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Radial Solutions of the Dirichlet Problem for the Prescribed Mean Curvature Equation in a Robertson-Walker Spacetime

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Abstract

We consider the prescribed mean curvature problem of spacelike graphs in Robertson-Walker spacetimes of flat fiber with homogeneous Dirichlet conditions on an Euclidean ball. Under reasonable assumptions, it is shown that every possible solution must be radially symmetric. Besides, an existence result for a singular nonlinear equation is proved by making use of the classical Schauder fixed point Theorem. The results are applied to realistic examples of Robertson-Walker spacetimes.

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1 Introduction

Let $B(R)$ be the Euclidean ball, centered at $0 \in \mathbb{R}^n$ with radius R. Let $I \subseteq \mathbb{R}$ be an open interval with $0 \in I$, and let $f \in C^\infty(I)$ be a positive function. For a given smooth radially symmetric function $H : I \times B(R) \to \mathbb{R}$, we study in this paper the existence of positive, radial solutions of the following

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quasilinear elliptic problem

$$\begin{gathered}
\operatorname{div}\left(\frac{\nabla u}{f(u)\sqrt{f(u)^2-|\nabla u|^2}}\right)+\frac{f'(u)}{\sqrt{f(u)^2-|\nabla u|^2}}\left(n+\frac{|\nabla u|^2}{f(u)^2}\right)=nH \quad \text{in} \quad B(R),\\
|\nabla u|<f(u),\\
u=0 \quad \text{in} \quad \partial B(R).
\end{gathered} \tag{1.1}$$

The approach to this PDE is motivated by Lorentzian Geometry, specifically by the problem of the mean curvature prescription. Explicitly, every solution of (1.1) defines a spacelike graph on a ball of the fiber of the Robertson-Walker spacetime, $\mathcal{M} = I \times_f \mathbb{R}^n$ (see Section 2) where the function H prescribes the mean curvature.

In the same way as in Riemannian Geometry, constant mean curvature spacelike hypersurfaces in Lorentzian manifolds are characterized as critical points of the functional 'area', under certain 'volume constraints' [6], [11]. This situation is a particular case of our mean curvature prescription problem (see [10, 7] and the references therein). However, the interest of this problem is not only geometric. Namely, spacelike hypersurfaces with constant mean curvature in spacetimes have special importance in Physics. These are used as initial condition for solving Einstein's equations, because they acquire a more pleasant appearance (see, for example, [9]). Other relevant results about foliations of the spacetime through constant mean curvature hypersurfaces are given in [9], [18]. Specially relevant is the study of spacelike hypersurfaces in Robertson-Walker spacetimes modelling relativistic universes, where matter and energy evolute as a perfect fluid (see [19, 21] for more information).

Moreover, the sign of the mean curvature operator has a physical meaning. A spacelike hypersurface has associated a family of *instantaneous observers*, the future-pointing timelike unit normal vectors, i.e., the *normal observers*. The mean curvature measures, intuitively, *how normal observers get away with respect to the next one, when it is averaged in all spacial directions*. The result in this paper may be contemplated then as prescribing locally the behaviour of normal instantaneous observers.

In the latter years, many researchers have worked on the prescribed mean curvature problem in the Riemannian ambient (specially in the Euclidean space). In the Lorentzian ambient, the efforts have mainly focused in the Lorentz-Minkowski spacetime. In this context, it is remarkable the celebrated paper of Barnitk and Simon [2], where a kind of "universal existence result" is proved for the Dirichlet problem. More recently, the interest is focused on the existence of positive solutions, by using a combination of variational techniques, critical point theory, sub-supersolutions and topological degree (see for instance [3, 4, 5, 12, 13, 14] and the references therein). Up to our knowledge, the problem of the existence of prescribed mean curvature graphs for Robertson-Walker spacetimes has not been considered before. In this context, the uniqueness problem for constant mean curvature has been studied in more depth (see for instance [1, 8]).

