Entire spherically symmetric spacelike graphs with prescribed mean curvature function in Schwarzschild and Reissner-Nordström spacetimes

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Abstract

For each spacetime of a family of static spacetimes, we prove the existence of entire spherically symmetric spacelike graphs with prescribed mean curvature function. In particular, classical Schwarzschild and Reissner-Nordström spacetimes are considered. In both cases, the entire spacelike graph asymptotically approaches the event horizon. Spacelike graphs of constant mean curvature remain as a particular situation in the existence results, obtaining explicit expressions for the solutions. The proof of the results is based on the analysis of the associated homogeneous Dirichlet problem on an Euclidean ball together with the obtention of a suitable bound for the lenght of the gradient of a solution which permits the prolongability to the whole space.

2010 MSC: 35J93, 35J25, 35A01, 53C42, 53C50.
Keywords: Entire graph, quasilinear elliptic equation, Dirichlet boundary condition, singular ϕ-Laplacian, prescribed mean curvature function, Schwarzschild spacetime, Reissner-Nordström spacetime.

*The first and third authors are partially supported by Spanish MICINN Grant with FEDER funds MTM2011-23652. The second author by Spanish MICINN Grant with FEDER funds MTM2010-18099 and by the Junta de Andalucía Regional Grant P09-FQM-4496.

1 Introduction

In this paper we investigate the existence of solutions of the following prescribed mean curvature spacelike graph equation

$$\begin{gathered} \frac{1}{f}\,\mathrm{div}\left(\frac{f^2\,\nabla u}{\sqrt{1-f^2|\nabla u|^2}}\right)+\frac{\langle \nabla u\,,\,\nabla f\rangle}{\sqrt{1-f^2|\nabla u|^2}}=\frac{n}{f^2}\,H(x,u), \quad x\in M, \\ |\nabla u|<\frac{1}{f}, \end{gathered} \tag{1}$$

where the Riemannian manifold M is either $\mathbb{R}^n$ or $\mathbb{R}^n\setminus\overline{B}_0(a)$, $a\geq 0$, endowed with a radial metric

$$\langle\ ,\ \rangle=E^2(r)dr^2+r^2d\Theta^2,$$

being $E(r)>0$, $d\Theta^2$ the usual metric of the sphere $\mathbb{S}^{n-1}$. Finally, $f\in C^\infty(a,+\infty)$ is a positive function and $H: M\times\mathbb{R}\to\mathbb{R}$ is a given smooth radially symmetric function.

The approach to this PDE is motivated by Lorentzian Geometry, specifically by the problem of the mean curvature prescription. Explicitly, every solution of (1) defines a spacelike graph in a standard static spacetime, $\mathcal{M}:=M\times_f I$, and the function H prescribes the mean curvature.

In this paper we deal with graphs in static spacetimes with respect to a family of observers (see Section 2), to which spatial universe always looks the same. There are many relevant examples of this kind of spacetimes. Of special importance are (in addition to Lorentz-Minkowski spacetime) *Schwarzschild* and *Reissner-Nordström* spacetimes. Both models describe a universe where there is only a spherically symmetric non-rotating mass, as a star or a black hole. In the first model, the mass has not electric charge, while in the second it is uniformly charged (in fact, Reissner-Nordström spacetime may be seen as an extension of Schwarzschild one). They have one or two *event horizons* and an inevitable singularity at the center of the mass (see [7, Chap. 3] and [7, Chap. 5] for details and physical interpretations).

In general, a spacelike hypersurface has associated a family of instantaneous observers, the future and unit normal vectors. They induce a family of normal observers near the spacelike hypersurface. The mean curvature function intuitively measures *how normal observers get away with respect to a given one, when it is averaged in all spatial directions.* Here, we are interested in prescribing how these normal observers get away (see [16], [17]).

In the latter years, many researchers have worked on the prescribed mean curvature problem on spacelike hypersurfaces in Lorentzian manifolds, especially in the Lorentz-Minkowski spacetime $\mathbb{L}^{n+1}$. In this context, it is remarkable the celebrated paper of Bartnik and Simon [2], where a kind of "universal existence result" is proved for the Dirichlet problem. More recently, the interest is focused on the existence of positive solutions, by using a combination of variational techniques, critical point theory, sub-supersolutions and topological degree (see for instance [3, 4, 5, 11, 12, 13] and the references therein). The Dirichlet problem in a more general spacetime was solved by Gerhardt [14].

In contrast, the number of references devoted to the study of entire spacelike graphs in the Lorentz-Minkowski spacetime with constant or prescribed mean curvature is considerably lower. In this setting, the study of entire constant mean curvature spacelike graphs developed in [19] is motivated by the remarkable Calabi-Bernstein property in the maximal case, i.e., when mean curvature identically vanishes. Namely, Calabi [9] has shown for $n \leq 4$ and latter Cheng and Yau [10] for all n, that a entire maximal graph in $\mathbb{L}^{n+1}$ must be a spacelike hyperplane. Treibergs proved the existence of entire graphs of constant mean curvature with certain asymptotic conditions in $\mathbb{L}^{n+1}$ [19]. Later, Bartnik and Simon [2, Th. 4.4] extended this result to a more general mean curvature function, but related references concerning the prescribed curvature problem for entire graphs are rare. Up to our knowledge, in the latter years only [1], [6] treat this problem by using a variational approach for very concrete prescribed curvatures.

From the analysis of the related bibliography, it seems that the problem of the existence of prescribed mean curvature entire graphs with radial symmetry for static spacetimes has not been considered before.

The main findings of this paper can be summarized as follows.

Theorem 1.1 *Let $\mathbb{R}^n \times_f \mathbb{R}$ be a standard static spacetime, endowed with the spherically symmetric metric*

$$E^2(r)dr^2 + r^2 d\Theta^2 - f^2(r)dt^2,$$

and let $H : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a radially symmetric continuous function. Then, there exists a spherically symmetric entire spacelike graph with mean curvature function H. Moreover, for each $R > 0$, the graph may be chosen such that its intersection with $t = 0$ is a ball of radius R.

