Dynamics of interacting vortices on trapped Bose-Einstein condensates

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Joint work with:

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Publications:

- P.J. Torres, R. Carretero-González, S. Middelkamp, P. Schmelcher, D.J. Frantzeskakis and P.G. Kevrekidis, *Vortex Interaction Dynamics in Trapped Bose-Einstein Condensates*, Communications on Pure and Applied Analysis 10 (2011) 1589-1615.
- S. Middelkamp, P. J. Torres, P. G. Kevrekidis, D.J. Frantzeskakis, R. Carretero-González, P. Schmelcher, D.V. Freilich, and D.S. Hall, *Guiding-center dynamics of vortex dipoles in Bose-Einstein condensates.*, Phys. Rev. A, Vol. 84, 011605(R) (2011).
- P. J. Torres, P. G. Kevrekidis, D.J. Frantzeskakis, R. Carretero-González, P. Schmelcher, and D.S. Hall, *Dynamics of Vortex Dipoles in Confined Bose-Einstein Condensates.*, Phys. Lett. A, Volume 375, Issue 33 (2011), pp. 3044-3050

Physical background

- A Bose-Einstein condensate (BEC) is a state of matter of a dilute gas of weakly interacting bosons confined in an external potential and cooled to temperatures very near absolute zero (0 K or -273.15 -C). Under such conditions, a large fraction of the bosons occupy the lowest quantum state of the external potential, at which point quantum effects become apparent on a macroscopic scale.

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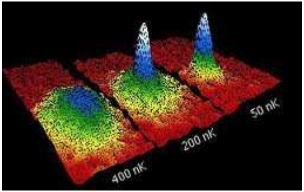
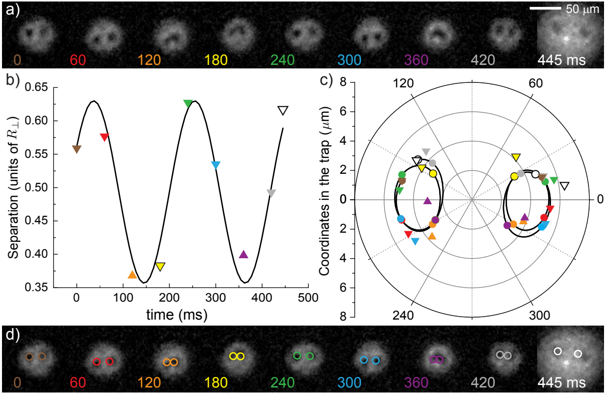
- Theoretically predicted by Satyendra Nath Bose and Albert Einstein in 1924-25. First experiment performed in 1995 by Eric Cornell and Carl Wieman (Nobel Prize in Physics 2001).



Physical background

Vortices can be created on a BEC.



Physical background

The dynamical evolution of such vortices is a natural question

The model for n vortices

$$\dot{x}_k = -c(r_k, t)\, S_k y_k - b \sum_{j \neq k} S_j \frac{y_k - y_j}{r_{jk}^2},$$

$$\dot{y}_k = +c(r_k, t)\, S_k x_k + b \sum_{j \neq k} S_j \frac{x_k - x_j}{r_{jk}^2}, \qquad k = 1 \ldots n,$$

where
$(x_k, y_k) \equiv$ coordinate of vortex k
$S_k = \pm 1$ charge of vortex k
$r_{jk} = \sqrt{(x_j - x_k)^2 + (y_j - y_k)^2} \equiv$ separation between vortex j and vortex k
$r_k = \sqrt{x_k^2 + y_k^2} \equiv$ distance of vortex k to the center.
$c(r_k, t) \equiv$ trap coefficient, positive and T- periodic in time.

The case $c(r_i, t) \equiv c(t)$. Two co-rotating vortices.

