Existence and nonexistence results for a singular boundary value problem arising in the theory of epitaxial growth

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Abstract

The existence of stationary radial solutions to a partial differential equation arising in the theory of epitaxial growth is studied. It turns out that the existence or not of such solutions depends on the size of a parameter that plays the role of the velocity at which mass is introduced into the system. For small values of this parameter we prove existence of solutions to this boundary value problem. For large values of the same parameter we prove nonexistence of solutions. We also provide rigorous bounds for the values of this parameter which separate existence from

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nonexistence. The proofs come as a combination of several differential inequalities and the method of upper and lower functions applied to an associated two-point boundary value problem.

1 Introduction

Epitaxial growth is a technique used in the semiconductor industry for the growth of thin films [1]. It is employed for growing crystal structures by means of the deposition of a given material under high vacuum conditions. In epitaxial growth it is quite usual finding a mounded structure generated along the surface evolution rather than a flat surface [7]. A number of models has been considered in order to explain this phenomenology. These models have been usually introduced either as a discrete probabilistic system or as a differential equation [1]. In this work we are interested in this second type of modeling approach.

Here we will focus on the rigorous mathematical analysis of ordinary differential equations related to models which have been introduced in the context of epitaxial growth. The mathematical description of epitaxial growth uses the function

$$\phi : \Omega \subset \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}, \tag{1.1}$$

which describes the height of the growing interface in the spatial point $x \in \Omega \subset \mathbb{R}^2$ at time $t \in \mathbb{R}^+$. Although this theoretical framework can be extended to any spatial dimension N, we will concentrate here on the physical situation $N = 2$. A basic modeling assumption is of course that ϕ is a well defined function, a fact that holds in a reasonably large number of cases [1]. The macroscopic description of the growing interface is given by a partial differential equation for ϕ which is usually postulated using phenomenological and symmetry arguments [1, 8]. There are many such equations in the theory of non-equilibrium surface growth. We will focus on the following one

$$\partial_t \phi = \kappa_1 \det\left(D^2\phi\right) - \kappa_2 \Delta^2 \phi + \xi(x,t). \tag{1.2}$$

This partial differential equation has been considered in the physical literature as a possible continuum description of epitaxial growth [3, 4]. At the mathematical level we can consider it a parabolic problem whose evolution is dictated by a Monge-Ampère term stabilized by a fourth order viscosity.

In this work we are concerned with the stationary version of (1.2), which reads

$$\begin{cases} \Delta^2\phi = \det\left(D^2\phi\right) + \lambda F, & \text{on } \Omega \subset \mathbb{R}^2, \\ \text{boundary} \quad \text{conditions}, & \end{cases} \tag{1.3}$$

after getting rid of the equation constant parameters by means of a trivial re-scaling of field and coordinates. Our last assumption is that the forcing term $F(x)$ is time independent. The constant λ is a measure of the speed at which new particles are deposited, and for physical reasons we assume $\lambda \geq 0$ and $F(x) \geq 0$. We will devote our efforts to rigorously clarify the existence and nonexistence of solutions to this elliptic problem when set on a radially symmetric domain.

2 Radial Solutions

As already outlined in the previous section, our goal will be determining the existence of radially symmetric solutions of boundary value problem (1.3). We set the problem on the unit disk. In a previous work we have numerically found, for a constant forcing term $F(x) \equiv 1$, the existence of two solutions for small values of λ and nonexistence of any solution for sufficiently large values of λ [5]. In this paper we will still assume the forcing term is constant, and we will find rigorous bounds for the values of λ separating existence from nonexistence. We note that we could also build the existence/nonexistence theory for arbitrary forcing terms $F(x)$ with radial symmetry employing the same methods. However, in this case, of course, we would loose accuracy on the estimates of the bounds of λ.