Theorem 1.2 *Let $\mathcal{M}$ be either the Schwarzschild exterior spacetime or the Reissner-Nordström exterior spacetime with exterior radius $a > 0$, and let $H : \mathcal{M} \longrightarrow \mathbb{R}$ be a spherically symmetric and bounded continuous function. Then, there exists a spherically symmetric entire spacelike graph with mean curvature function H that approaches the event horizon as $r \to a$. Moreover, for each $R > a$, the graph may be chosen such that its intersection with $t = 0$ is a ball of radius R.*

The proofs of the previous theorems are based on the following existence results for the associated Dirichlet problems on a ball, which are interesting by themselves.

Theorem 1.3 *Let $\mathbb{R}^n \times_f \mathbb{R}$ be a standard static spacetime, endowed with the spherically symmetric metric*

$$E^2(r)dr^2 + r^2 d\Theta^2 - f^2(r)dt^2.$$

Let $B = B_0(R)$ be the Euclidean ball with radius R centered at $0 \in \mathbb{R}^n$, and let $H : \overline{B} \times \mathbb{R} \to \mathbb{R}$ be a spherically symmetric continuous function. Then, there exists a spherically symmetric spacelike graph with mean curvature function H defined on $\overline{B}$ and supported on the slice $t = 0$.

Theorem 1.4 *Let $\mathcal{M}$ be either the Schwarzschild exterior spacetime or the Reissner-Nordström exterior spacetime (with radius a), and let $H : \overline{A}(a, R) \times \mathbb{R} \to \mathbb{R}$ be a spherically symmetric and bounded smooth function, where $\overline{A}(a, R)$ is the closed annulus $a \leq |x| \leq R$. Then, there exists a spherically symmetric spacelike graph with mean curvature function H, which touches the slice $t = 0$ on the boundary $|x| = R$, and approaches the event horizon as $|x| \to a$. Moreover, the graph is radially decreasing on the annulus $\overline{A}(a, R)$ and it intersects the slice $t = 0$ only at the boundary $|x| = R$.*

Remark 1.5 In Theorem 1.4, we have chosen the presented formulation for simplicity. An inspection of the proof will provide more details about the geometry of the solutions. One of the main features is that the hyperbolic angle of the graph with the slice $t = 0$ can be prescribed. More precisely, there exists χ_0 such that for every $\chi < \chi_0$, the graph given by Theorem 1.4 can be chosen so that the hyperbolic angle with the slice $t = 0$ is χ. The value of χ_0 can be explicitly calculated (see the inequality (25)).

The content of the paper is organized as follows. In Section 2 we deal with geometrical preliminaries and notation. Section 3 is devoted to the construction of the elliptic differential equation to be studied in the paper. Section 4 is devoted to prove Theorems 1.3 and 1.4 by using classical arguments from Fixed Point Theory. Once the problem is written as fixed point problem for a suitable operator, the proof of Theorem 1.3 follows from a basic application of Schauder Fixed Point Theorem. Meanwhile, the proof of Theorem 1.4 is more involved because the metric associated to the Schwarzschild or Reissner-Nordström exterior spacetime presents a singularity. Such a singularity implies that the coefficients in the corresponding differential equation are singular. This technical problem is solved by a suitable transformation and an approximation argument. Finally, Section 5 is devoted to the proofs of Theorems 1.1 and 1.2, consisting on a simple prolongability argument on the solutions of the Dirichlet problem.

2 Preliminaries

In a spacetime $\mathcal{M}$, a vector field Q which is unit timelike and future pointing is called a *reference frame* [17, Def. 2.31]. Each integral curve of Q represents an observer in $\mathcal{M}$. So, the choice of a reference frame in a spacetime gives a distinguished family of observers in $\mathcal{M}$. A spacetime is said to be *static* relative to a reference frame Q if Q is irrotational (i.e., $\mathrm{curl}(Q)(X,Y) = 0$ for any $X, Y \in Q^{\perp}$) and there is $f \in C^{\infty}(\mathcal{M})$ such that the vector field fQ is Killing. Observe that Q is irrotational if and only if the distribution $Q^{\perp}$ is integrable [16, Prop. 12.30]. Therefore, given an event $p \in \mathcal{M}$ there exists a unique (inextensible) spacelike hypersurface $\mathcal{S}$ of $\mathcal{M}$ such that $p \in \mathcal{S}$ and $T_q\mathcal{S} = Q_q^{\perp}$, for all $q \in \mathcal{S}$ (in fact, $\mathcal{S}$ is an inextensible leaf of the foliation $Q^{\perp}$ through $p \in \mathcal{M}$). The spacelike hypersurface $\mathcal{S}$ may be interpreted as the spatial universe of each observer in Q which intersects $\mathcal{S}$ at an instant of its proper time.

On the other hand, any (local) flow $\{\phi_t\}$ of the Killing vector field fQ consists of (local) isometries of $\mathcal{M}$ which preserve restspaces of observers in q, i.e., if S is an integral leaf of the foliation $Q^{\perp}$ through $p \in \mathcal{M}$, then $\phi_t(S)$ is an integral leaf of $Q^{\perp}$ through $\phi_t(p)$. Therefore, the spatial universe always looks the same for the observers in Q at least locally. Geometrically, for each $p \in \mathcal{M}$ there exists an open neighbourhood U of p such that $\phi : S \times I \longrightarrow U$, $(q,t) \mapsto \phi_t(q)$ is a diffeomorphism, where S is a leaf of $Q^{\perp}$. Using now that fQ is Killing we have that $f\big(\phi(q,t)\big)$ is independent of t. Moreover, if g denotes the Lorentzian metric of $\mathcal{M}$, then on $S \times I$ we can write

$$\phi^* g = \pi_S^* g_S - h_S(\pi_S)^2 \pi_I^* dt^2,$$

where π_S, π_I are the projections onto S and I respectively, g_S denotes the Riemannian metric on S obtained by restriction of g and $h_S \in C^{\infty}(S)$ satisfies $h_S(q) = f\big(\phi(q,t)\big) > 0$ for all $q \in S$. Note that in the spacetime $(S \times I\,, \phi^* g)$ the role of the Killing vector field fQ is played by the coordinate vector field ∂_t, in fact $\phi_*(\partial_t) = fQ$.