Fix $n = 2$, $S_1 = S_2 = 1$.

$$\dot{x}_1 = -c(t)\, y_1 - b\, \frac{y_1 - y_2}{r^2}, \qquad \dot{x}_2 = -c(t)\, y_2 - b\, \frac{y_2 - y_1}{r^2}$$

$$\dot{y}_1 = +c(t)\, x_1 + b\, \frac{x_1 - x_2}{r^2}, \qquad \dot{y}_2 = +c(t)\, x_2 + b\, \frac{x_2 - x_1}{r^2},$$

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Change:

$$s_1 = x_1 + x_2 \qquad d_1 = x_1 - x_2$$

$$s_2 = y_1 + y_2 \qquad d_2 = y_1 - y_2.$$

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Gives:

$$s_1' = -c(t)s_2$$

$$s_2' = c(t)s_1$$

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Solution:

$$s_1 = A \cos\left(C(t) + B\right)$$

$$s_2 = A \sin\left(C(t) + B\right),$$

with $C(t) = \int_0^t c(s) ds$.

The case $c(r_i, t) \equiv c(t)$. Two co-rotating vortices.

$$\begin{aligned}
\dot{d}_1 &= -c(t)\, d_2 - \frac{2bd_2}{(d_1^2 + d_2^2)}, \\
\dot{d}_2 &= +c(t)\, d_1 + \frac{2bd_1}{(d_1^2 + d_2^2)}.
\end{aligned}$$

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Change to polar coordinates:

$$d_1 = r\cos\varphi, \quad d_2 = r\sin\varphi,$$

From $r^2 = d_1^2 + d_2^2$, we get

$$r\dot{r} = d_1\dot{d}_1 + d_2\dot{d}_2 = 0.$$

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Hence, r is a constant of motion.

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The angular component $\varphi = \arctan\left(d_2/d_1\right)$ yields

$$\dot{\varphi} = \frac{d_1\dot{d}_2 - d_2\dot{d}_1}{r^2} = \frac{2b}{r} + c(t).$$

Since r is constant,

$$\varphi(t) = \frac{2b}{r}t + C(t) + D,$$

where D is an arbitrary constant.

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where D is an arbitrary constant.
The general solution reads

$$\begin{aligned} d_1 &= R \cos\left(\frac{2b}{R} t + C(t) + D\right), \\ d_2 &= R \sin\left(\frac{2b}{R} t + C(t) + D\right), \end{aligned}$$

where D and R are arbitrary constants.
Hence, the case of two corotating vortices is explicitly solvable. Generically, one finds quasi-periodic solutions and there is a sequence of periodic solutions that can be obtained by fine-tuning R.

The case $c(r_i, t) \equiv c(t)$. Two co-rotating vortices.

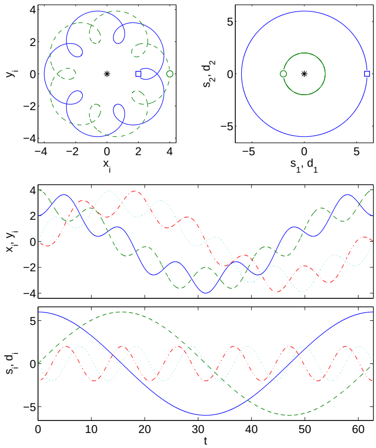

Figure: Periodic orbit corresponding to two vortices with *constant* trapping coefficient $c(t) = 0.1$, vortex-vortex interaction coefficient $b = 1$. The vortices are initially placed at $(x_1(0), y_1(0)) = (2, 0)$ (see [blue] square) and $(x_2(0), y_2(0)) = (4, 0)$ (see [green] circle).

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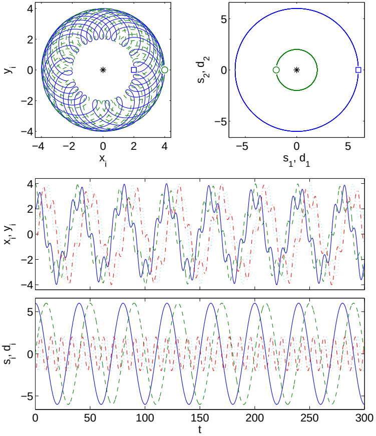

Figure: Same as before but for a quasi-periodic orbit for a constant trapping coefficient $c(t) = \pi/20$.

