ that $\|v\| < 1$. By the conclusion of [24], we know that $H : \mathbb{R}\times\overline{\mathcal{B}} \longrightarrow \overline{\mathcal{B}}$ is completely continuous, where $\mathcal{B} = \{v \in X : \|v\| < 1\}$.

We claim that $H = o(\|v\|)$ near $v = 0$ uniformly on bounded $\lambda$ intervals. Let

$$\widetilde{\xi}(r,u) = \max_{|s|\le u} |\xi(r,s)|,\ \ r \in (0,1).$$

Then $\widetilde{\xi}$ is nondecreasing with respect to $u$ and

$$\lim_{u\to 0^+} \frac{\widetilde{\xi}(r,u)}{u} = 0. \tag{2.6}$$

It follows from (2.6) that

$$\left|\frac{\xi(r,v)}{\|v\|}\right| \le \frac{\widetilde{\xi}(r,v)}{\|v\|} \le \frac{\widetilde{\xi}\,(r,\|v\|_\infty)}{\|v\|} \le \frac{\widetilde{\xi}\,(r,\|v'\|_\infty)}{\|v\|} \le \delta\frac{\widetilde{\xi}\,(r,\|v\|\,\delta)}{\delta\|v\|} \to 0 \text{ as } \|v\| \to 0$$

uniformly in $r \in (0,1)$. So, we have that

$$\frac{Ea\left(1-f^2v'^2\right)^{3/2}\xi(r,v)}{\|v\|} \to 0 \text{ as } \|v\| \to 0$$

uniformly in $r \in (0,1)$. Clearly, one has that $1/(r+\epsilon) \le 1/\epsilon$ for any $r \in [0,1]$ and

$$\frac{v\left[\left(1-f^2v'^2\right)^{3/2}-1\right]}{\|v\|} \to 0,\ \ \frac{a'f^3v'^3}{\|v\|} \to 0,\ \ \frac{af^3v'^3}{\|v\|} \to 0 \text{ as } \|v\| \to 0$$

uniformly in $r \in (0,1)$. It follows the claim as desired.

Applying Lemma 2.3 with $\mathcal{O} = \mathbb{R}\times\mathcal{B}$, there exist two continua, $\mathscr{C}_\epsilon^+$ and $\mathscr{C}_\epsilon^-$, of solution set of problem (2.5) emanating from $(\lambda_1(\epsilon), 0)$ which satisfy one of the following three properties:

(i) $\mathscr{C}_\epsilon^+$ and $\mathscr{C}_\epsilon^-$ are both unbounded in $\overline{\mathcal{O}}$,
(ii) $\mathscr{C}_\epsilon^+$ and $\mathscr{C}_\epsilon^-$ both meets $\mathbb{R}\times\partial\mathcal{B}$,
(iii) $\mathscr{C}_\epsilon^+ \cap \mathscr{C}_\epsilon^- \ne \{(\lambda_1(\epsilon), 0)\}$.

By Lemma 2.1, we have $\|v\| < 1$ for any $(\lambda, v) \in \mathscr{C}_\epsilon^+ \cup \mathscr{C}_\epsilon^-$. Thus, the second alternative does not occur. From the variation of constants formula, we can see that problem (2.2) has only trivial solution if $\lambda = 0$. Since 0 is not an eigenvalue of problem (2.4), we have that $\mathscr{C}_\epsilon^\pm \cap (\{0\}\times X) = \emptyset$.

Let $\varphi_1^\epsilon$ be an eigenfunction corresponding to $\lambda_1(\epsilon)$. Lemma 1.24 of [24] implies that if $(\lambda, v) \in \mathscr{C}_\epsilon^+ \cup \mathscr{C}_\epsilon^-$ and is near $(\lambda_1(\epsilon), 0)$, then $v = \alpha\varphi_1^\epsilon + w$ with $w = o(|\alpha|)$ for $\alpha$ near 0. So, there exists an open neighborhood $\mathscr{N}$ of $(\lambda_1(\epsilon), 0)$ such that

$$\left(\left(\mathscr{C}_\epsilon^\pm \setminus \{(\lambda_1(\epsilon), 0)\}\right)\cap\mathscr{N}\right) \subset \left(\mathbb{R}\times P^\pm\right).$$

We claim that $\mathscr{C}_\epsilon^\pm \subset ((\mathbb{R}\times P^\pm)\cup\{(\lambda_1(\epsilon), 0)\})$. Suppose, by contradiction, that there exists $(\lambda, v) \in ((\mathscr{C}_\epsilon^\pm \setminus \{(\lambda_1(\epsilon), 0)\})\cap(\mathbb{R}\times\partial P^\pm))$ such that $(\lambda, v)$ is the limit in $\mathbb{R}\times X$ of $(\lambda_n, v_n) \in \mathbb{R}\times P^\pm$. Thus, $v$ has either an interior zero in $[0,1)$ or $v'(1) = 0$. It follows from Lemma 2.2 that $v \equiv 0$. So $\lambda = \lambda_j$ is an eigenvalue of problem (2.4) for some $j > 1$. By Lemma 1.24 of [24], we have that $v_n = \alpha\varphi_j^\epsilon + w_n$ with $w_n = o(|\alpha|)$ for $\alpha$ near 0, where $\varphi_j^\epsilon$ is an eigenfunction corresponding to $\lambda_j(\epsilon)$. It is well known that $\varphi_j^\epsilon$ changes its sign (see [20, 29]). Thus, $v_n$ must change its sign for $n$ large enough, which is a contradiction. Therefore, we verify this claim. It follows that $\mathscr{C}_\epsilon^+ \cap \mathscr{C}_\epsilon^- = \{(\lambda_1(\epsilon), 0)\}$. So, both $\mathscr{C}_\epsilon^+$ and $\mathscr{C}_\epsilon^-$ are unbounded in $(0, +\infty) \times \mathcal{B}$. Obviously, the projection of $\mathscr{C}_\epsilon^\pm$ on $(0, +\infty)$ is unbounded. Therefore, we have that $(\lambda_1(\epsilon), +\infty) \subseteq \mathrm{pr}_{\mathbb{R}}(\mathscr{C}_\epsilon^\nu)$.

Next, we show the asymptotic behavior of $v_\lambda$ as $\lambda \to +\infty$ for $(\lambda, v_\lambda) \in \mathscr{C}_\epsilon^\nu \setminus \{(\lambda_1(\epsilon), 0)\}$. Suppose, by contradiction, that there exist a constant $\delta_0 > 0$ and $(\lambda_n, v_n) \in \mathscr{C}_\epsilon^\nu \setminus \{(\lambda_1(\epsilon), 0)\}$ with $\lambda_n \to +\infty$ as $n \to +\infty$ such that $\|v_n\|^2 \le 1 - \delta_0^2$ for any $n \in \mathbb{N}$. Without loss of generality, we assume that $(\lambda_n, v_n) \in \mathscr{C}_\epsilon^+ \setminus \{(\lambda_1(\epsilon), 0)\}$. Note that $(\lambda_n, v_n)$ satisfies the following problem

$$\begin{cases} -\left((r+\epsilon)^{N-1}\phi(fv')\right)' = \dfrac{a'}{a}(r+\epsilon)^{N-1}\phi(fv') + \lambda N(r+\epsilon)^{N-1}m(r)E(r)v, \\ v'(0) = v(1) = 0, \end{cases} \tag{2.7}$$

where

$$m(r) = \frac{-H(r, v(r))}{v(r)}.$$

Our assumptions of $H$ imply that there exists a positive constant $\rho_0$ such that

$$m(r) \ge \rho_0$$

for any $r \in (0, 1)$. By some elementary calculations, we find that problem (2.7) is equivalent to

$$\begin{cases} -\left(\frac{(r+\epsilon)^{N-1}a^2v'}{E\sqrt{1-f^2v'^2}}\right)' = \lambda N(r+\epsilon)^{N-1}a(r)m(r)E(r)v, \\ v'(0) = v(1) = 0. \end{cases} \tag{2.8}$$