Motivated by the last observation, a spacetime $\mathcal{M}$ is said to be a *standard static spacetime* if it is a warped product $M \times_f I$ where (M, g) is a Riemannian manifold, I is an open interval of $\mathbb{R}$ endowed with the metric $-dt^2$, and $f : M \longrightarrow \mathbb{R}^+$ is a smooth function. In other words, $\mathcal{M} = M \times I$ with the metric

$$\overline{g} := \pi_M^* g + f^2(\pi_M)\pi_I^*(-dt^2) \equiv g - f^2(x)dt^2. \tag{2}$$

Observe that the vector field ∂_t is timelike which determines the time-orientation on $\mathcal{M}$. Thus, if $\phi : S \longrightarrow \mathcal{M}$ is a (connected) spacelike hypersurface in $\mathcal{M}$, the time-orientability of $\mathcal{M}$ allows us to construct a globally defined, unit, timelike vector field, N, normal to S in the same time-orientation of ∂_t.

There is a remarkable family of spacelike hypersurfaces in the standard static spacetime $\mathcal{M}$. Namely, the so called spacelike slices $\{t = t_0\}$, which are totally geodesic and, therefore, its mean curvature function vanishes.

Among the spacelike hypersurfaces, the spacelike graphs on domains of the base M appear in a natural way. We will denote by $\Sigma_u(U)$ the graph of $u \in C^\infty(U)$ such that $u(U) \subseteq I$, i.e.,

$$\Sigma_u(U) = \{(x, u(x)) \ : \ x \in U\}.$$

From (2) the induced metric on $\Sigma_u(U)$ is given on U by $g - f^2(x)du^2$. Therefore, it is spacelike whenever the following inequality holds

$$|\nabla u| < \frac{1}{f} \qquad \text{on} \quad U. \tag{3}$$

3 The mean curvature function

In this section we construct the equation which satisfies a smooth function $u : U \subset M \longrightarrow \mathbb{R}$ defining a spacelike graph $\Sigma_u(U)$ in $\mathcal{M}$ with mean curvature function H.

From now on, we denote by

$$g' = \frac{1}{f^2}\, g - dt^2,$$

and we will point with a superscript $'$ the geometric quantities related with metric g' (∇', H', etc). Note that the Lorentzian metric (2) may be now written as $\overline{g} = f^2 g'$. Denote by $\overline{\nabla}$ the Levi-Civita connection and the gradient operator associated to $\overline{g}$.

The extrinsic geometry of $\Sigma_u(U)$ in $\mathcal{M}$ is completely codified by the shape operator $\overline{A}$, which is defined by

$$\overline{A}(X) = -\overline{\nabla}_X \overline{N},$$

for all $X \in \mathfrak{X}(\Sigma_u)$, where $\overline{N}$ is the unit normal vector field on $\Sigma_u(U)$ in the time orientation of $\mathcal{M}$.

Observe that if N' is the unit normal vector field on $\Sigma_u(U)$ on $(M \times I, g')$, then $N' = f\overline{N}$.

Our strategy now is to use the well-known relation between the Levi-Civita connections of two conformal metrics to get the following equation which gives the shape operator $\overline{A}$ from the shape operator A' of $\Sigma_u(U)$ in $(M \times I, g')$. Making use of the cited formulae,

$$\overline{A}(X) = fA'(X) - g'(\nabla' f, N')X - 2g'(\nabla' f, X)N' + g'(X, N')\nabla' f.$$

Taking traces in both members we obtain

$$nH = nfH' + (n+1)g'(\nabla' f, N').$$

It is not difficult to compute that the unit future pointing vector field N' is

$$N' = \frac{1}{\sqrt{1-f^2|\nabla u|^2}}\big(f^2\nabla u + \partial_t\big),$$

where ∇ and $|\cdot|$ are the gradient operator and the modulus associated to the metric g on M.

Therefore, we obtain the relation

$$H = fH' + \frac{n+1}{n}\frac{f^2 g(\nabla u, \nabla f)}{\sqrt{1-f^2|\nabla u|^2}}. \tag{4}$$

Now, we have to compute H'. In order to do that, consider the Riemannian metric $g^* := \frac{1}{f^2}g$ on M. It is well-known that the mean curvature function H' may be expressed by

$$H' = \frac{1}{n}\mathrm{div}^*\left(\frac{\nabla^* u}{\sqrt{1-|\nabla^* u|^{*2}}}\right),$$

where $\nabla^* u$ means the g*-gradient of u, $|\nabla^* u|^{*2} = g^*(\nabla^* u, \nabla^* u)$ and div^* represents the divergence operator corresponding to g^*.

On the other hand, taking into account the formulae which relates the divergence operators associated to two conformal metrics, we have

$$nH' = \mathrm{div}\left(\frac{f^2\nabla u}{\sqrt{1-f^2|\nabla u|^2}}\right) - \frac{nf}{\sqrt{1-f^2|\nabla u|^2}}g(\nabla u, \nabla f)\,. \tag{5}$$

From the last equality and equation (4) we conclude

$$\frac{1}{f}\mathrm{div}\left(\frac{f^2\nabla u}{\sqrt{1-f^2|\nabla u|^2}}\right) + \frac{g(\nabla u\,,\,\nabla f)}{\sqrt{1-f^2|\nabla u|^2}} = \frac{n}{f^2}H(x,u), \quad x\in U. \tag{6}$$

Note that (1) is obtained as a particular case of (6). In fact, we will consider from now on that M is either $\mathbb{R}^n$ or $\mathbb{R}^n\setminus\overline{B}_0(a)$, where $B_0(a)$ is the euclidean ball centered at 0 with radius $a\geq 0$. As it is shown later, this choice cover the most important explamples from a physical point of view.