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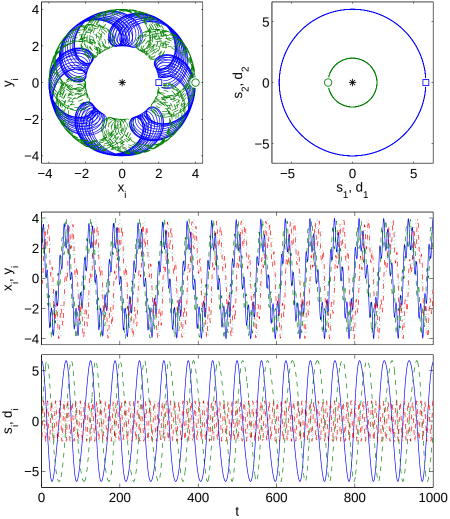

Figure: A quasi-periodic orbit for $c(t) = 0.1(1 + \epsilon \sin(\omega t))$ and $b = 1$ with $\epsilon = 0.25$ and $\omega = \pi/30$.

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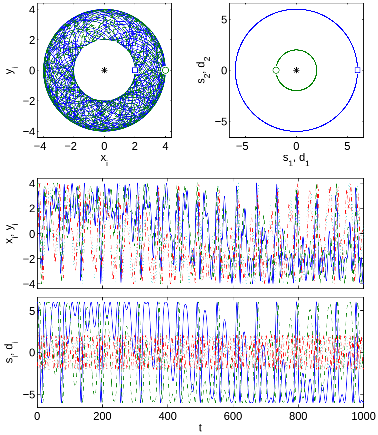

Figure: Same as in Fig. 3 but for a larger perturbation strength of $\epsilon = 2.5$. The apparent complex motion displayed in the top-left panel (in original coordinates) is nothing but a quasi-periodic orbit that is better elucidated in transformed coordinates in the bottom panel.

The case $c(r_i, t) \equiv c(t)$. Two counter-rotating vortices.

Fix $n = 2$, $S_1 = 1, S_2 = -1$.

$$\dot{x}_1 = -c(t)\, y_1 + b\, \frac{y_1 - y_2}{r^2}, \qquad \dot{x}_2 = c(t)\, y_2 - b\, \frac{y_2 - y_1}{r^2}$$

$$\dot{y}_1 = +c(t)\, x_1 - b\, \frac{x_1 - x_2}{r^2}, \qquad \dot{y}_2 = -c(t)\, x_2 + b\, \frac{x_2 - x_1}{r^2},$$

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$$\begin{aligned} \dot{s}_1 &= +2b\frac{d_2}{r^2} - c(t)\, d_2, \\ \dot{s}_2 &= -2b\frac{d_1}{r^2} + c(t)\, d_1, \end{aligned}$$

and

$$\begin{aligned} \dot{d}_1 &= -c(t)\, s_2, \\ \dot{d}_2 &= +c(t)\, s_1. \end{aligned}$$

The case $c(r_i, t) \equiv c(t)$. Two counter-rotating vortices.

Changing time $\tau = C(t) \equiv \int_0^t c(s)ds$ yields the equivalent system

$$\begin{aligned} \dot{s}_1 &= +f(\tau)\frac{d_2}{r^2} - d_2, \\ \dot{s}_2 &= -f(\tau)\frac{d_1}{r^2} + d_1, \\ \dot{d}_1 &= -s_2, \\ \dot{d}_2 &= +s_1, \end{aligned} \tag{1}$$

where $f(\tau) \equiv 2b/c(t(\tau))$.