The main aim of this article is to use an approach based on the Schauder fixed point Theorem (see for instance [15]) to deal with the the existence problem. It should be noted that our results do not follow directly from the obtained ones previously when $\mathcal{M}$ is Minkowski spacetime ([3] and references therein). In fact, we will deal here with an equation with an extra singular term with respect to the considered in Minkowski spacetime. Besides, we give conditions only on the prescription function (not on the warping function) which ensure *a priori* radial symmetry of all the (possible) positive solutions of the equation (3.2). In other words, we will prove that *the symmetry of the base domain 'spreads to solutions'*. To carry out this aim, we will take advantage of the results obtained in 1979 by B. Gidas, W. Ni and L. Nirenberg in [16] about symmetry of the solutions of certain nonlinear differential equations. The method used by the three authors had yet been invented by Alexandroff almost thirty years before, when he proved successfully that the round spheres are the only connected, compact hypersurfaces embedded in the Euclidean space with constant mean curvature. Indeed, currently this technique is known as 'Alexandroff's reflection method' and its use is very extended in the field of the elliptic PDE's and Geometric Analysis. In our case, we are able to use a truncature argument exposed in [12] and apply directly the results of [16].

The structure of this paper is as follows. Section 2 and 3 are devoted to provide some preliminaries and fix the precise set up of the problem. Section 4 is devoted to study the radial symmetry of the possible solutions of (1.1) under suitable assumptions on the prescribed mean curvature function. Thus, we arrive that any positive solution of (1.1) must be radially symmetric (Theorem 4.1). By using the classical Schauder fixed point Theorem, a general existence result is proved in Section 5 (Theorem 5.5).

The main findings of this paper can be summarized as follows.

Theorem 1.1 *Let $I \times_f \mathbb{R}^n$ be a Robertson-Walker spacetime, and let $B = B_0(R)$ be the Euclidean ball with radius R centred at $0 \in \mathbb{R}^n$. Assume $I_f^+(R) \subset I$, where*

$$I_f(R) := \left[-\int_{-R}^{0} f(\varphi^{-1}(s))ds \ , \ \int_{0}^{R} f(\varphi^{-1}(s))ds \right],$$

and suppose that the following inequality holds

$$\max_{\mathbb{R}^+ \cap I_f(R)} |f'| < \frac{1}{R}.$$

For each radially symmetric smooth function $H : I \times \overline{B} \to \mathbb{R}$ such that

$$H(t,r) \leq \frac{f'}{f}(t) \quad \text{and} \quad f'(t) \geq 0, \qquad \text{for any} \quad r \in]0,R[\ \ , \ \ t \in I_f(R),$$

there exists a spacelike graph with mean curvature function H defined on $\overline{B}$, supported on the slice $t = 0$ and only touching it on the boundary $\{0\} \times \partial B$, and forming a non-zero hyperbolic angle with ∂_t. Moreover, if H is increasing in the second variable, such a spacelike graph must be radially symmetric.

It should be pointed out that the assumptions in this result have a reasonable physical interpretation. In fact, the inequality $f'(t) \geq 0$ means that the divergence in the spacetime $I \times_f \mathbb{R}^n$ of the *reference frame* ∂_t is nonnegative, which indicates that the comoving observers are on average spreading apart [21, p.121] and therefore, for these observers, the universe is really expanding whenever $f'(t) > 0$. On the other hand, the inequality $H(t,r) \leq (f'/f)(t)$ expresses an above control of the prescription function by the *Hubble function* f'/f of the spacetime $I \times_f \mathbb{R}^n$.

Note that previous inequality is not a comparison assumption between extrinsic quantities of two spacelike hypersurfaces of $\mathcal{M}$ (the right member corresponds to a spacelike slice which changes when changes the point at the graph). This kind of inequality has been used to characterize some spacelike slices of certain $I \times_f \mathbb{R}^n$ when $n = 2$ [20].

Moreover, the family of Robertson-Walker spacetimes where the result applies is very wide, and contains relevant relativistic spacetimes. Indeed, it includes the Lorentz-Minkowski spacetime ($f = 1$, $I = \mathbb{R}$), the Einstein-De Sitter spacetime ($I =]-t_0, +\infty[$, $f(t) = (t + t_0)^{2/3}$, with $t_0 > 0$), and the steady state spacetime ($I = \mathbb{R}$, $f(t) = e^t$), which is an open subset of the De Sitter spacetime.