We will consider in M a spherically symmetric metric, with the following form

$$g \equiv \langle\,,\rangle = E^2(r)dr^2 + r^2 d\Theta^2. \tag{7}$$

According to [16, Chap. 13], the spacetime $\mathcal{M}$ is spherically symmetric in the sense that for any $\psi\in O(3)$, the map $(t,x)\mapsto\big(t,\psi(x)\big)$ is an isometry of $\overline{g}$. Thus, endowed with $\overline{g}$, $\mathcal{M}$ becomes a spherically symmetric static spacetime (see [16, pag. 365] for more details).

In this setting, it is natural to consider a spacelike graph which inherits the symmetry assumption of $\mathcal{M}$, so we will assume $u(x)\equiv u(r)$ and $f(x)\equiv f(r)$. In this case, we clearly have

$\nabla u = (u'/E^2)\,\partial_r$ and $\nabla f = (f'/E^2)\,\partial_r$ and equation (1) may be rewritten as the following ODE

$$\begin{aligned} \frac{1}{r^{n-1}E(r)}\left(\frac{r^{n-1}E(r)(f/E)^2u'}{\sqrt{1-(f/E)^2u'^2}}\right)' + \frac{1}{E^2}\frac{f'(r)f(r)u'}{\sqrt{1-(f/E)^2u'^2}} &= \frac{n}{f}H(r,u) \qquad \text{in} \quad (a,+\infty), \\ |u'| < E/f \qquad\qquad & \qquad\qquad\qquad \text{in} \quad (a,+\infty). \end{aligned} \tag{8}$$

4 Existence results of the associated Dirichlet problem

Our ultimate objective is to prove the existence of entire radial spacelike graphs. To this aim, we will first study a Dirichlet problem that may be interesting by itself. Thus, in this section, we deal with the Dirichlet problem on $U = B_0(R)$, in the case of $M = \mathbb{R}^n$, or $U = B_0(R)\setminus\overline{B}_0(a)$, with $0 < a < R$, if we consider $M = \mathbb{R}^n\setminus\overline{B}_0(a)$. In both cases, we impose the condition $u(R) = 0$.

Before of distinguishing the two cases, the following considerations are pertinent. Let us define the functional spaces

$$\widehat{C}^1[a,R] := \left\{u \in C^1[a,R] \ : \ u(R) = 0\right\},$$

and

$$\widehat{C}^1(a,R] := \left\{u \in C^1(a,R] \ : \ u(R) = 0\right\},$$

and the linear operator

$$\begin{aligned} T : \widehat{C}^1[a,R] &\longrightarrow \widehat{C}^1(a,R] \\ T[u](r) &:= -\int_r^R \frac{E}{f}(s)\,u'(s)ds. \end{aligned} \tag{9}$$

This operator is inyective but, in general, is not onto on $\widehat{C}^1(a,R]$. From now on, we consider solutions which belong to the image of T. Calling W to this image, the restriction $T : \widehat{C}^1[a,R] \longrightarrow W \subset \widehat{C}^1(a,R]$ is invertible and the inverse is given by

$$T^{-1}[u](r) = -\int_r^R \frac{f}{E}(s)\,u'(s)ds.$$

Now, introducing the change of variable

$$v(r) = T^{-1}[u](r), \tag{10}$$

equation (8) is rewritten as

$$\begin{aligned} \left(r^{n-1}\phi(v')\right)' + 2\frac{f'}{f}(r)\left(r^{n-1}\phi(v')\right) &= nr^{n-1}\frac{E}{f^2}(r)H(r,T[v]) \qquad \text{in} \quad (a,R), \\ |v'| < 1 \qquad\qquad & \qquad\qquad\qquad \text{in} \quad (a,R), \end{aligned} \tag{11}$$

where $\phi(s) = \dfrac{s}{\sqrt{1-s^2}}$.

In the following, we distinguish two cases according to the behaviour of the limits of $f(r)$ and $E(r)$ when $r \to a^+$. In the first case (regular case below), $\lim_{r\to a^+} f(r)$ and $\lim_{r\to a^+} E(r)$ are finite and positive, and the problem for $M = \mathbb{R}^n$ will be essentially the same that for $M = \mathbb{R}^n \setminus \overline{B}_0(0)$. In the second case (singular case), $\lim_{r\to a^+} f(r) = 0$ and $\lim_{r\to a^+} E(r) = +\infty$.

4.1 The regular case. Proof of Theorem 1.3.

In the first situation under study, we assume that $a \geq 0$ and the following hypotheses

(A1) $f,\, E : [a, +\infty) \longrightarrow \mathbb{R}^+$ are continuous functions.

(A2) $H(r, u)$ is continuous in $[a, R] \times \mathbb{R}$.

Theorem 1.3 is a direct consequence of the following result.

Theorem 4.1 *Assume (A1) and (A2). Then, the problem (11) has at least one radially symmetric solution v such that $v'(a) = 0$, $v(R) = 0$.*

The proof uses a fixed point argument. Taking

$$w(r) := r^{n-1}\phi\big(v'(r)\big),$$

equation (11) is transformed into

$$\begin{gathered} w'(r) + 2\,\frac{f'}{f}(r)w(r) = nr^{n-1}\frac{E}{f^2}(r)H(r, T[v]) \quad \text{in} \quad (a, R), \\ w(0) = 0. \end{gathered} \tag{12}$$

Observe that condition $|v'| < 1$ is necessary to have w well-defined. Recall that, from the variation of constants formula, the linear equation

$$\begin{gathered} w'(r) + h(r)w(r) = \varphi(r) \quad \text{in} \quad (a, R), \\ w(0) = 0, \end{gathered} \tag{13}$$

with $h, \varphi \in C^1[a, R]$, has a unique solution given by

$$w(r) = \int_a^r \varphi(t) e^{-\int_t^r h(s)ds} dt.$$

In the case of (12),

$$e^{-\int_t^r h(s)ds} = \frac{f^2(t)}{f^2(r)},$$

hence,

$$r^{n-1}\phi(v') = \int_a^r \varphi(t)\frac{f^2(t)}{f^2(r)} dt.$$

After some easy computations, it turns out that a solution of problem (12) must verify

$$v = \mathcal{A}[v], \tag{14}$$

where $\mathcal{A} : C^1[a,R] \longrightarrow C^1[a,R]$ is defined as

$$\mathcal{A}[v](r) := -\int_r^R \phi^{-1}\left[\frac{1}{\tau^{n-1}f^2(\tau)}\int_a^\tau nt^{n-1}E(t)H(t,T[v])dt\right]d\tau.$$