The case $c(r_i, t) \equiv c(t)$. Two counter-rotating vortices.

$$\begin{aligned} \ddot{d}_1 + d_1 &= f(\tau)\frac{d_1}{r^2}, \\ \ddot{d}_2 + d_2 &= f(\tau)\frac{d_2}{r^2}. \end{aligned} \tag{2}$$

Recalling that $r = \sqrt{d_1^2 + d_2^2}$, one notes that this Newtonian system has a radial symmetry. This type of systems plays a central role in Celestial Mechanics.

The case $c(r_i, t) \equiv c(t)$. Two counter-rotating vortices.

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Change to polar coordinates: $d_1 = r\cos(\varphi)$, $\qquad d_2 = r\sin(\varphi)$, leads to

$$\ddot{r} + r = \frac{\ell^2}{r^3} + \frac{f(\tau)}{r}, \tag{3}$$

$$\dot{\varphi} = \frac{\ell}{r^2}, \tag{4}$$

where $\ell \equiv r^2\dot{\varphi} = d_1\dot{d}_2 - d_2\dot{d}_1 = r_1^2 - r_2^2 =$ const. is the angular momentum of the solution (d_1, d_2).

The case $c(r_i, t) \equiv c(t)$. Two counter-rotating vortices.

Equation (3) is decoupled and can be studied separately. If $r(t+T) = r(t)$:

$$\varphi(\tau) = \int_0^\tau \frac{\ell}{r^2} ds$$

and

$$r(t+T) = r(t) \qquad \varphi(t+T) = \varphi(t) + \theta,$$

where $\theta \equiv \varphi(T) = \int_0^T \ell/r^2 ds$ is the rotation number. Coming back to the Cartesian coordinates $\boldsymbol{d} = (d_1, d_2)$ and using the more convenient complex notation,

$$\boldsymbol{d}(t+T) = e^{i\theta} \boldsymbol{d}(t).$$

Therefore, from a T-periodic solution of (3) we get a quasi-periodic solution of the original system. If $\theta = 0$ (stationary case) the solution d is T-periodic, whereas if $\theta = 2\pi/k$ then d is kT-periodic (subharmonic of order k).

The case $c(r_i, t) \equiv c(t)$. Two counter-rotating vortices.

$$\ddot{r} + r = \frac{\ell^2}{r^3} + \frac{f(\tau)}{r}$$

For simplicity, take $c(t) = 1 + \epsilon \sin(\omega t)$.

Theorem

Assume that $\omega > 2$. Then, for any $\ell \geq 0$, (3) has a T-periodic solution such that

$$r(t) > r_* = \sqrt{\frac{2b}{1+\epsilon}} \cos\left(\frac{\pi}{\omega}\right).$$

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Sketch of the proof.
We summarize the technique employed in [Torres, JDE, 2003].

- $\omega > 2$ implies that the linear operator $Lr := \ddot{r} + r$ has a positive Green's function $G(t,s)$ such that

$$m := \cot\left(\frac{\pi}{\omega}\right) \leq G(t,s) \leq M := 1/\sin\left(\frac{\pi}{\omega}\right).$$

- A T-periodic solution of Eq. (3) is a fixed point of the operator

$$\mathcal{A}r = \int_0^T G(\tau,s)\left[\frac{\ell^2}{r^3} + \frac{f(\tau)}{r}\right] ds.$$

- Apply Krasnoselskii's fixed point Theorem for cone compression-expansion with the cone

$$P = \left\{u \in X : \min_x u \geq \frac{m}{M}\|u\|\right\},$$

where X the Banach space of the continuous and T-periodic functions with the norm of the supremum.

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In consequence, there is a continuous branch of T-periodic solutions (ℓ, r_ℓ) of Eq. (3). Now we analyze the stability of such solutions.

Theorem

Take $\ell \geq 0$. Then, there exists an explicitly computable ω_ℓ such that if $\omega > \omega_\ell$ then r_ℓ is linearly stable as a T-periodic solution of Eq. (3).