2 Preliminaries

First of all, we are going to introduce the ambient spacetimes where our spacelike graphs are embedded. We consider the Euclidean space $(\mathbb{R}^n, \langle\ ,\ \rangle)$, and let $I \subseteq \mathbb{R}$ be a open interval in $\mathbb{R}$ with the metric $-dt^2$. Throughout this paper we will denote by $(\mathcal{M}, g)$ the $(n+1)$-dimensional product manifold $I \times \mathbb{R}^n$ endowed the Lorentzian metric

$$g = \pi_I^*(-dt^2) + f^2(\pi_I)\pi_F^*(\langle\, ,\rangle) \equiv -dt^2 + f^2(t)\langle\ ,\ \rangle, \tag{2.1}$$

where $f > 0$ is a smooth function on I, and π_I and π_F denote the projections onto I and $\mathbb{R}^n$ respectively. Thus, $(\mathcal{M}, g)$ is a Lorentzian warped product with base, I fiber $\mathbb{R}^n$ and warping function f, and we will denote it by $I \times_f M$. We will refer $\mathcal{M}$ as a Robertson-Walker (RW) spacetime.

Given an n-dimensional manifold S, an immersion $\phi : S \to \mathcal{M}$ is said to be *spacelike* if the Lorentzian metric given by (2.1) induces, via ϕ, a Riemannian metric g_s on S. In this case, S is called a spacelike hypersurface.

Observe that the vector field $\partial_t := \partial/\partial t \in \mathfrak{X}(\mathcal{M})$ is timelike and unit which determines a time-orientation on $\mathcal{M}$. Thus, if $\phi : S \longrightarrow \mathcal{M}$ is a (connected) spacelike hypersurface in $\mathcal{M}$, the time-orientability of $\mathcal{M}$ allows us to define $N \in \mathfrak{X}^\perp(S)$ as the only globally defined, unit timelike vector field normal to S in the same time-orientation of ∂_t.

There is a remarkable family of spacelike hypersurfaces in the RW spacetime $\mathcal{M}$. Namely, the level hypersurfaces of projection function t. They are also called *spacelike slices*. Each spacelike slice $t = t_0$ is umbilical and its mean curvature is $f'(t_0)/f(t_0)$.

Among the spacelike hypersurfaces, the spacelike graphs on domains of the fiber $\mathbb{R}^n$, appear in a natural way. We will denote by Σ_u the graph defined from $u \in C^\infty(U)$ such that $u(U) \subseteq I$, i.e., $\Sigma_u = \{(x, u(x)) : x \in U\}$. The spacelike condition is expressed as follows

$$|\nabla u| < f(u) \qquad \text{in} \quad U. \tag{2.2}$$

For a spacelike graph Σ_u, the unit timelike normal vector field in the same time orientation of ∂_t it is given by

$$N = \frac{f(u)}{\sqrt{f(u) - |\nabla u|^2}} \left(\frac{1}{f^2(u)} \nabla u + \partial_t \right),$$

The corresponding mean curvature associated to N, is defined by

$$\operatorname{div}\left(\frac{\nabla u}{f(u)\sqrt{f(u)^2 - |\nabla u|^2}} \right) + \frac{f'(u)}{\sqrt{f(u)^2 - |\nabla u|^2}} \left(n + \frac{|\nabla u|^2}{f(u)^2} \right),$$

which can be seen as a quasilinear elliptic operator Q, because of (2.2). Hence, our prescription problem is translated into the equation

$$Q(u)(x) = nH(u, x). \tag{2.3}$$

3 Set up

Note that Q is a quasilinear elliptic operator defined only on smooth functions which satisfy (2.2). In order to face our problem, the first step is to perform a suitable variable change in (2.3) to make it easier. Indeed, consider

$$v = \varphi(u), \qquad \text{where} \qquad \varphi(t) = \int_0^t \frac{ds}{f(s)}.$$

Clearly, φ is a diffeomorphism from I to another open interval J in $\mathbb{R}$. Consequently, it follows that $\nabla v = \dfrac{1}{f(u)}\nabla u$. Therefore, $|\nabla u| < f(u)$ holds if and only if $|\nabla v| < 1$. It is clear that u is radially symmetric if and only if v is also radially symmetric.