Multiplying the first equation of problem (2.8) by $\varphi_1^\epsilon$, we obtain after integration by parts that

$$\begin{aligned} \frac{\lambda_1(\epsilon)}{\delta_0}\int_0^1 (r+\epsilon)^{N-1}a(r)E(r)v_n\varphi_1^\epsilon\,dr &= \frac{1}{\delta_0}\int_0^1 (r+\epsilon)^{N-1}afv_n'(\varphi_1^\epsilon)'\,dr \\ &\ge \int_0^1 (r+\epsilon)^{N-1}\frac{afv_n'(\varphi_1^\epsilon)'}{\sqrt{1-f^2|v_n'|^2}}\,dr \\ &= \lambda_n N\int_0^1 (r+\epsilon)^{N-1}a(r)m(r)E(r)v_n\varphi_1^\epsilon\,dr \\ &\ge \lambda_n N\rho_0\int_0^1 (r+\epsilon)^{N-1}a(r)E(r)v_n\varphi_1^\epsilon\,dr. \end{aligned}$$

It follows that $\lambda_n \le \lambda_1(\epsilon)/(N\delta_0\rho_0)$, which is a contradiction.

(b) For any $n \in \mathbb{N}$, define

$$H^n(r, s) = \begin{cases} -ns, & s \in \left[-\frac{1}{n}, \frac{1}{n}\right], \\ n\left(H\left(r, \frac{2}{n}\right) + 1\right)\left(s - \frac{1}{n}\right) - 1, & s \in \left(\frac{1}{n}, \frac{2}{n}\right), \\ n\left(-H\left(r, -\frac{2}{n}\right) + 1\right)\left(s + \frac{1}{n}\right) + 1, & s \in \left(-\frac{2}{n}, -\frac{1}{n}\right), \\ H(r, s), & s \in \left(-\infty, -\frac{2}{n}\right] \cup \left[\frac{2}{n}, +\infty\right). \end{cases}$$

Clearly, we see that $\lim_{n\to+\infty} H^n(r,s) = H(r,s)$ and $H_0^n = n$. Consider the following problem

$$\begin{cases} -\left((r+\epsilon)^{N-1}\phi\left(fv'\right)\right)' = \dfrac{a'}{a}(r+\epsilon)^{N-1}\phi\left(fv'\right) - \lambda N(r+\epsilon)^{N-1}EH^n\left(r,v\right), \\ v'(0) = v(1) = 0. \end{cases} \tag{2.9}$$

By the conclusion of (a), there exist two sequences unbounded continua $\mathscr{C}_{\epsilon,n}^+$ and $\mathscr{C}_{\epsilon,n}^-$ of the set of nontrivial solutions of problem (2.9) emanating from $(\lambda_1(\epsilon)/n, 0)$ and joining to $(+\infty, 1)$ such that

$$\mathscr{C}_{\epsilon,n}^\pm \subseteq \left(\left(\mathbb{R}\times P^\pm\right)\cup\{(\lambda_1(\epsilon)/n,0)\}\right).$$

Taking $z^* = (0,0)$, clearly, one has that $z^* \in \liminf_{n\to+\infty}\mathscr{C}_{\epsilon,n}^\pm$. The compactness of $\Psi$ implies that $\left(\cup_{n=1}^{+\infty}\mathscr{C}_{\epsilon,n}^\pm\right)\cap\mathbb{B}_R$ is pre-compact, where $\mathbb{B}_R = \{z\in\mathbb{R}\times X : \|z\| < R\}$ for any $R > 0$. By Theorem 2.1 of [11], $\mathscr{C}_\epsilon^\pm = \limsup_{n\to+\infty}\mathscr{C}_{\epsilon,n}^\pm$ is unbounded and connected such that $z^* \in \mathscr{C}_\epsilon^\pm$ and $(+\infty,1)\in\mathscr{C}_\epsilon^\pm$.

For any $(\lambda, v) \in \mathscr{C}_\epsilon^\pm$, the definition of superior limit (see [30]) implies that there exists a sequence $(\lambda_n, v_n) \in \mathscr{C}_{\epsilon,n}^\pm$ such that $(\lambda_n, v_n) \to (\lambda, v)$ as $n \to +\infty$, which implies that $v$ is a solution of problem (2.1). We claim that $\mathscr{C}_\epsilon^\pm \cap ((0,+\infty)\times\{0\}) = \emptyset$. Suppose, by contradiction, that there exists $\mu > 0$ such that $(\mu, 0) \in \mathscr{C}_\epsilon^\pm$. There exists $N_0 > 0$ such that $\mu > \lambda_1(\epsilon)/n$ for any $n > N_0$. It follows that $(\mu,0)\notin\mathscr{C}_{\epsilon,n}^\pm$ for any $n > N_0$. So, we have that $(\mu, 0)\notin\mathscr{C}_\epsilon^\pm$, which is a contradiction. Clearly, $v\in\overline{P^\pm}$ for any $(\lambda,v)\in\mathscr{C}_\epsilon^\pm\setminus\{(0,0)\}$. Further, by Lemma 2.2, we have $v \in P^\pm$ for any $(\lambda, v) \in \mathscr{C}_\epsilon^\pm \setminus \{(0, 0)\}$. By Lemma 2.1, we have that $\|v\| < 1$ for any $(\lambda, v) \in \mathscr{C}_\epsilon^+ \cup \mathscr{C}_\epsilon^-$.

(c) For any $n\in\mathbb{N}$, define

$$H_n(r,s) = \begin{cases} -\frac{1}{n}s, & s\in\left[-\frac{1}{n},\frac{1}{n}\right], \\ \left(H\left(r,\frac{2}{n}\right)+\frac{1}{n^2}\right)n\left(s-\frac{1}{n}\right)-\frac{1}{n^2}, & s\in\left(\frac{1}{n},\frac{2}{n}\right), \\ \left(-H\left(r,-\frac{2}{n}\right)+\frac{1}{n^2}\right)n\left(s+\frac{1}{n}\right)+\frac{1}{n^2}, & s\in\left(-\frac{2}{n},-\frac{1}{n}\right), \\ H(r,s), & s\in\left(-\infty,-\frac{2}{n}\right]\cup\left[\frac{2}{n},+\infty\right). \end{cases}$$

Then, we consider the following problem

$$\begin{cases} -\left((r+\epsilon)^{N-1}\phi\left(fv'\right)\right)' = \dfrac{a'}{a}(r+\epsilon)^{N-1}\phi\left(fv'\right) - \lambda N(r+\epsilon)^{N-1}E(r)H_n\left(r,v\right), \\ v'(0) = v(1) = 0. \end{cases} \tag{2.10}$$

It is easy to see that $\lim_{n\to+\infty} H_n(r,s) = H(r,s)$ and

$$\lim_{s\to0}\frac{H_n(r,s)}{s} = \frac{1}{n} \text{ uniformly in } r\in(0,1).$$

By the conclusion of (a), there exist two sequences unbounded continua $\mathscr{C}_{\epsilon,n}^+$ and $\mathscr{C}_{\epsilon,n}^-$ of one-sign solutions set of problem (2.10) in $\mathbb{R}\times X$ emanating from $(\lambda_1(\epsilon)n, 0)$ for any $n\in\mathbb{N}$ and joining to $(+\infty,1) := z_*$.