Proof. The linearized equation around r_ℓ is $\ddot{y} + a(\tau)y = 0$ with

$$a(\tau) = 1 + \frac{3\ell^2}{r_\ell^4} + \frac{f(\tau)}{r_\ell^2}.$$

Now, use $r(t) > r_*$ and apply Lyapunov's stability criterion.

The case $c(r_i, t) \equiv c(t)$. Two counter-rotating vortices.

It is possible to prove analytically the presence of KAM dynamics in Eq. (3), by using a Ortega's third order approximation method as in [Torres,Adv.Nonlinear Stud.,2002].

Theorem

Let us assume $\ell = 0$, $\omega > \sqrt{2}$, $\omega \neq \{3\sqrt{2}, 4\sqrt{2}\}$. Then the T-periodic solution r_0 of Eq. (3) is of twist type except possibly for a finite number of values of ϵ.

The case $c(r_i, t) \equiv c(t)$. Two counter-rotating vortices.

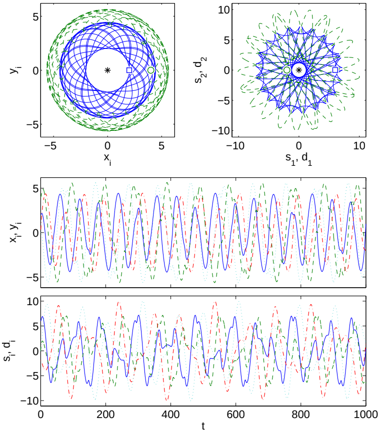

Figure: *Opposite* charge vortices $S_1 = 1$ (see solid [blue] line) and $S_2 = -1$ (see dashed [green] line) with *constant* trapping coefficient $c(t) = 0.1$, vortex-vortex interaction coefficient $b = 1$. The vortices are initially placed at $(x_1(0), y_1(0)) = (2, 0)$ (see [blue] square) and $(x_2(0), y_2(0)) = (4, 0)$ (see [green] circle) .

The case $c(r_i, t) \equiv c(t)$. Two counter-rotating vortices.

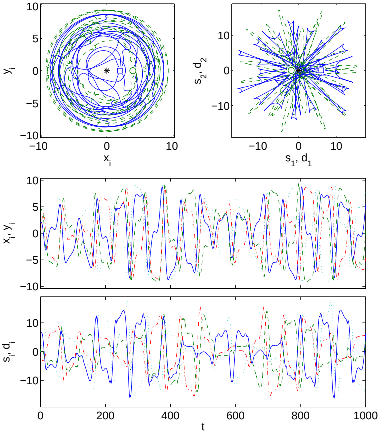

Figure: An irregular orbit corresponding to $c(t) = 0.1(1 + \epsilon \sin(\omega t))$ and $b = 1$ with $\epsilon = 1$ and $\omega = 0.12$.

The Fetter case.

In a more realistic situation, the precession frequency depends also on r. Fetter [2001] proposed

$$c(r,t) \equiv \Omega(r) = \frac{2\hbar\omega_r^2}{8\mu(1 - r^2/R_\perp^2)} \left(3 + \frac{\omega_r^2}{5\omega_z^2}\right) \ln\left(\frac{2\mu}{\hbar\omega_r}\right),$$

where ω_r (ω_z) is the radial (axial) trap frequency, $R_\perp$ is the Thomas-Fermi radial extent of the condensate (can be normalized to 1), and μ is the chemical potential.

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From now on,

$$\Omega(r) = \frac{\Omega_0}{1-r^2}.$$

The Fetter case.

The model is

$$\dot{x}_k = -S_k \Omega(r_k) y_k - \frac{b}{2} \sum_{j \neq k}^{n} S_j \frac{y_k - y_j}{r_{jk}^2},$$

$$\dot{y}_k = S_k \Omega(r_k) x_k + \frac{b}{2} \sum_{j \neq k}^{n} S_j \frac{x_k - x_j}{r_{jk}^2}.$$

or in complex variables

$$i\dot{z}_k = -S_k \Omega(r_k) z_k + \frac{b}{2} \sum_{j \neq k}^{n} S_j \frac{z_k - z_j}{r_{jk}^2}$$

Conserved quantities and integrabitlity.