Taking into account

$$\begin{aligned}
\operatorname{div}\left(\frac{\nabla u}{f(u)\sqrt{f(u)^2-|\nabla u|^2}}\right) &= \operatorname{div}\left(\frac{\nabla v}{f(u)\sqrt{1-|\nabla v|^2}}\right)\\
&= \frac{1}{f(u)}\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right)+\left\langle \nabla\frac{1}{f(u)},\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right\rangle\\
&= \frac{1}{f(u)}\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right)-\left\langle \frac{f'(u)}{f(u)^2}\nabla u,\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right\rangle\\
&= \frac{1}{f(u)}\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right)-\frac{f'(u)}{f(u)\sqrt{1-|\nabla v|^2}}|\nabla v|^2,
\end{aligned}$$

our equation is transformed in

$$Q(v) := \operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right)+\frac{nf'(\varphi^{-1}(v))}{\sqrt{1-|\nabla v|^2}} = nf(\varphi^{-1}(v))H(\varphi^{-1}(v),x). \tag{3.1}$$

Actually, the previous variable change is equivalent to consider the following conformal map

$$\begin{aligned}
\varphi\times Id : I\times_f \mathbb{R}^n &\longrightarrow (J\times\mathbb{R}^n, -ds^2+g)\\
(t,p) &\longmapsto (\varphi(t),p),
\end{aligned}$$

which has conformal factor $\dfrac{1}{f(t)}$. The Lorentzian product spacetime is in fact an open subset of Lorentz-Minkowski $\mathbb{L}^{n+1}$. In $\mathbb{L}^{n+1}$, the mean curvature function of the spacelike graph by v is

$$\frac{1}{n}\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right).$$

From now on, we will deal with equation (3.1). Next we fix some notation. We take a polar coordinate system centered at $0\in B(R)$ and write the Euclidean metric as usual as

$$dr^2+r^2d\Theta^2,$$

where $d^2\Theta$ is the canonical metric of the unit sphere $(n-1)$-dimensional. In addition, $H : I\times B(R)\longrightarrow \mathbb{R}$ will be a radially symmetric smooth function. In this work, we are interested in spacelike graphs defined on a closed ball of the fiber, whose boundary is supported on the slice $t=0$. In other words, the function v, which define the graph, is strictly positive in the open ball, and it is zero at the boundary.

Therefore, the problem is reduced to deal with the existence of a positive solutions of the following elliptic quasilinear differential equation

$$\begin{aligned}
\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right)+\frac{nf'(\varphi^{-1}(v))}{\sqrt{1-|\nabla v|^2}} = nH(\varphi^{-1}(v),|x|)f(\varphi^{-1}(v)) \quad &\text{in} \quad B(R)\\
|\nabla v|<1,&\\
v=0 \quad \text{in} \quad \partial B(R).&
\end{aligned} \tag{3.2}$$

4 A priori radial symmetry of positive solutions

The aim of this section is to provide sufficient conditions on the prescription function to ensure that any eventual positive solution of (3.2) must be radially symmetric. In fact, we can state the following result.

Theorem 4.1 *Let $I \times_f \mathbb{R}^n$ be a RW spacetime, and let $B = B_0(R)$ be the Euclidean ball with radius R centred at $0 \in \mathbb{R}^n$.. For each smooth radially symmetric function $H : I \times [0, R] \to \mathbb{R}$, $H = H(t, r)$, radially increasing in the second variable and which satisfies $H(0, r) \leq \dfrac{f'(0)}{f(0)}$ in ∂B, any positive solution v of equation* (3.2) *is radially symmetric. Moreover, $\dfrac{\partial v}{\partial r} < 0$ holds in ∂B.*

Remark 4.2 Geometrically, the last assertion means that the hyperbolic angle between the unit normal vector field N and ∂_t is nowhere zero at the points of the graph corresponding to $\{0\} \times \partial B$.

In order to use the Strong Maximum Principle (see for instance [17]) to derive a suitable Alexandroff reflection method, it is required that the involved differential operator is defined on $C^2(\overline{B}(R))$, and it must be uniformly elliptic. To this aim, we apply to our operator (3.1) a truncature argument first used in [12] for the Lorentz-Minskowski operator.

First of all, we rewrite our operator Q as

$$Q(v) = \operatorname{div}(h(|\nabla v|^2)\nabla v) + nh(|\nabla v|^2) f'(\varphi^{-1}(v)),$$

where $h(s) := \dfrac{1}{\sqrt{1-s}}$.