Taking $z^* = (+\infty, 0)$, clearly, we have that $z^* \in \liminf_{n\to+\infty}\mathscr{C}_{\epsilon,n}^\pm$ with $\|z^*\|_{\mathbb{R}\times X} = +\infty$. Let

$$S = \{(+\infty, v) : 0 < \|v\| < 1\}.$$

For fixed $n\in\mathbb{N}$, we claim that $\mathscr{C}_{\epsilon,n}^\pm\cap S = \emptyset$. Suppose, by contradiction, that there exists a sequence $(\lambda_m, v_m)\in\mathscr{C}_{\epsilon,n}^\pm$ such that $(\lambda_m, v_m)\to(+\infty, v_*)\in S$ with $\|v_*\|\in(0,1)$. Then by the argument as that of (a), we obtain that $\lambda_m \le c_n$ for some positive constant $c_n$, which is a contradiction. It follows that $\left(\cup_{n=1}^{+\infty}\mathscr{C}_{\epsilon,n}^\pm\right)\cap S = \cup_{n=1}^{+\infty}\left(\mathscr{C}_{\epsilon,n}^\pm\cap S\right) = \emptyset$. Letting $\mathscr{C}_\epsilon^\pm = \limsup_{n\to+\infty}\mathscr{C}_{\epsilon,n}^\pm$, since $\mathscr{C}_\epsilon^\pm \subseteq \cup_{n=1}^{+\infty}\mathscr{C}_{\epsilon,n}^\pm$, we have that $\mathscr{C}_\epsilon^\pm\cap S = \emptyset$. Therefore, we have that $\mathscr{C}_\epsilon^\pm\cap\{\infty\} = \{z_*, z^*\}$.

Now we show that $\mathscr{C}_\epsilon^\pm \setminus \{\infty\} \neq \emptyset$. It is enough to show that the projection of $\mathscr{C}_\epsilon^\pm$ on $\mathbb{R}$ is nonempty. From the argument of (a), we have known that $\mathscr{C}_{\epsilon,n}^\pm$ has unbounded projection on $\mathbb{R}$ for any fixed $n \in \mathbb{N}$. By Proposition 2 of [12], for any fixed $\sigma > 0$ there exists an $N_1 > 0$ such that for every $n > N_1$, $\mathscr{C}_{\epsilon,n}^\pm \subset V_\sigma\left(\mathscr{C}_\epsilon^\pm\right)$, where $V_\sigma\left(\mathscr{C}_\epsilon^\pm\right)$ denotes the $\sigma$-neighborhood of $\mathscr{C}_\epsilon^\pm$ in $\mathbb{R} \times X$. It follows that

$$(\lambda_1(\epsilon)n, +\infty) \subseteq \mathrm{pr}_{\mathbb{R}}\left(\mathscr{C}_{\epsilon,n}^\pm\right) \subseteq \mathrm{pr}_{\mathbb{R}}\left(V_\sigma\left(\mathscr{C}_\epsilon^\pm\right)\right).$$

So, we have that $(n\lambda_1(\epsilon) + \sigma, +\infty) \subseteq \mathrm{pr}_{\mathbb{R}}\left(\mathscr{C}_\epsilon^\pm\right)$, which implies $\mathscr{C}_\epsilon^\pm \setminus \{\infty\} \neq \emptyset$. Using Lemma 3.1 of [14], we obtain that $\mathscr{C}_\epsilon^\pm$ is connected. By an argument similar to that of (b), we can show that $\mathscr{C}_\epsilon^\pm \cap ([0, +\infty) \times \{0\}) = \emptyset$ and $v$ is one-sign solution of problem (2.1) for any $(\lambda, v) \in \mathscr{C}_\epsilon^\pm$. ■

Note that the monotonicity of $a$ is only used to obtain the asymptotic behavior of $v_\lambda$ as $\lambda \to +\infty$ for $(\lambda, v_\lambda) \in \mathscr{C}_\epsilon^\nu \setminus \{(\lambda_1(\epsilon), 0)\}$. If we don't care the asymptotic behavior, this condition can be removed.

## 3. **Convergence of solutions as $\epsilon \to 0$.**

From Theorem 2.1, we can obtain the existence and multiplicity of solutions of the approximation problem (2.2). To study the convergence of solutions of problem (2.2) as $\epsilon \to 0^+$, we first provide a boundary derivative a priori bound estimate as follows.

**Proposition 3.1.** *For any solution $v$ of problem* (2.2)*, there exists $\gamma \in (0,1)$ such that $f(1)\left|v'(1)\right| \leq \gamma$.*

**Proof.** Let $u = -\int_r^1 f(\tau)v'(\tau)\,d\tau$. Then, clearly, one has that $u' = fv'$ and $v = -\int_r^1 u'(\tau)/f(\tau)\,d\tau := T[u]$. So, we have that

$$\begin{cases} -\dfrac{1}{(r+\epsilon)^{N-1}}\left(\dfrac{(r+\epsilon)^{N-1}u'}{\sqrt{1-u'^2}}\right)' - \dfrac{a'u'}{a\sqrt{1-u'^2}} = -\lambda NEH\left(r, T[u]\right), \, r \in (0,1), \\ u'(0) = u(1) = 0. \end{cases}$$

Consider the following elliptic operator

$$Q_u(w)(r) := -\frac{1}{(r+\epsilon)^{N-1}}\left(\frac{(r+\epsilon)^{N-1}w'}{\sqrt{1-w'^2}}\right)' - \frac{a'u'}{a\sqrt{1-w'^2}}.$$

Define $w^+ : [1/2, 1] \to \mathbb{R}$ by

$$w^+(r) = \int_0^{1-r} \frac{1}{\sqrt{1+\beta(t)}}\,dt,$$

where $\beta(t) = \alpha e^{\mu t}$ with positive constants $\alpha$ and $\mu$ which will be determined later.

Clearly, $w^+(r)$ is decreasing and

$$\left(w^+\right)'(r) = -\frac{1}{\sqrt{1+\beta(1-r)}}.$$

Then, we have that

$$Q_u\left(w^+\right)(r) = \frac{1}{\sqrt{\beta(1-r)}}\left[\frac{N-1}{r+\epsilon} + \frac{\mu}{2} - \frac{a'u'}{a}\sqrt{1+\alpha e^{\mu(1-r)}}\right].$$

For any $\mu > 0$, choose $\alpha \in (0, e^{-\mu}]$. It follows that, for any $r \in [1/2, 1]$,

$$Q_u\left(w^+\right)(r) \geq \left[\frac{N-1}{r+\epsilon} + \frac{\mu}{2} - \sqrt{2}a^0\right],$$

where $a^0 = \max_{[1/2,1]} \left(|a'|/a\right)$.