This operator can be written as

$$\mathcal{A} = K \circ \phi^{-1} \circ S \circ N_H,$$

where

$$\begin{array}{c} S : C[a,R] \longrightarrow C[a,R] \\ S[v](r) = \dfrac{1}{r^{n-1}}\int_a^r t^{n-1}E(t)v(t)dt \quad (r \in (a,R]), \quad S[v](a) = 0, \end{array} \tag{15}$$

$$\begin{array}{c} K : C[a,R] \longrightarrow C^1[a,R] \\ K[v](r) = -\int_r^R v(t)dt. \end{array} \tag{16}$$

and N_H is the Nemytskii operator associated to H,

$$N_H : C^1[a,R] \longrightarrow C[a,R], \quad N_H[v] = H(\cdot, T[v]). \tag{17}$$

From assumptions $(A1)$, $(A2)$, S and N_H are continuous and, from the compactness of K, we deduce that $\mathcal{A}$ is a continuous and compact operator in the Banach space $C^1[a,R]$ (endowed with its usual norm $\|v\| = \|v\|_\infty + \|v'\|_\infty$). In conclusion, we can state the following lemma.

Lemma 4.2 *A function $v \in C^1[a,R]$ is a solution of equation* (12) *if and only if v is a fixed point of the nonlinear compact continuous operator $\mathcal{A}$.*

Remark 4.3 Note that the image of the operator $\mathcal{A}$ is contained in $C^2[a,R]$, so the fixed points (solutions of (12)) will be of class C^2. Moreover, using the regularity theorem for elliptic nonlinear operators, (see [15, Chap. 4]) we conclude that, if the prescription function H is of class C^∞, then the solutions will also be infinitely derivable.

Observe that fixed points of $\mathcal{A}$ always verify the boundary conditions $v'(a) = 0$ and $v(R) = 0$. Define the set

$$\mathcal{B} := \{v \in C^1[a,R] \ : \ \|v\|_\infty < R-a, \ \|v'\|_\infty < 1\}.$$

By using that $\phi^{-1}(\mathbb{R}) = (-1\,,1)$, one gets

$$\|\mathcal{A}(v)\|_\infty < R-a \quad \text{and} \quad \|\big(\mathcal{A}(v)\big)'\|_\infty < 1 \qquad \text{for all} \quad v \in \mathcal{B}.$$

These inequalities implies that $\mathcal{A}(\overline{\mathcal{B}}) \subset \mathcal{B}$. Since $\overline{\mathcal{B}}$ is contractible to a point, and $\mathcal{A}$ is a continuous and compact operator, the Schauder Fixed Point Theorem applies, finishing the proof of Theorem 4.1.

Remark 4.4 The proof of Theorem 1.3 follows immediately by taking $a = 0$. With the same arguments, we could impose a different constraint in the derivative of v, for instance, to prescribe a fixed value of $v'(a)$ or $v'(R)$. By simplicity, we have just considered the condition $v'(a) = 0$, which is the most important example.

Remark 4.5 If $H \leq 0$, from the fixed point formulation (14), we easily deduce that v is decreasing and positive in $[a,R]$. The same conclusion is reached for u. Since the slices $\{t = t_0\}$ are totally geodesic, the hypothesis $H \leq 0$ is interpreted by saying that the mean curvature prescription function is less than the mean curvature of the slices.

4.2 The singular case. Proof of Theorem 1.4.

The main motivation of this paper is to study the Schwarzschild and Reissner-Nordström spacetimes, which play a central role in General Relativity (see for instance [7], [16], [17]). The *Schwarzschild exterior spacetime* models the exterior region of a spacetime where there is only a spherically symmetric non-rotating star without charge. Such a spacetime is defined by the metric

$$E^2(r)dr^2 + r^2 d\Theta^2 - f^2(r)dt^2$$

where

$$f(r) = \sqrt{1 - \frac{2m}{r}} \qquad \text{and} \qquad E(r) = \frac{1}{\sqrt{1 - \frac{2m}{r}}}.$$

Here, m is interpreted as the mass of a star (or black hole) in certain unit system. The value of the radius $r = 2m$ is known as *Schwarzschild radius*. When this radius is bigger than the radius of the star, we are in presence of a *Schwarzschild black hole*.

A generalization of the latter example is the *Reissner-Nordström exterior spacetime*, in which the mass has non-zero electric charge. In this case, we have

$$f(r) = \sqrt{1 - \frac{2m}{r} + \frac{r_Q^2}{r^2}} \qquad \text{and} \qquad E(r) = \frac{1}{\sqrt{1 - \frac{2m}{r} + \frac{r_Q^2}{r^2}}},$$

where $r_Q > 0$ is a characteristic length relative to the charge Q of the mass. Our interest lies in the region where $r > m + \sqrt{m^2 - r_Q^2}$, i.e., outside of the exterior event horizon (recall that, in this spacetime, there are two horizons, in the physical and realistic case $m > r_Q$).

Our idea is to treat both spacetimes in the same way. To this aim, let us consider $a > 0$ and continuous functions f, $E : (a, +\infty) \longrightarrow \mathbb{R}^+$, $H : [a, R] \times \mathbb{R} \to \mathbb{R}$ such that

(B1) $\lim_{r \to a^+} f(r) = 0$ and $\lim_{r \to a^+} E(r) = +\infty$, but $E(r)$ is integrable in $[a, R]$.

(B2) H is bounded in $[a, R] \times \mathbb{R}$, i.e., there exists a constant $C > 0$ such that $|H(r, t)| < C$ for any $r \in [a, R]$ and $t \in \mathbb{R}$.