Fix $v \in C^2(\overline{B}(R))$ a positive solution of (3.2), and let $m := \max_{\overline{B}(R)} |\nabla v| < 1$. We define the truncated function $\overline{h}$,

$$\overline{h}(s) = \begin{cases} h(s) & \text{if} \quad s \leq m^2, \\ \alpha(s) & \text{if} \quad m^2 < s < 1, \\ c & \text{if} \quad s \geq 1, \end{cases}$$

where the function $\alpha : \mathbb{R} \to \mathbb{R}^+$ and the constant c are such that $\overline{h} \in C^\infty(\mathbb{R})$ is increasing and positive. Observe that both $\overline{h}$ and $\overline{h}'$ are bounded on all $\mathbb{R}$. We introduce a new operator, denoted by Q_v, as follows,

$$w \longmapsto Q_v(w) = \operatorname{div}(\overline{h}(|\nabla w|^2)\nabla w) + n\overline{h}(w) f'(\varphi^{-1}(w)), \tag{4.1}$$

where $w \in C^2(\overline{B}(R))$. Note that $Q_v(w) = Q(w)$ whenever $|\nabla w| \leq |\nabla v|$. It is not difficult to compute the principal symbol of Q_v (see [17, chap. 1]) and to prove that Q_v is uniformly elliptic.

Now the Strong Maximum Principle may be applied to Q_v and then, the proof of Theorem 4.1 follows from [16, Cor. 1].

5 Existence result

We have proved in Section 4 that under some conditions every eventual positive solution v of problem (3.2) must be radially symmetric. The purpose of this section is provide sufficient conditions for the existence of such radially symmetric solutions. Passing to polar coordinates, the equation is reduced to the following ODE with mixed boundary conditions

$$\begin{gathered}\frac{1}{r^{n-1}}\left(r^{n-1}\phi(v')\right)' + \frac{nf'(\varphi^{-1}(v))}{\sqrt{1-v'^2}} = nH(\varphi^{-1}(v), r)f(\varphi^{-1}(v)),\\ |v'| < 1 \quad \text{in} \quad (0,R),\\ v'(0) = 0 = v(R),\end{gathered} \tag{5.1}$$

where $\phi(s) := \dfrac{s}{\sqrt{1-s^2}}$.

We fix some notation which will be used in the rest of the section. Let C be the Banach space of the real continuous functions in $[0,R]$, with the maximum norm, and C^1 the space of continuously differentiable functions with its usual norm $\|v\| = \|v\|_\infty + \|v'\|_\infty$. We write $B_{\rho,\gamma} = \{v \in C^1 : \|v\|_\infty < \rho, \|v'\|_\infty < \gamma\}$.

Our first step is to associate a *fixed point operator* $\mathcal{N}$ to problem (5.1). We start by defining

$$S : C \longrightarrow C^1,$$

$$S(v)(r) = \frac{1}{r^{n-1}}\int_0^r t^{n-1}v(t)dt \quad (r \in (0,R]), \quad S(v)(0) = 0,$$

$$K : C^1 \longrightarrow C^1,$$

$$K(v)(r) = \int_r^R v(t)dt.$$

An easy checking shows that, for each $h \in C$, the mixed problem

$$\left(r^{n-1}\phi(v')\right)' + r^{n-1}h = 0, \quad v'(0) = v(R) = 0,$$

has a unique solution given by

$$v = K \circ \phi^{-1} \circ S(h).$$

Now, we consider the Nemytskii operator

$$N_F : B_{\alpha,1} \subset C^1 \longrightarrow C, \quad N_F(v) = F(\cdot, v, v'),$$

where $F : [0,R] \times \varphi(I) \times (-1,1) \to \mathbb{R}$ is given by

$$F(r,s,t) = -nH(\varphi^{-1}(s), r)f(\varphi^{-1}(s)) + \frac{n(f' \circ \varphi^{-1})(s)}{\sqrt{1-t^2}}.$$

Obviously, N_F is continuous and $N_F(\overline{B}_{\rho,\gamma})$ is a bounded subset of C for all $\rho > 0$ and $0 \leq \gamma < 1$. Moreover, from the compactness of K we deduce the compactness of $K \circ \phi^{-1} \circ S : C \to C^1$ in $\overline{B}_{\rho,\gamma}$ for all $\rho > 0$ and $0 \leq \gamma < 1$. In this way, solving the problem (5.1) is equivalent to find the fixed points of $\widehat{\mathcal{N}}$.