Let

$$\theta = \max\left\{ -\lambda NEH\left(r,t\right) : r \in \left[\frac{1}{2}\,,\,1\right], t \in [-\delta,\delta] \right\}$$

and choose $\mu$ large enough such that

$$\frac{\mu}{2} \geq \sqrt{2}a^0 + \theta.$$

So, we obtain that, for any $r \in [\,1/2\,,\,1\,]$,

$$Q_u\left(w^+\right)(r) \geq Q_u(u)(r).$$

If $w^+(1/2) \geq u(1/2)$, then by the Comparison Principle of [21, Theorem 4.4], we have that

$$w^+(r) \geq u(r),\;\; r \in \left[\frac{1}{2}, 1\right].$$

Since $w^+(1) = u(1) = 0$, we conclude that

$$u'(1) \geq \left(w^+\right)'(1) = -\frac{1}{\sqrt{1+\alpha}}.$$

From now on, we assume that $w^+(1/2) < u(1/2)$. Set $K := \max_{[1/2,1]} |u'|$. By Lemma 2.1, one has that $K < 1$. For any fixed $r_0 \in ((1+K)/2, 1)$, we have that

$$\frac{r_0 - \frac{1}{2}}{u\left(\frac{1}{2}\right) - w^+\left(r_0\right)} \geq \frac{r_0 - \frac{1}{2}}{u\left(\frac{1}{2}\right) - u\left(1\right)} \geq \frac{r_0 - \frac{1}{2}}{\frac{K}{2}} > 1.$$

Then, choose $\alpha_u > 0$ such that

$$\alpha_u < \min\left\{ \alpha, \left( \left( \frac{r_0 - \frac{1}{2}}{u\left(\frac{1}{2}\right) - w^+\left(r_0\right)} \right)^2 - 1 \right) e^{-\frac{\mu}{2}} \right\}$$

and define

$$\overline{\alpha}(s) = \begin{cases} \alpha & \text{if } s \in [0, 1-r_1]\,, \\ h(s) & \text{if } s \in (1-r_1, 1-r_0]\,, \\ \alpha_u & \text{if } s \in \left(1-r_0, \frac{1}{2}\right] \end{cases}$$

for any fixed $r_1 \in (r_0, 1)$, where $h$ is a decreasing function such that $\overline{\alpha}$ being differentiable. Define $w_u^+ : [1/2, 1] \to \mathbb{R}$ by

$$w_u^+(r) = \int_0^{1-r} \frac{1}{\sqrt{1+\overline{\alpha}(t)e^{\mu t}}}\, dt.$$

It follows that

$$w_u^+\left(\frac{1}{2}\right) = \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1+\overline{\alpha}(t)e^{\mu t}}}\, dt \geq \int_{1-r_0}^{\frac{1}{2}} \frac{1}{\sqrt{1+\alpha_u(t)e^{\frac{\mu}{2}}}}\, dt > u\left(\frac{1}{2}\right) - w^+\left(r_0\right).$$

Letting $r_0 \to 1$, we obtain $w_u^+(1/2) \geq u(1/2)$. It follows from the monotonicity of $Q_u\left(w^+\right)$ with respect to $\alpha$, we have that

$$Q_u\left(w_u^+\right)(r) \geq Q_u\left(w^+\right)(r).$$

Then, reasoning as the above case, we can derive that

$$u'(1) \geq \left(w_u^+\right)'(1) = -\frac{1}{\sqrt{1+\alpha}}.$$

Similarly, considering $-w^+(r)$ and $-w_u^+(r)$, we can obtain that

$$u'(1) \leq \frac{1}{\sqrt{1+\widehat{\alpha}}}$$

for some positive constant $\widehat{\alpha}$. Taking

$$\gamma = \max\left\{\frac{1}{\sqrt{1+\widehat{\alpha}}}, \frac{1}{\sqrt{1+\alpha}}\right\},$$

we have that

$$|u'(1)| \leq \gamma.$$

Therefore, we obtain that $f(1)\,|v'(1)| \leq \gamma$. ∎

The following convergence conclusion is our main result of this section.

**Theorem 3.1.** *For any one-sign solution $(\lambda, v_\epsilon)$ of problem (2.2) with any fixed $\lambda$, up to a subsequence, the limit of $v_\epsilon$ exists as $\epsilon \to 0^+$ which is denoted by $v$, and $(\lambda, v)$ is the solution of problem (1.6).*

**Proof.** By Lemma 2.1, we can easily verify that $\|v_\epsilon\|_\infty < \delta$ and $\|v_\epsilon'\|_\infty < \delta$. By the Arzelà-Ascoli Theorem, up to a subsequence, there exists $v \in C[0,1]$ such that $v_\epsilon \to v$ in $C[0,1]$ as $\epsilon \to 0^+$. Note that

$$\begin{aligned} -v_\epsilon'' &= -\lambda N \frac{E^2}{a} H(r, v_\epsilon)\left(1 - f^2 v_\epsilon'^2\right)^{3/2} + \frac{N-1}{r+\epsilon} v_\epsilon' \left(1 - f^2 v_\epsilon'^2\right) + \\ &\quad \frac{f' v_\epsilon'}{f}\left(1 - f^2 v_\epsilon'^2\right) + f f' v_\epsilon'^3 + \frac{a' v_\epsilon'}{a}\left(1 - f^2 v_\epsilon'^2\right) := \mathcal{G}_\epsilon(r, v_\epsilon, v_\epsilon'), \end{aligned} \tag{3.1}$$

which implies that $|v_\epsilon''|$ is uniformly bounded on $[\kappa, 1]$ with respect to $\epsilon$ for any fixed $\kappa \in (0,1]$. Thus, $\{v_\epsilon'\}$ is equicontinuous and uniformly bounded on $[\kappa, 1]$. Applying the Arzelà-Ascoli Theorem again, we obtain that $v_\epsilon' \to w$ in $C[\kappa, 1]$ as $\epsilon \to 0^+$. Using the Dominated Convergence Theorem, we have that

$$v_\epsilon = \int_1^r v_\epsilon'(\tau)\,d\tau \to \int_1^r w(\tau)\,d\tau = v$$

for any $r \in [\kappa, 1]$. It follows that $w = v'$ and $v_\epsilon \to v$ in $C^1[\kappa, 1]$ as $\epsilon \to 0^+$.

We shall show that $(\lambda, v)$ is the solution of problem (1.6). Firstly, integrating equation (3.1) from $\kappa$ to 1, we derive that

$$v_\epsilon'(\kappa) - v_\epsilon'(1) = \int_\kappa^1 \mathcal{G}_\epsilon(\tau, v_\epsilon, v_\epsilon')\,d\tau.$$

Obviously, $\mathcal{G}_\epsilon$ is uniformly bounded on $[\kappa, 1]$ with respect to $\epsilon$ and

$$\begin{aligned} \lim_{\epsilon \to 0^+} \mathcal{G}_\epsilon(r, v_\epsilon, v_\epsilon') &= -\lambda N \frac{E^2}{a} H(r, v)\left(1 - f^2 v'^2\right)^{3/2} + \frac{N-1}{r} v'\left(1 - f^2 v'^2\right) \\ &\quad + \frac{f' v'}{f}\left(1 - f^2 v'^2\right) + f f' v'^3 + \frac{a' v'}{a}\left(1 - f^2 v'^2\right) := \mathcal{G}(r, v, v') \end{aligned}$$

for any $r \in [\kappa, 1]$. Then, by the Lebesgue Dominated Convergence Theorem, we obtain that

$$v'(\kappa) - v'(1) = \int_\kappa^1 \mathcal{G}(\tau, v, v')\,d\tau.$$

In view of the arbitrariness of $\kappa$, it follows that

$$-v'' = \mathcal{G}(r, v, v'),\ r \in (0,1). \tag{3.2}$$

Since $v_\epsilon(1) = 0$, we have that $v(1) = 0$. So, it suffices to verify the existence of $\lim_{r\to 0^+} v'(r) := v'(0)$ and $v'(0) = 0$.