In this context, the most geometrically natural and physically relevant condition on u is $\lim_{r \to a^+} u(r) = +\infty$ (and, therefore, $\lim_{r \to a^+} u'(r) = -\infty$). Physically, it may be interpreted as our spacelike graph tends to the *event horizon* (see again [7], [16], [17]). From the mathematical standpoint, we are looking for blow-up solutions of equation (8). Let us see how such condition can be guaranteed.

Lemma 4.6 *Let us assume the condition*

($B3$) *E/f is not integrable on $[a, R]$.*

Then,

$$\lim_{r \to a^+} v'(r) = -1, \tag{18}$$

implies

$$\lim_{r \to a^+} u(r) = +\infty.$$

Proof. From the transformation (10),we obtain

$$u(r) = \int_r^R \frac{E(s)}{f(s)} v'(s) ds,$$

and the result is trivial in view of $(B3)$.

$\square$

As a first step in the proof of Theorem 1.4, we are going to fix the hyperbolic angle between our graph and the observers in the reference frame $\frac{1}{f}\partial_t$ (called Schwarzschild observers). Analytically, this is equivalent to fix the value of $u'(R)$. Therefore, we are interested in proving the existence of solutions of the following problem,

$$\begin{aligned}
\left(r^{n-1}\phi(v')\right)' + 2\,\frac{f'}{f}(r)\left(r^{n-1}\phi(v')\right) = nr^{n-1}\frac{E}{f^2}(r)H(r,Tv) \qquad &\text{in} \quad (a,R),\\
v(R) = 0, \quad v'(R) = k, \qquad & \\
\lim_{r\to a^+} v'(r) = -1, \qquad &
\end{aligned} \tag{19}$$

where k is a constant, $|k| < 1$.

Theorem 1.4 is a direct consequence of the following result.

Theorem 4.7 *Let us assume* $(B1)$, $(B2)$ *and* $(B3)$. *Then, there exists* $k_0 < 0$ *such that, for any* $-1 < k \le k_0$, *problem* (19) *has at least one solution.*

In order to prove this theorem, we consider an intermediate proposition by adding a technical assumption to be deleted later.

Proposition 4.8 *Let us assume* $(B1) - (B3)$ *and the additional condition*

$(B4)$ *H has compact support contained in* $[a,R] \times [-j,j]$, *for some natural* j.

Then, there exists $k_0 < 0$ *such that, for any* $-1 < k \le k_0$, *problem* (19) *has at least one solution.*

Proof. By defining $w(r) := r^{n-1}\phi\big(v'(r)\big)$, problem (19) is transformed into

$$\begin{aligned}
w'(r) + 2\,\frac{f'}{f}(r)w(r) = nr^{n-1}\frac{E}{f^2}(r)H(r,Tv) \qquad &\text{in} \quad (a,R),\\
|v'| < 1 \qquad &\text{in} \quad (a,R),\\
w(R) = -A/f^2(R), \qquad & \\
\lim_{r\to a^+} w(r) = -\infty, \qquad &
\end{aligned} \tag{20}$$

where $A = -R^{n-1}\phi(k)f^2(R)$.

Let us consider the linear problem

$$w'(r) + 2\,\frac{f'}{f}(r)w(r) = \varphi(r) \qquad \text{in} \quad (a,R), \tag{21}$$

$$w(R) = -A/f^2(R), \tag{22}$$

$$\lim_{r\to a^+} w(r) = -\infty, \tag{23}$$

where φ is an arbitrary continuous function defined on (a, R). Applying the variation of constants formula, the unique solution of the initial value problem (21)-(22) is

$$w(r) = -\frac{A}{f^2(r)} - \frac{1}{f^2(r)} \int_r^R \varphi(s) f^2(s) ds.$$

by using $(B1)$ and $(B2)$, the limit condition (23) is satisfied if A is chosen such that

$$A > n\,C \int_a^R r^{n-1} E(r) dr. \tag{24}$$

The relation between k and the hyperbolic angle $\chi \in \mathbb{R}$ between the Schwarzschild observers and the normal vector field $\overline{N}$ (see Section 3) is given by

$$\sinh(\chi) = \phi(k).$$

Since by definition $A = -R^{n-1}\phi(k) f^2(R)$, condition (24) holds if and only if

$$\phi(k) = \sinh(\chi) < -\frac{n\,C}{f^2(R)} \frac{1}{R^{n-1}} \int_a^R r^{n-1} E(r) dr, \tag{25}$$

or equivalently, $k < k_0$ where

$$k_0 := \phi^{-1}\left(-\frac{n\,C}{f^2(R)} \frac{1}{R^{n-1}} \int_a^R r^{n-1} E(r) dr \right).$$

In this way, we define the nonlinear operator $\mathcal{N} : X \longrightarrow X$,

$$\mathcal{N}[v](r) := \int_r^R \phi^{-1}\left[\frac{1}{\tau^{n-1} f^2(\tau)} \left(A + \int_\tau^R n t^{n-1} E(t) H(t, T[v]) dt \right) \right] d\tau, \tag{26}$$

where

$$X = \{v \in C^1[a, R] \,:\, v(R) = 0, v'(a) = -1\}.$$

A function $v \in C^1[a, R]$ is a solution of problem (19) if and only if v is a fixed point of the nonlinear operator $\mathcal{N}$. In the next lemma, we are going to prove that such operator is continuus and compact.

Lemma 4.9 *Assume* $(B1)-(B4)$. *Then,* $\mathcal{N}$ *is a compact and continuous nonlinear operator.*

Proof. Let us write

$$\mathcal{N} = K \circ \mathcal{S} \circ N_H,$$

where the operators K and N_H are defined in the previous subsection by (16) and (17) respectively, and the operator $\mathcal{S} : C^1[a, R] \longrightarrow C^1[a, R]$ has the expression

$$\mathcal{S}[v](r) := \phi^{-1}\left[\frac{1}{r^{n-1} f^2(r)} \left(A + \int_r^R n t^{n-1} E(t) v(t) dt \right) \right]. \tag{27}$$

The operator K is continuous and compact, so we only have to verify the continuity of the operator $\mathcal{N}$.