Lemma 5.1 *A function $v \in C^1$ is a solution of equation* (5.1) *if and only if v is a fixed point of the nonlinear compact operator*

$$\widehat{\mathcal{N}} : B_{\rho,\gamma} \subset C^1 \longrightarrow C^1, \quad \widehat{\mathcal{N}} = K \circ \phi^{-1} \circ S \circ N_F. \tag{5.2}$$

Remark 5.2 Note that the image of the operator $\widehat{\mathcal{N}}$ is contained in $C^2[0,R]$, so the fixed points (solutions of the equation (5.1)) will be of class C^2. Moreover, using the regularity theorem for elliptic nonlinear operators, (see [17, Chapter 4]) we conclude that, if the prescription function H is of class C^∞, then the solutions will also be infinitely derivable.

Note that fixed points of $\widehat{\mathcal{N}}$ always verify the restrictions $v'(0) = v(R) = 0$. Therefore, we will consider the Banach subspace of C^1 which satisfy these boundary conditions. Our aim is to search a suitable subset to apply the Schauder point fixed theorem. Let us define the set

$$\mathcal{B}(\gamma) = \{v \in \overline{B}_{R,\gamma} : \ v'(0) = 0 = v(R)\}.$$

Since the graph associated to v is spacelike, i.e., $\|v'\|_\infty < 1$, we deduce that $\|v\|_\infty < R$. So, the image of v is in $[-R,R]$ or, equivalently, the image of $u = \varphi^{-1}(v)$ is contained in $\varphi^{-1}([-R,R])$. Hence, this observation gives us a height bound of the spacelike graphs. In order to restrict the operator $\widehat{\mathcal{N}}$ to $\mathcal{B}(\gamma)$, we impose the first assumption on the interval I in our RW spacetime

(A1) $[-R,R] \subset \varphi(I)$, i.e., $I_f(R) := \left[-\int_{-R}^{0} f(\varphi^{-1}(s))ds \ , \ \int_0^R f(\varphi^{-1}(s))ds\right] \subset I.$

Basically, (A1) says that the interval I must be sufficiently big to contain the highest or lowest possible spacelike graph.

Now, the compact operator $\widehat{\mathcal{N}}$ restricted to $\mathcal{B}(\gamma)$ will be denoted by $\mathcal{N} : \mathcal{B}(\gamma) \to C^1$. It is possible write it explicitly as follows

$$\mathcal{N}(v)(r) = \int_r^R \phi^{-1}\left[\frac{1}{s^{n-1}}\int_0^s \tau^{n-1}F(\tau,v,v')d\tau\right]ds.$$

By using that $\phi^{-1}(\mathbb{R}) = (-1,1)$, one gets

$$\|\mathcal{N}(v)\|_\infty < R \quad \text{for all} \quad v \in \mathcal{B}(\gamma). \tag{5.3}$$

On the other hand, deriving $\mathcal{N}(v)$

$$\mathcal{N}'(v)(s) = -\phi^{-1}\left[\frac{1}{s^{n-1}}\int_0^s \tau^{n-1}F(\tau,v,v')d\tau\right].$$

Then, taking into account that ϕ is an odd and increasing homeomorphism, we have

$$\begin{aligned}\|\mathcal{N}'(v)\|_\infty &\leq \phi^{-1}\left(\max_{[0,R]}\left[\left(h^* + \frac{g*}{\sqrt{1-\gamma^2}}\right)\frac{n}{s^{n-1}}\int_0^s \tau^{n-1}d\tau\right]\right) = \\ &= \phi^{-1}\left(\left[h^* + \frac{g*}{\sqrt{1-\gamma^2}}\right]R\right),\end{aligned} \tag{5.4}$$

for every $v \in \mathcal{B}(\gamma)$, where we have defined

$$h^* = \max\{|H(r,\varphi^{-1}(s))f(\varphi^{-1}(s))| : \ r \in [0,R], \ s \in [-R,R]\},$$

$$g^* = \max\{|(f' \circ \varphi^{-1})(s)| : \ s \in [-R,R]\}.$$

At this point, the second assumption on the warping function f is imposed.

(A2) The absolute value of the expansion, f', along the temporal interval $I_f(R)$ is lower than $\dfrac{1}{R}$.