From problem (2.2), we can see that

$$\begin{aligned}\frac{fv''_\epsilon}{1-f^2v'^2_\epsilon} &= \lambda NEH\left(r,v_\epsilon\right)\sqrt{1-f^2v'^2_\epsilon} - \frac{N-1}{r+\epsilon}fv'_\epsilon \\ &\quad -f'v'_\epsilon - \frac{f^2f'v'^3_\epsilon}{1-f^2v'^2_\epsilon} - \frac{a'fv'_\epsilon}{a}.\end{aligned}$$

It follows that

$$\begin{aligned}\frac{fv''_\epsilon + f'v'_\epsilon}{1-f^2v'^2_\epsilon} &= \lambda NEH\left(r,v_\epsilon\right)\sqrt{1-f^2v'^2_\epsilon} - \frac{N-1}{r+\epsilon}fv'_\epsilon \\ &\quad -f'v'_\epsilon - \frac{f^2f'v'^3_\epsilon}{1-f^2v'^2_\epsilon} - \frac{a'fv'_\epsilon}{a} + \frac{f'v'_\epsilon}{1-f^2v'^2_\epsilon} \\ &= \lambda NEH\left(r,v_\epsilon\right)\sqrt{1-f^2v'^2_\epsilon} - \frac{N-1}{r+\epsilon}fv'_\epsilon - \frac{a'fv'_\epsilon}{a}.\end{aligned}$$

It is easy to verify that

$$\left(\frac{1}{2}\ln\left(\frac{1+fv'_\epsilon}{1-fv'_\epsilon}\right)\right)' = \frac{fv''_\epsilon + f'v'_\epsilon}{1-f^2v'^2_\epsilon}.$$

Define $\psi : (-1,1) \to \mathbb{R}$ by

$$\psi(s) = \frac{1}{2}\ln\left(\frac{1+s}{1-s}\right).$$

Clearly, $\psi$ is an increasing diffeomorphism. So, we obtain that

$$\left(\psi\left(fv'_\epsilon\right)\right)' = \lambda NEH\left(r,v_\epsilon\right)\sqrt{1-f^2v'^2_\epsilon} - \frac{N-1}{r+\epsilon}fv'_\epsilon - \frac{a'fv'_\epsilon}{a}.$$

Integrating the above equation between 0 and 1, we have that

$$\int_0^1 \frac{N-1}{r+\epsilon}fv'_\epsilon\,dr = -\psi\left(f(1)v'_\epsilon(1)\right) + \int_0^1\left(\lambda NEH\left(r,v_\epsilon\right)\sqrt{1-f^2v'^2_\epsilon} - \frac{a'fv'_\epsilon}{a}\right)dr.$$

By Proposition 3.1, the right term of the above equation is uniformly bounded with respect to $\epsilon$. Hence, there exists a positive constant $C$ which is independent on $\epsilon$ such that

$$\left|\int_0^1 \frac{fv'_\epsilon}{r+\epsilon}\,dr\right| \le C.$$

Without loss of generality, from now on, we assume that $v_\epsilon$ is positive on $[0,1)$. In view of Lemma 2.2, we have that

$$\int_0^1 \frac{-fv'_\epsilon}{r+\epsilon}\,dr = \int_0^1 \frac{f\left|v'_\epsilon\right|}{r+\epsilon}\,dr = \left|\int_0^1 \frac{fv'_\epsilon}{r+\epsilon}\,dr\right| \le C.$$

By virtue of the Fatou Lemma, we infer that

$$\int_0^1 \frac{-fv'}{r}\,dr \le C,$$

which shows that

$$\frac{-fv'}{r}$$

is an integrable function on $(0,1]$. It implies that

$$\frac{-fv'}{r}\left(1-f^2v'^2\right)$$

is also an integrable function on $(0,1]$.

For any $r\in(0,1]$, integrating equation (3.2) from $r$ to 1, we get that

$$\begin{aligned} v'(r)-v'(1) &= \int_r^1 \mathcal{G}\left(\tau,v,v'\right)\,d\tau \\ &= \int_r^1 \frac{N-1}{\tau}v'\left(1-f^2v'^2\right)\,d\tau-\lambda N\int_r^1 \frac{E^2}{a}H\left(\tau,v\right)\left(1-f^2v'^2\right)^{3/2}\,d\tau \\ &\quad+\int_r^1\left(\frac{f'v'}{f}\left(1-f^2v'^2\right)+ff'v'^3+\frac{a'v'}{a}\left(1-f^2v'^2\right)\right)\,d\tau. \end{aligned}$$

Now, we can see that the limit of the right term exists as $r\to0^+$. Therefore, we verify the existence of $\lim_{r\to0^+}v'(r)$, which is denoted by $v'(0)$. Then, by integrability of $-\left(fv'\right)/r$ and $f(0)>0$, we derive that $v'(0)=0$, which is just our desired conclusion. ■

4. **Proofs of Theorems 1.1 and 1.2.** From Theorem 3.1, for any solution $v_\varepsilon$ of problem (2.2), we have that $v=\lim_{\epsilon\to0^+}v_\varepsilon$ is a solution of problem (1.6). It follows that $\mathscr{C}^\nu:=\limsup_{\varepsilon\to0^+}\mathscr{C}_\varepsilon^\nu$ is the solution set of problem (1.6), where $\mathscr{C}_\varepsilon^\nu$ is obtained in Theorem 2.1. To study the structure of $\mathscr{C}^\nu$, we first present the following eigenvalue result.

**Lemma 4.1.** *Passing to a subsequence, we have that* $\lim_{\epsilon\to0^+}\lambda_1(\varepsilon)=\lambda_1$.

**Proof.** Let $\mu_1$ be the first eigenvalue of

$$\begin{cases} -\left((r+1)^{N-1}\dfrac{a^2(r)}{E(r)}u'\right)'=\lambda r^{N-1}a(r)E(r)u,\ r\in(0,1),\\ u'(0)=u(R)=0. \end{cases}$$

By the Comparison Theorem for eigenvalues of [29, p. 276], we have that

$$\lambda_1(\varepsilon)<\mu_1.$$

Up to a subsequence, we have $\lim_{\varepsilon\to0^+}\lambda_1(\varepsilon)=\mu$ for some $\mu\in[0,\mu_1]$.

Let $\varphi_{\varepsilon,1}$ be the positive eigenfunction with $\left\|\varphi_{\varepsilon,1}'\right\|_\infty=1$ corresponding to $\lambda_1(\varepsilon)$. Passing if necessary to a subsequence, there exists $\varphi_1\in C[0,1]$ such that

$$\lim_{\varepsilon\to0^+}\left\|\varphi_{\varepsilon,1}-\varphi_1\right\|_\infty=0.$$

Note that

$$-\left((r+\epsilon)^{N-1}\frac{a^2(r)}{E(r)}\varphi_{\varepsilon,1}'\right)'=\lambda_1(\epsilon)(r+\epsilon)^{N-1}a(r)E(r)\varphi_{\varepsilon,1},\ r\in(0,1).$$

It follows that

$$-\varphi_{\varepsilon,1}''=\frac{N-1}{r+\epsilon}\varphi_{\varepsilon,1}'+\frac{2a'E-E'a}{aE}\varphi_{\varepsilon,1}'+\lambda_1(\epsilon)\frac{E^2}{a}\varphi_{\varepsilon,1},\ r\in(0,1). \tag{4.1}$$

Consequently, for any $\kappa\in(0,1]$ and $r\in[\kappa,1]$, we have that

$$\left|\varphi_{\varepsilon,1}''\right|\le\frac{N-1}{\kappa}+C_1\mu_1+C_2$$

for some positive constants $C_i$, $i=1,2$, which are independent on $\epsilon$. Reasoning as that of Theorem 3.1, we have that $\varphi_{\varepsilon,1}$ converges to $\varphi_1$ in $C^1[\kappa,1]$. Integrating equation (4.1) from $\kappa$ to 1, we obtain that

$$\varphi_{\varepsilon,1}'(\kappa)-\varphi_{\varepsilon,1}'(1)=\int_\kappa^1\left(\frac{N-1}{r+\epsilon}\varphi_{\varepsilon,1}'+\frac{2a'E-E'a}{aE}\varphi_{\varepsilon,1}'+\lambda_1(\epsilon)\frac{E^2}{a}\varphi_{\varepsilon,1}\right)dr.$$

Applying the Lebesgue Dominated Convergence Theorem, we have that

$$\varphi_1'(\kappa)-\varphi_1'(1)=\int_\kappa^1\left(\frac{N-1}{r}\varphi_1'+\frac{2a'E-E'a}{aE}\varphi_1'+\mu\frac{E^2}{a}\varphi_1\right)dr.$$

It follows that

$$-\left(r^{N-1}\frac{a^2(r)}{E(r)}\varphi_1'\right)'=\mu r^{N-1}a(r)E(r)\varphi_1,\ r\in(0,1). \tag{4.2}$$

Noting $\varphi_1(1)=0$, it is sufficient to show that $\varphi_1'(0)=0$.