The system is hamiltonian. Define

$$H(z_1, \dots, z_n) = -\frac{\Omega_0}{2} \sum_{k=1}^{n} \ln(1 - r_k^2) + \frac{b}{4} \sum_{k=1}^{n} \sum_{j \neq k}^{n} S_j S_k \ln(r_{jk}^2)$$

where $\Omega_0 \equiv \Omega(0)$. Then,

$$\begin{aligned} S_k \dot{x}_k &= -\frac{\partial H}{\partial y_k}, \\ S_k \dot{y}_k &= \frac{\partial H}{\partial x_k} \qquad \text{for every } k = 1, \dots, n. \end{aligned}$$

H is a the first integral, i.e., a first conserved quantity along the orbits of the system.
A second integral of motion is

$$V = \sum_{k=1}^{n} S_k r_k^2,$$

which is a form of inertial momentum.

Conserved quantities and integrabitlity.

The presence of two conserved quantities or dynamical invariants guarantees integrability in the classical Liouville sense for the case $n = 2$. This implies that the energy level sets are compact and the phase space is foliated by invariant tori. On each of these, the motion is quasi-periodic with two frequencies.

The Fetter case with $n = 2, S_1 = 1, S_2 = -1$.

Theorem

The two vortices never collide, i.e., they are separated by a computable minimal distance.

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Taking exponentials on the hamiltonian, we have

$$(1 - r_1^2)^{\Omega_0}(1 - r_2^2)^{\Omega_0} r_{12}^{b/2} = C^2 > 0 \qquad \forall t,$$

where $C^2 = (1 - r_1(0)^2)^{\Omega_0}(1 - r_2(0)^2)^{\Omega_0} r_{12}(0)^{b/2}$. Note that $0 < C^2 < 2^{b/2}$ because $0 \le r_i(0)^2 < 1$ $(i = 1, 2)$ and $0 < r_{12}(0) < 2$. Then,

$$r_{12}(t) > C^{4/b} \qquad \forall t,$$

The Fetter case with $n = 2, S_1 = 1, S_2 = -1$.

Theorem

There is an equilibrium, which is unique up to rotations, given by

$$(x_1^0, y_1^0) = \left(\sqrt{\frac{b}{4\Omega_0 + b}}, 0\right), \qquad (x_2^0, y_2^0) = \left(-\sqrt{\frac{b}{4\Omega_0 + b}}, 0\right)$$

The Fetter case with $n = 2, S_1 = 1, S_2 = -1$.

To emphasize the rotational invariance of our system, it is convenient to work on the polar coordinates (r_1, r_2, θ). The (unique) equilibrium is $(r_1^0, r_2^0, \theta_0) = (\sqrt{\frac{b}{4\Omega_0 + b}}, \sqrt{\frac{b}{4\Omega_0 + b}}, \pi)$.

Theorem

The equilibrium (r_1^0, r_2^0, θ_0) is stable (in the sense of Lyapunov).

The Fetter case with $n = 2, S_1 = 1, S_2 = -1$.

The hamiltonian H is still a conserved quantity which in the new variables reads

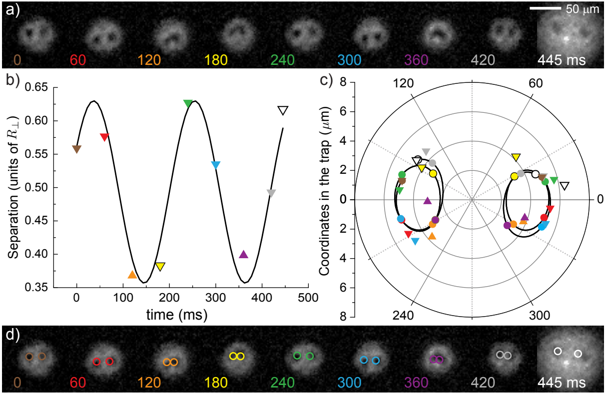
$$H(r_1, r_2, \theta) = -\frac{1}{2}\left[\Omega_0 \ln(1 - r_1^2) + \Omega_0 \ln(1 - r_2^2) + \frac{b}{2}\ln(r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta)\right].$$