This is equivalent to say that $\left|(f' \circ \varphi^{-1})(s)\right| = \left|\frac{\left(f \circ \varphi^{-1}\right)'}{f \circ \varphi^{-1}}(s)\right| < \frac{1}{R}$ for all $s \in [-R, R]$, or more simply $g^* < \frac{1}{R}$. Using this hypothesis, we can take a $\gamma \in (0, 1)$ sufficiently close to 1 such that

$$R\left[h^* + \frac{g^*}{\sqrt{1-\gamma^2}}\right] \leq \phi(\gamma).$$

Introducing this inequality in (5.4),

$$\|\mathcal{N}'(v)\|_\infty \leq \gamma.$$

This last inequality, together with (5.3), implies that $\mathcal{N}(\mathcal{B}(\gamma)) \subset \mathcal{B}(\gamma)$. Since $\mathcal{B}(\gamma)$ is contractible to a point, and $\mathcal{N}$ is a continuous and compact operator, the Schauder Point Fixed theorem applies, leading to the following result.

Proposition 5.3 *Assume* $(A1)$ *and* $(A2)$. *Then, problem* (3.2) *has at least one radially symmetric solution.*

Note that the solution given in previous result is not necessarily positive. To assure the positivity of the solutions we need an additional condition.

Proposition 5.4 *Assume that*

(A3) $H(t, r) \leq \frac{f'}{f}(t)$ and $f'(t) \geq 0$ for all $r \in [0, R]$ and $t \in I_f(R)$.

Then, any v *not identically zero solution of* (5.1) *verifies* $v > 0$ *on* $[0, R)$.

Proof. First, note that condition (A3) implies that F is nonnegative in $[0, R] \times [-R, R] \times [0, \gamma]$. From the equality

$$v'(r) = -\phi^{-1}\left[\frac{n}{r^{n-1}} \int_0^r \tau^{n-1} F(\tau, v, v') d\tau\right], \tag{5.5}$$

and taking into account that ϕ is an odd increasing diffeomorphism, we deduce that v is decreasing. Since $v(R) = 0$, we have $v \geq 0$ on $[0, R]$. v is a solution identically zero if and only if $H(0, r) = \frac{f'}{f}(0)$ for all $r \in [0, R]$. If v does not vanished identically, then $v(0) > 0$ and there exists $r_0 \in (0, R)$ where $v'(r_0) < 0$. Then, from (5.5) we get

$$\int_0^{r_0} \tau^{n-1} F(\tau, v, v') d\tau > 0.$$

Since $F(\tau, v, v') \geq 0$ for all $\tau \in [0, R]$, this implies

$$\int_0^r \tau^{n-1} F(\tau, v, v') d\tau > 0 \qquad \text{for all } r > r_0.$$

From (5.5), we get $v'(r) < 0$ on $[r_0, R]$ and therefore, we conclude that $v > 0$ on $[0, R)$. □

Summarizing in a more geometric perspective, we can state the following result.

Theorem 5.5 *Let $I \times_f \mathbb{R}^n$ be a Robertson-Walker spacetime and $B = B(R)$ the Euclidean ball centred in $0 \in \mathbb{R}^n$ with radius R. Assume that $I_f(R) \subset I$, where*

$$I_f(R) = \left[-\int_{-R}^{0} f(\varphi^{-1}(s))ds,\ \int_0^R f(\varphi^{-1}(s))ds \right].$$

Let $H : I \times \overline{B} \to \mathbb{R}$ be a smooth radially symmetric function. Suppose that the following inequality holds

$$\max_{I_f(R)} |f'| < \frac{1}{R}.$$

Then, there exists at least one spacelike graph defined on $\overline{B}$ with mean curvature function H, supported on the slice $\{t = 0\}$. Moreover, if

$$H(t, r) \leq \frac{f'}{f}(t) \quad \text{and} \quad f'(t) \geq 0 \qquad \text{for all} \quad r \in [0, R] \ \text{ and } \ t \in I_f(R).$$

then the graph is either a slice or is above of $\{t = 0\}$ and only touches it on the boundary $\{0\} \times \partial B$.

As a final remark, observe that in the previous result, if (A3) is assumed from the beginning, every eventual solution is a priori positive, therefore condition (A2) can be weakened to $\mathbb{R}^+ \cap I_f(R) = \left[0\ ,\ \int_0^R f(\varphi^{-1}(s))ds\right]$. Thus, Theorem 1.1 is a direct consequence of Theorems 4.1 and 5.5.

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