The first step is to prove that $\mathcal{S}$ is continuous. By $(B2)$, the image of N_H is bounded, then it is sufficient to prove that $\mathcal{S}$ is continuous on a certain closed ball B_ρ of $C^1[a,R]$, with arbitrary radius ρ. So, let $\{v_k\}$ be a sequence converging to v uniformly in B_ρ. The objective is to see that $\mathcal{S}[v_k]$ converges uniformly to $\mathcal{S}[v]$].

Denote by $g_k(r) := \int_r^R nt^{n-1}E(t)v_k(t)dt$. Since

$$|g_k(r) - g_l(r)| \leq \|v_k - v_l\|_\infty \int_a^R nt^{n-1}E(t)dt,$$

from the uniform convergence of $\{v_k\}$ and Cauchy criterion, we deduce that $\{g_k\}$ is also uniformly convergent. The Dominated Convergence Theorem ensures that the limit is

$$g := \int_r^R nt^{n-1}E(t)v(t)dt.$$

Let us denote $\varrho = \rho\int_a^R nt^{n-1}E(t)dt$ and $x_k(r) := \big(r, g_k(r)\big)$. With this notation, we can write

$$\mathcal{S}[v_k](r) = F\big(x_k(r)\big),$$

where $F : [a,R]\times[-\varrho,\varrho] \longrightarrow \mathbb{R}$ is a continuous function. From the uniform convergence of v_k and g_k, we deduce the uniform convergence of $\{x_k\}$ to (id, g) (for any fixed norm in $\mathbb{R}^3$). Hence, since F is uniformly continuous, because of compactness of $[a,R]\times[-\varrho,\varrho]$, we conclude that $\mathcal{S}[v_k]$ converges uniformly to $\mathcal{S}[v]$.

It remains to prove the continuity of Nemytskii operator N_H. At this point, the hypothesis $(B4)$ is crucial. Note that the boundedness of H is not enough (for instance, $H(r,s) = \sin(s)$).

Let $\{v_k\} \subset X$ be a sequence which converges to $v \in X$ (in the usual C^1-norm). We have to prove that $H(r, T[v_k](r)) \longrightarrow H(r, T[v](r))$ uniformly on $[a,R]$. The uniform convergence on any compact set in $(a,R]$ follows from applying Ascoli-Arzela Theorem, once it is observed that the derivative of $\phi^{-1}(s) = \frac{s}{\sqrt{1+s^2}}$ is a bounded function. On the other hand, $v'(a) = v_k'(a) = -1$ and condition $(B3)$ implies that

$$\lim_{r\to a^+} T[v_k](r) = \lim_{r\to a^+} T[v](r) = +\infty.$$

By condition $(B4)$, this means that

$$\lim_{r\to a^+} H(r, T[v_k](r)) = \lim_{r\to a^+} H(r, T[v](r)) = 0.$$

Therefore, the pointwise convergence at $r = a$ is ensured.

From $(B1)$ and using that $v'(a) = -1$, we may take $\overline{r} \in (a,R)$ such that, for any $r \in [a,\overline{r}]$, $|-1-v'(r)| < 1/3$ and the following inequality holds

$$-\int_r^R \frac{E}{f}(t)v'(t)dt > j+1. \tag{28}$$

Taking $0 < \varepsilon < \min\{\frac{1}{3}, -1/\int_{\overline{r}}^R \frac{E}{f}(t)dt\}$ there exists k_0 such that, for all $k > k_0$, $\|v_k' - v'\|_\infty < \varepsilon$, hence we have

$$-\int_{\overline{r}}^R \frac{E}{f}(t)v_k'(t)dt > j+1-\varepsilon\int_{\overline{r}}^R \frac{E}{f}(t)dt.$$

Since $v_k' < 0$ on $[a, \overline{r}]$, we obtain

$$-\int_r^R \frac{E}{f}(t) v_k'(t) dt > j.$$

As a consequence of $(B4)$, from the latter inequality and (28) we conclude that

$$H(r, Tv_k) = H(r, Tv) = 0 \quad \text{on} \quad [a, \overline{r}].$$

Thus, the uniform convergence is trivial on the compact set $[a, \overline{r}]$ and the proof is finished.

$\square$

Now, fixed an hyperbolic angle χ satisfying (25) (so, fixed the corresponding k), one gets that the image of $\mathcal{N}$ is contained in the closed and convex set $\overline{\mathcal{D}_k}$ defined by

$$\overline{\mathcal{D}} = \{v \in X \, : \, v'(R) = k, \; \|v\|_\infty \leq (R - a), \; \|v'\|_\infty \leq 1\}.$$

Then, a basic application of the Schauder fixed point theorem finishes the proof of Proposition 4.8.

$\square$

Now, as a final step for the proof of Theorem 4.7, we are going to remove assumption $(B4)$ by means of a truncation argument. Let $h_j : \mathbb{R} \longrightarrow [0, 1]$ be a smooth function such that is equal to 1 on $[-j + 1, j - 1]$ and vanishes outside of the interval $(-j, j)$, $j > 1$. Then, we construct the sequence of functions

$$H_j : [a, R] \times \mathbb{R} \longrightarrow \mathbb{R}, \qquad H_j(r, s) := H(r, s)\, h_j(s),$$

which converges pointwise to the function H. Note that each H_n satisfies the assumption $(B4)$. Therefore, by using Proposition 4.8, we have a sequence $\{v_j\}_{j=1}^\infty$ of fixed points of the nonlinear operators

$$\mathcal{N}_j[v](r) := \int_r^R \phi^{-1} \left[\frac{1}{\tau^{n-1} f^2(\tau)} \left(A + \int_\tau^R n t^{n-1} E(t) H_j(t, Tv) dt \right) \right] d\tau, \tag{29}$$