The equilibrium (r_1^0, r_2^0, θ_0) is a critical point of H. The hessian matrix evaluated on (r_1^0, r_2^0, θ_0) is

$$\frac{1}{8}\begin{pmatrix} 7b + 12\Omega_0 + \frac{b^2}{\Omega_0} & b + 4\Omega_0 & 0 \\ b + 4\Omega_0 & 7b + 12\Omega_0 + \frac{b^2}{\Omega_0} & 0 \\ 0 & 0 & b \end{pmatrix}.$$

One can prove easily that this matrix is positive-definite by Sylvester's criterion. Hence H attains a minimum at (r_1^0, r_2^0, θ_0), H is a Lyapunov function and the equilibrium is, in fact, stable.

The Fetter case with $n = 2, S_1 = 1, S_2 = -1$.



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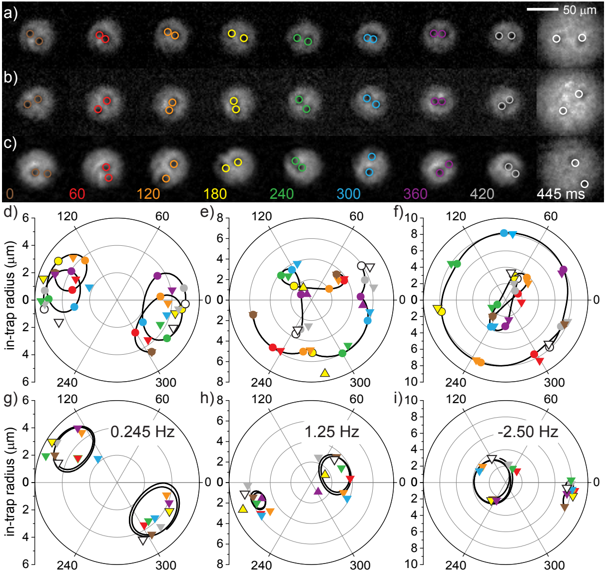

Guiding Center Equilibria and their Stability

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We adopt a trial solution of the form

$$z_1(t) = r_1 \exp(i\omega t) \qquad \text{and} \qquad z_2(t) = r_2 \exp(i\omega t + i\pi) = -r_2 \exp(i\omega t). \quad (5)$$

where r_1 and r_2 are now constant, as is ω, the precession frequency of the guiding center.

Guiding Center Equilibria and their Stability

Fix

$$\alpha = \frac{1}{1-r_1^2}, \quad \beta = \frac{1}{1-r_2^2}, \qquad \text{and} \qquad \gamma = \frac{1}{2r_{12}^2} = \frac{1}{2s^2}, \tag{6}$$

where $s = r_{12} = r_1 + r_2$ is the separation distance between the two guiding centers.

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After some algebra, one finds that the precession frequency should be

$$\omega = \frac{1}{2}\left[\Omega_0(\alpha-\beta) + \gamma b_0\left(\frac{r_1}{r_2} - \frac{r_2}{r_1}\right)\right]. \tag{7}$$

Note that if $r_2 = r_1$, then $\alpha = \beta$ and $\omega = 0$, as we expect.

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The stability of the guiding centers can be done as in the case of the equilibrium by passing to a rotational frame.

Open problems.

- The Fetter case with two corotating vortices ($n = 2, S_1 = S_2 = 1$) can be handled in a similar way.
- Dynamics of many vortices, both in the Fetter and non-Fetter case.
- Collisions.
- Effect of a more complicated precession frequency (for instance non symmetric).

Thank you for your attention!!!