Integrating equation (4.1) from 0 and 1, we have that

$$(N-1)\int_0^1\frac{\varphi_{\varepsilon,1}'}{r+\varepsilon}\,dr=-\varphi_{\varepsilon,1}'(1)-\int_0^1\lambda_1(\varepsilon)\frac{E^2}{a}\varphi_{\varepsilon,1}\,dr-\int_0^1\frac{2a'E-E'a}{aE}\varphi_1'\,dr.$$

It follows that

$$(N-1)\left|\int_0^1\frac{\varphi_{\varepsilon,1}'}{r+\varepsilon}\,dr\right|\le C_3$$

for some positive constant $C_3$ independing on $\epsilon$. It is easy to verify that $\varphi_{\varepsilon,1}$ is decreasing in $[0,1]$. So, we have that

$$(1-N)\int_0^1\frac{\varphi_{\varepsilon,1}'}{r+\varepsilon}\,dr=(N-1)\int_0^1\frac{\left|\varphi_{\varepsilon,1}'\right|}{r+\varepsilon}\,dr=(N-1)\left|\int_0^1\frac{\varphi_{\varepsilon,1}'}{r+\varepsilon}\,dr\right|\le C_3.$$

Using the Fatou Lemma, we have that $(1-N)\varphi_1'/r$ is integrable on $(0,1]$. Since $\left\|\varphi_{\varepsilon,1}'\right\|_\infty=1$, one has that $|\varphi_1'(r)|\le 1$ for any $r\in(0,1]$. It follows from equation (4.2) that

$$-\varphi_1''=\frac{N-1}{r}\varphi_1'+\frac{2a'E-E'a}{aE}\varphi_1'+\mu\frac{E^2}{a}\varphi_1,\ r\in(0,1).$$

Integrating it between $r$ and 1 with any $r>0$, we obtain that

$$\varphi_1'(r)-\varphi_1'(1)=\int_r^1\left((N-1)\frac{\varphi_1'}{\tau}+\frac{2a'E-E'a}{aE}\varphi_1'+\mu\frac{E^2}{a}\varphi_1\right)d\tau.$$

Therefore, the limit of the right term exists when $r$ tends to 0. Then, by integrability of $\varphi_1'/r$, we conclude that $\lim_{r\to0^+}\varphi_1'(r)=0$. Clearly, we have that $\varphi_1$ is nonnegative with $\|\varphi_1'\|_\infty=1$. Therefore, we obtain that $\lim_{\varepsilon\to0^+}\lambda_1(\varepsilon)=\mu=\lambda_1$. ■

To give the proof of Theorem 1.1, we take $\varepsilon=1/n$ and rewrite $\mathscr{C}_\varepsilon^\nu$ by $\mathscr{C}_n^\nu$.

**Proof of Theorem 1.1.** (a) By Lemma 4.1 and Theorem 2.1, we know that $(\lambda_1,0)\in\liminf_{n\to+\infty}\mathscr{C}_n^\nu$. The compactness of $\Psi$ implies that $\left(\cup_{n=1}^{+\infty}\mathscr{C}_n^\nu\right)\cap\mathbb{B}_R$ is pre-compact. Applying Theorem 2.1 of [11], $\mathscr{C}^\nu=\limsup_{n\to+\infty}\mathscr{C}_n^\nu$ is connected. Since $(+\infty,1)\in\mathscr{C}_n^\nu$ for every $n$, we have $(+\infty,1)\in\mathscr{C}^\nu$. So, $\mathscr{C}^\nu$ joins $(\lambda_1,0)$ to $(+\infty,1)$. From Theorem 2.1 and Theorem 3.1, we see that $v\in\overline{P^\nu}$ for any $(\lambda,v)\in\mathscr{C}^\nu$ with $\lambda\neq\lambda_1$. By Lemma 2.2, $v\equiv0$ if $v\in\partial P^\nu$. Define

$$\mathcal{F}(\lambda,v)=\frac{1}{r^{N-1}E}\left(\frac{r^{N-1}av'}{E\sqrt{1-f^2v'^2}}\right)'+\frac{a'v'}{E^2\sqrt{1-f^2v'^2}}-\lambda NH(r,v)$$

for $v \in \mathcal{B}$. Then, by some elementary calculations, we obtain that

$$\mathcal{F}_v(\lambda, 0)v = \lim_{t\to 0^+} \frac{\mathcal{F}(\lambda, tv)}{t} = \left(r^{N-1}\frac{a^2(r)}{E(r)}v'\right)' + \lambda r^{N-1}a(r)E(r)v.$$

By the Implicit function theorem, $\lambda$ must be an eigenvalue of problem (1.7). So, $\lambda = \lambda_j$ for some $j > 1$. Then, by an argument similar to that of Theorem 2.1, we can get a contradiction. So, we have that $\mathscr{C}^\nu \subseteq (\mathbb{R} \times P^\nu) \cup \{(\lambda_1, 0)\}$. Lemma 2.1 implies that $(\lambda_1, +\infty) \subseteq \mathrm{pr}_{\mathbb{R}}(\mathscr{C}^\nu)$.

(b) Clearly, one has that $(0, 0) \in \mathscr{C}^\nu$. Applying Theorem 2.1 of [11], we have that $\mathscr{C}^\nu$ is connected. By Theorem 2.1, we know that $\mathscr{C}^\nu$ joins $(0, 0)$ to $(+\infty, 1)$. Reasoning as that of (a), we see that $(\lambda, 0)$ is a bifurcation of problem (1.6) if and only if $\lambda H_0$ is an eigenvalue of problem (1.7), which combining $H_0 = +\infty$ implies that $\mathscr{C}^\nu \cap ((0, +\infty) \times \{0\}) = \emptyset$. Furthermore, by Lemmas 2.1–2.2 and an argument similar to that of (a), we can obtain the desired conclusions.

(c) It is easy to see that $z^* := (+\infty, 0) \in \mathscr{C}^\nu$ and $z_* := (+\infty, 1) \in \mathscr{C}^\nu$. It follows from $\mathscr{C}_n^\nu \cap S = \emptyset$ for each $n$ that $\mathscr{C}^\nu \cap S = \emptyset$. So, we have that $\mathscr{C}^\nu \cap \{\infty\} = \{z_*, z^*\}$. The fact of $\mathscr{C}_n^\nu \setminus \{\infty\} \neq \emptyset$ combining Theorem 3.1 implies that $\mathscr{C}^\nu \setminus \{\infty\} \neq \emptyset$. By Lemma 3.1 of [14], we obtain that $\mathscr{C}^\nu$ is connected. Since $H_0 = 0$, reasoning as that of (b), we can show that $\mathscr{C}^\nu \cap ([0, +\infty) \times \{0\}) = \emptyset$. Then, the desired conclusions can be derived from Lemmas 2.1–2.2 immediately. ∎

Finally, we present the proof of Theorem 1.2.