i.e.,

$$v_j = \mathcal{N}_j[v_j]. \tag{30}$$

It is immediate that $\|v_j\|_\infty < R - a$ and $\|v_j'\|_\infty \leq 1$. Thus, from Ascoli-Arzela Theorem, there exists a subsequence and a function $v \in C[a, R]$ such that

$$\{v_j\} \longrightarrow v \qquad \text{uniformly on } [a, R].$$

Choose an arbitrary closed interval $[c, d] \subset (a, R]$. Performing into (29)-(30) and using again that the derivative of $\phi^{-1}(s)$ is a bounded function, it is easy to check that there exists a constant L, depending only on the interval $[c, d]$, such that $\|v_j''\|_\infty < L$. Hence, the family $\{v_j'\}_{j=1}^\infty$ is equicontinuous on $[c, d]$. Since $|v_j'| < 1$ we can apply the Ascoli-Arzela Theorem and conclude that there exists a continuous $v \in C^1[c, d]$ such that

$$\{v_j'\} \longrightarrow v' \qquad \text{in} \quad C^1[c, d].$$

In order to prove Theorem 4.7, we only have to see that v is a fixed point of the nonlinear operator defined by (26). Taking limits in (30) and using the C^1-convergence of $\{v_j\}_{j=1}^{\infty}$ on compacts in $(a, R]$ we get

$$v(r) = \mathcal{N}[v](r) \qquad r \in (a, R].$$

Moreover, $v(a) = \mathcal{N}[v](a)$ holds because the function $r \longmapsto nr^{n-1}E(r)H\big(r, Tv(r)\big)$ is integrable on $[a, R]$ (although it is not assured the existence of limit of this function when $r \to a$). Therefore, v is a fixed point of $\mathcal{N}$, or equivalently, v is a solution of problem (19).

To conclude the proof of Theorem 4.7, we only have to observe that any choice of χ satisfying (25) implies that the solution as a fixed point of the non linear operator $\mathcal{N}$ is a decreasing and non-negative function in $(a, R]$.

Remark 4.10 In the particular case of a constant mean curvature H, the operator $\mathcal{N}$ provides an explicit integral expression of a radially symmetric spacelike graph with constant mean curvature tending to the event horizon. In particular, we obtain maximal graphs different from the slices.

Remark 4.11 Imposing $\lim\limits_{r \to a^+} v'(r) = +1$, instead of (18), we obtain the existence of non-positive and increasing solutions in (a, R) which approach the event horizon in the past of the Schwarzschild observers. The arguments of the proof remain unchaged.

5 Proof of the main result

Finally, we will prove that any graph on a ball obtained in the previous section can be extended on $\mathbb{R}^n$ or $\mathbb{R}^n \setminus B_0(a)$, depending on the case. For this, we need the following lemma.

Lemma 5.1 *Let $v \in C^2[a, b]$ be a solution of* (11). *Then $|v'| < 1$ on $(a, b]$.*

Proof. On (a, b) the solutions verify $|v'| < 1$. We only have to prove the inequality at $r = b$. Suppose that

$$\lim_{r \to b^-} |v'(r)| = 1 \qquad \text{and} \qquad \lim_{r \to b^-} |\phi\big(v'(r)\big)| = \infty.$$

For r sufficiently close to b it is easy to see

$$\frac{\big[r^{n-1}\phi\big(v'(r)\big)\big]'}{\big[r^{n-1}\phi\big(v'(r)\big)\big]} = -2\frac{f'}{f}(r) + n\frac{E}{f^2}(r)\frac{H\big(r, T(v(r))\big)}{\phi\big(v'(r)\big)}.$$

Let $\overline{r} \in (a, b)$ be such that $|v'(\tau)| > 0$ for any $\tau \in (\overline{r}, b)$. Integrating the last equality, we have

$$\log\big|r^{n-1}\phi\big(v'(r)\big)\big| - \log\big|\overline{r}^{n-1}\phi\big(v'(\overline{r})\big)\big| = \int_{\overline{r}}^{r} \left(-2\frac{f'}{f}(\tau) + n\frac{E}{f^2}(\tau)\frac{H\big(\tau, T(v(\tau))\big)}{\phi\big(v'(\tau)\big)}\right) d\tau.$$

Taking limits, $r \to b^-$, we check that the left member tends to infinity while the right one is finite. Therefore, we deduce that $|\phi(v')|$ is bounded and, consequently, $\|v'\|_\infty$ must be strictly lower than 1. $\square$

To prove Theorem 1.1, once R is fixed, Theorems 1.3 and 1.4 provide a solution v of problem (11). Then, it suffices to guarantee that v can be continued until $+\infty$ as a solution. First, we rewrite equation (11), as a system of two ordinary differential equations of first order

$$\begin{aligned} v' &= z \\ z' &= \quad (1-z^2)\left(-(n-1)\frac{z}{r} - 2\frac{f'}{f}(r)z + n\frac{E}{f^2}(r)\sqrt{1-z^2}H(r,Tv)\right), \end{aligned}$$

which we can abbreviate as

$$\begin{bmatrix} v' \\ z' \end{bmatrix} = \mathcal{F}\big(r,(v,z)\big),$$

where $\mathcal{F} : \mathbb{R}^+ \times \mathbb{R} \times (-1\,,1) \longrightarrow \mathbb{R}^2$.

Let (a,b) be the maximal interval of definition of v. Suppose that $b < +\infty$. By the standard prolongability theorem of ordinary differential equations (see for instance [18, Section 2.5]), we have that the graph $\{\big(r, v(r), v'(r)\big) \ : \ r \in [a+(R-a)/2, b)\}$ goes out of any compact subset of $\mathbb{R}^+ \times \mathbb{R} \times (-1,1)$. However $|v(r)| < b$ then we know that $v(r) \in [-b, R]$. Moreover, by Lemma 5.1, $|v'(r)| < \rho < 1$. Therefore, the graph can not go out of the compact subset $[a+(R-a)/2\,, b] \times [-b, R] \times [-\rho, \rho]$ contained in the domain of $\mathcal{F}$. This is a contradiction and then $b = +\infty$.

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