**Proof of Theorem 1.2.** Assume that $v$ be any one-sign solution of problem (1.6) with some $\lambda > 0$. Multiplying the first equation of problem (1.6) by $v$, we obtain after integrations by parts that

$$\begin{aligned} \int_0^1 r^{N-1} f v'^2 \, dr &\leq \int_0^1 r^{N-1} \frac{f v'^2}{\sqrt{1 - f^2 v'^2}} \, dr \\ &= \int_0^1 \frac{a'}{a} r^{N-1} \phi\left(f v'\right) v \, dr + \lambda N \int_0^1 r^{N-1} E(r) \frac{-H(r, v)}{v} v^2 \, dr \\ &\leq \lambda N \varrho \int_0^1 r^{N-1} E(r) v^2 \, dr \leq \frac{\lambda \varrho N}{\mu_1} \int_0^1 r^{N-1} f v'^2 \, dr, \end{aligned}$$

where $\mu_1$ is the first eigenvalue of

$$\begin{cases} -\left(r^{N-1} f(r) u'\right)' = \lambda r^{N-1} E(r) u, \ r \in (0, 1), \\ u'(0) = u(1) = 0. \end{cases}$$

It follows that $\lambda \geq \mu_1/(\varrho N)$. ∎

5. **Entire radially spacelike graphs.** In this section, we study the existence of entire radially spacelike graphs of equation (1.4) in Schwarzschild spacetime with a parameter, i.e., the following equation

$$\begin{cases} -\dfrac{1}{r^{N-1}E}\left(\dfrac{r^{N-1} a v'}{E\sqrt{1 - f^2 v'^2}}\right)' - \dfrac{a' v'}{E^2\sqrt{1 - f^2 v'^2}} = -\lambda N H\left(r, v\right), \, r \in (0, +\infty), \\ v'(0) = 0, \end{cases} \tag{5.1}$$

where $\lambda$ is a nonnegative parameter.

**Theorem 5.1.** *Besides the assumptions of Theorem* 1.1, *assume that* $a : \mathbb{R}^N \to \mathbb{R}$

*is a smooth positive radially symmetric function,* $H : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ *is a continuous function and is radially symmetric with respect to* $x$.

(a) *If* $H_0 = 1$, *then there exist at least two entire radially symmetric spacelike graphs of equation* (5.1) *for any* $\lambda \in (\lambda_1, +\infty)$.

(b) *If* $H_0 = +\infty$, *for any* $\lambda \in (0, +\infty)$, *there exist at least two entire radially symmetric spacelike graphs of equation* (5.1).

(c) *If* $H_0 = 0$, *there exists* $\lambda^* > 0$ *such that equation* (5.1) *has at least four entire radially symmetric spacelike graphs for any* $\lambda \in (\lambda^*, +\infty)$.

*In addition, the spacelike slice* $t = 0$ *intersects these graphs in the unit ball.*

**Proof.** We only give the proof of (a) because the proofs of other cases are completely analogous. Theorem 1.1 provides at least two solutions $v^+$ and $v^-$ of problem (1.6) such that $\nu v^\nu > 0$ in $(0, 1)$. It suffices to show that $v^\nu$ can be extended to $(0, +\infty)$ such that they are also solutions of equation (5.1). We consider the following system

$$\begin{cases} v' = z, \\ z' = \lambda N \dfrac{E^2}{a} \left(1 - f^2 z^2\right)^{\frac{3}{2}} H(r, v) + \dfrac{N-1}{r} z \left(f^2 z^2 - 1\right) + \left(\dfrac{f'}{f} + \dfrac{a'}{a}\right) z \left(f^2 z^2 - 1\right) - f f' z^3, \end{cases}$$

which we can abbreviate

$$\begin{pmatrix} v' \\ z' \end{pmatrix} = \mathcal{F}(r, (v, z)), \tag{5.2}$$

where $\mathcal{F} : \mathbb{R}_+ \times \mathbb{R} \times (-\delta, \delta) \to \mathbb{R}^2$ is continuous.

We can see that $(v^\nu, z)$ is a solution of system (5.2) on $(0, 1)$ such that $\|v\|_\infty < \delta$. By Lemma 2.1, we can derive that $|z| = |v'| < \delta$. Thus, for some sequence $r_m \in [1/2, 1)$ converging to 1, we have $(v^\nu, z) \in [-\delta, \delta] \times [-\delta, \delta]$. By Corollary 2.15 of [27], there exists an extension of $(v^\nu, z)$ to the interval $(0, 1 + \varepsilon)$ for some $\varepsilon > 0$.

Let $[0, b)$ with $b > 1$ be the maximal interval of definition of $(v^\nu, z)$. Suppose that $b < +\infty$. By Corollary 2.16 of [27], the solution $(v^\nu, z)$ must eventually leave every compact set $C$ with $[1/2, b] \times C \subset \mathbb{R}_+ \times \mathbb{R} \times (-\delta, \delta)$ as $r$ approaches $b$. By an argument similar to that of Lemma 2.1, we can show that $f \left|(v^\nu)'\right| < 1$ on $[0, b]$. It follows that $|z| < \delta$ for any $r \in [1/2, b]$. It is easy to see that $|v^\nu| < b\delta$ for any $r \in [1/2, b]$. So, $(v^\nu, z)$ cannot go out of the compact subset $[-b\delta, b\delta] \times [-\delta, \delta]$ as $r$ approaches $b$. This contradiction indicates $b = +\infty$. Therefore, $v^\nu$ can be extended a radially entire of equation (1.4) in $\mathbb{R}^N$.■

Similarly, we consider the existence of radially spacelike graphs of equation (1.4) on exterior domain, i.e., the following equation

$$\begin{cases} -\dfrac{1}{r^{N-1} E} \left(\dfrac{r^{N-1} a v'}{E\sqrt{1 - f^2 v'^2}}\right)' - \dfrac{a' v'}{E^2 \sqrt{1 - f^2 v'^2}} = -\lambda N H\left(r, v\right), \, r \in \left(R_1, +\infty\right), \\ v'\left(R_1\right) = 0. \end{cases} \tag{5.3}$$

Using Corollary 1.1 and an argument similar to that of Theorem 5.1, we can obtain the following corollary.

**Corollary 5.1.** *Besides the assumptions of Corollary* 1.1, *assume that* $a : \mathbb{R}^N \setminus \overline{B}_{R_1}(0) \to \mathbb{R}$ *is a smooth positive radially symmetric function,* $H : \left(\mathbb{R}^N \setminus \overline{B}_{R_1}(0)\right) \times \mathbb{R} \to \mathbb{R}$ *is a continuous function and is radially symmetric with respect to* $x$.

(a) *If* $H_0 = 1$, *then there exist at least two entire radially symmetric spacelike graphs of equation* (5.3) *for any* $\lambda \in (\lambda_1, +\infty)$.

(b) *If $H_0 = +\infty$, for any $\lambda \in (0, +\infty)$, there exist at least two entire radially symmetric spacelike graphs of equation* (5.3).

(c) *If $H_0 = 0$, there exists $\lambda^* > 0$ such that equation* (5.3) *has at least four entire radially symmetric spacelike graphs for any $\lambda \in (\lambda^*, +\infty)$.*

*In addition, the spacelike slice $t = 0$ intersects these graphs in a ball of radius $R_2$.*

Consider

$$a(r) = \sqrt{1 - \frac{2m}{r}} \quad \text{and} \quad E(r) = \frac{1}{\sqrt{1 - \frac{2m}{r}}},$$

where $m$ is the mass of a star or black hole in certain unit system. Taking $R_1 > 2m$ and $\Omega = \mathbb{R}^N \setminus \overline{B}_{R_1}(0)$, $\mathcal{M}$ is the Schwarzschild exterior spacetime which models the exterior region of a spacetime where there is only a spherically symmetric non-rotating star without charge. The value of the radius $r = 2m$ is known as Schwarzschild radius. When $r < 2m$, we are in presence of a Schwarzschild black hole. It is not difficult to verify that $a$ and $E$ satisfy the assumptions of Corollary 1.1 and Corollary 5.1. So, the conclusions of Corollary 5.1 can be used on the Schwarzschild exterior spacetime.

Another example is the Reissner-Nordström exterior spacetime in which the mass has non-zero electric charge [7, 26]. In such case, we have

$$a(r) = \sqrt{1 - \frac{2m}{r} + \frac{r_Q^2}{r^2}} \quad \text{and} \quad E(r) = \frac{1}{\sqrt{1 - \frac{2m}{r} + \frac{r_Q^2}{r^2}}},$$

where $r_Q > 0$ is a characteristic length relative to the charge $Q$ of the mass. Taking $R_1 > m + \sqrt{m^2 - r_Q^2}$ and $\Omega = \mathbb{R}^N \setminus \overline{B}_{R_1}(0)$, $\mathcal{M}$ is the Reissner-Nordström exterior spacetime, which can be seen a generalization of the Schwarzschild exterior spacetime. The value of $r = m + \sqrt{m^2 - r_Q^2}$ is the exterior event horizon. Clearly, $a$ and $E$ still satisfy the assumptions of Corollary 1.1 and Corollary 5.1. Therefore, the conclusions of Corollary 5.1 can also be applied to the Reissner-Nordström exterior spacetime.

# Appendix: derivation of equation (1.4)

For the convenience of readers, we establish here equation (1.4) again by slightly different strategies from [19]. For $u \in C^2(\Omega)$ consider its graph $\Sigma_u = \{(u(x), x)) : x \in \Omega\}$ in the standard static spacetime $(M, \overline{g})$, where $\overline{g}$ is given in (1.1). Assume $\Sigma_u$ is spacelike, that it $|\nabla u| < 1/a$ holds everywhere on $\Omega$. Replace now the metric $\overline{g}$ by

$$g^* = -dt^2 + g' \quad \text{where} \quad g' = \frac{1}{a^2}\, g, \tag{5.4}$$

being $g$ the usual Riemannian metric of $\mathbb{R}^N$ induced on $\Omega$. Note that

$$g^* = \frac{1}{a^2}\, \overline{g}, \tag{5.5}$$

thus these Lorentzian metrics on $M$ are pointwise conformally equivalent, Moreover, $g'$ and $g$ are Riemannian metrics on $\Omega$ also pointwise conformally equivalent.

The graph $\Sigma_u$ is also spacelike in $(M, g^*)$, which is a Lorentz product manifold. The unit timelike normal vector field, pointing to future, on $\Sigma_u$ in $(M, g^*)$ is

$$\xi^* = \frac{1}{\sqrt{1 - |\nabla' u|'^2}} \left( 1 \,,\, \nabla' u \right), \tag{5.6}$$

where $\nabla' u$ means the $g'$-gradient of $u$ and $|\nabla' u|'^2 = g'(\nabla' u, \nabla' u)$ and therefore $|\nabla' u|' < 1$ holds on all $\Omega$. Comparing this expression for $\xi^*$ with the one for the unit timelike normal vector field $\xi$ on $\Sigma_u$ in $(M, \overline{g})$ given in (1.3), we have

$$\xi^* = a\,\xi. \tag{5.7}$$

A direct computation shows that the mean curvature function $H^*$ of $\Sigma_u$ in $(M, g^*)$ with respect to (5.6) satisfies

$$\mathrm{div}'\left(\frac{\nabla' u}{\sqrt{1 - |\nabla' u|'^2}}\right) = NH^*, \tag{5.8}$$

where div' represents the divergence operator in $(\Omega, g')$.

From the relationship (5.4) we have

$$\nabla' u = a^2\,\nabla u \quad \text{and} \quad |\nabla' u|'^2 = a^2\,|\nabla u|^2. \tag{5.9}$$

On the other hand, we have

$$\mathrm{div}'(X) = \mathrm{div}(X) - N\,g(X, \nabla \log a) = \mathrm{div}(X) - N\frac{1}{a}\,g(X, \nabla a), \tag{5.10}$$

for any vector field $X$ on $\Omega$. Now, from (5.9) and (5.10), we can rewrite (5.8) as follows

$$\mathrm{div}\left(\frac{a^2 \nabla u}{\sqrt{1 - a^2|\nabla u|^2}}\right) - N\frac{a\,g(\nabla u, \nabla a)}{\sqrt{1 - a^2|\nabla u|^2}} = NH^*,$$

that is,

$$a\,\mathrm{div}\left(\frac{a\nabla u}{\sqrt{1 - a^2|\nabla u|^2}}\right) + \frac{a\,g(\nabla u, \nabla a)}{\sqrt{1 - a^2|\nabla u|^2}} - N\frac{a\,g(\nabla u, \nabla a)}{\sqrt{1 - a^2|\nabla u|^2}} = NH^*, . \tag{5.11}$$

Denote by $\alpha^*$ (resp. $\overline{\alpha}$) the second fundamental form of $\Sigma_u$ in $(M, g^*)$ (resp. of $\Sigma_u$ in $(M, \overline{g})$). Taking into account (5.5), we have

$$\alpha^*(X, Y) = \overline{\alpha}(X, Y) + \frac{1}{a}\,\overline{g}(X, Y)(\overline{\nabla} a)^{\perp}$$

for all tangent $X, Y$, where $(\overline{\nabla} a)^{\perp}$ is the tangent component of the $\overline{g}$-gradient of the function $a$ (considered on $M$) (see for instance [16, p. 132]). Equivalently,

$$-\,g^*(\alpha^*(X, Y), \xi^*)\,\xi^* = -\overline{g}(\overline{\alpha}(X, Y), \xi)\,\xi - \frac{1}{a}\,\overline{g}(X, Y)\overline{g}(\overline{\nabla} a, \xi)\xi. \tag{5.12}$$

On the other hand, from (1.3) we obtain

$$\overline{g}(\overline{\nabla} a, \xi) = \frac{a\,g(\nabla u, \nabla a)}{\sqrt{1 - a^2|\nabla u|^2}}. \tag{5.13}$$

Using (5.6) and (5.13) in (5.12) we get

$$A^* = a\,A \;+\; \frac{a\,g(\nabla u, \nabla a)}{\sqrt{1 - a^2|\nabla u|^2}}\,Id, \tag{5.14}$$

where $A^*$ (respe. A) is the shape operator of $\Sigma_u$ in $(M, g^*)$ (resp. in $(M, \overline{g})$) relative to $\xi^*$ (resp. relative to $\xi$) and $Id$ is the identity transformation. Taking in mind that the corresponding mean curvatures functions are given by $H^* = -(1/N)\,\mathrm{trace}(A^*)$ and $H = -(1/N)\,\mathrm{trace}(A)$, formula (5.14) gives

$$H^* = a\,H \;-\; \frac{a\,g(\nabla u, \nabla a)}{\sqrt{1 - a^2|\nabla u|^2}}\,. \tag{5.15}$$

Finally, equation (1.4) is obtained by substitution of (5.15) in (5.11).

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