In more precise terms, we look for solutions of the form $\phi(x) = \tilde{\phi}(|x|)$ on Ω. If $r = |x|$, by means of a direct substitution we find

$$\frac{1}{r}\left\{r\left[\frac{1}{r}\left(r\tilde{\phi}'\right)'\right]'\right\}' = \frac{1}{r}\,\tilde{\phi}'\tilde{\phi}'' + \lambda F(r), \tag{2.1}$$

where $' = \frac{d}{dr}$. In the first case we consider homogeneous Dirichlet boundary conditions for problem (1.3). This translates to the conditions $\tilde{\phi}'(0) = 0$, $\tilde{\phi}(1) = 0$, $\tilde{\phi}'(1) = 0$, and $\lim_{r\to 0} r\tilde{\phi}'''(r) = 0$; the first one imposes the existence of an extremum at the origin and the second and third ones are the actual boundary conditions. The fourth boundary condition is technical and imposes higher regularity at the origin. If this condition were removed this would open the possibility of constructing functions $\tilde{\phi}(r)$ whose second derivative had a peak at the origin. This would in turn imply the presence of a measure at the origin when calculating the fourth derivative of such an $\tilde{\phi}(r)$, so this type of function cannot be considered as an acceptable solution of (2.1) whenever $F(r)$ is a function. Throughout this work we will assume $F \equiv 1$ as already mentioned.

Integrating once equation (2.1) against the measure $r\,dr$ and using boundary condition $\lim_{r\to 0} r\tilde{\phi}'''(r) = 0$ yields

$$r\left[\frac{1}{r}\left(r\tilde{\phi}'\right)'\right]' = \frac{1}{2}(\tilde{\phi}')^2 + \frac{1}{2}\lambda r^2. \tag{2.2}$$

By changing variables $w = r\tilde{\phi}'$ we find the equation

$$w'' - \frac{1}{r}\,w' = \frac{1}{2}\,\frac{w^2}{r^2} + \frac{1}{2}\,\lambda\,r^2, \tag{2.3}$$

subject to the conditions $w'(0) = 0$ and $w(1) = 0$. This is the equation that has been numerically integrated in [5]; now we proceed to summarize these results. We observed that for $\lambda = 0$ there are one trivial and one non-trivial solution. For $0 < \lambda < \lambda_0$ there are two non-trivial solutions which approach each other for increasing λ. For $\lambda > \lambda_0$ no more solutions were numerically found. The critical value of λ was numerically estimated to be $\lambda_0 \approx 169$. These results suggest no solutions exist for large enough λ.

In the second case we reconsider problem (1.3) but this time subject to homogeneous Navier boundary conditions. For the radial case we start as above, with equation (2.3), but now with the condition $w(1) = w'(1)$ arbitrary instead of $w(1) = 0$, what corresponds to homogeneous Navier boundary conditions. The results for this second case were qualitatively similar to those of the first case but quantitatively rather different [5]. In particular, we numerically observed that for $\lambda = 0$ there are one trivial and one non-trivial solutions. For $0 < \lambda < \lambda_0$ there are two non-trivial solutions which approach each other for increasing λ. For $\lambda > \lambda_0$ no more solutions were numerically found. The critical value of λ was numerically estimated to be $\lambda_0 \approx 11.34$.

3 Statement of the Boundary Value Problem

The rest of this work is devoted to rigorously justify the numerical facts summarized in the previous section. We will show how to prove the existence of solutions for small λ and nonexistence for large λ, as well as provide rigorous bounds for the values of this bifurcation parameter which separate existence from nonexistence. Our first step will be recasting the differential equation under study into a form more suitable for its mathematical analysis.

Let us consider again equation (2.3), which we write now in the following form

$$r^2 w'' - r w' = \frac{\lambda}{2} r^4 + \frac{w^2}{2} \qquad \text{for } r \in (0, 1] \tag{3.1}$$

with $\lambda \geq 0$, together with the boundary conditions

$$w'(0) = 0, \tag{3.2}$$

$$w(1) = 0, \tag{3.3}$$

corresponding to Dirichlet boundary conditions, respectively the conditions (3.2) and

$$w(1) = w'(1), \tag{3.4}$$

corresponding to Navier boundary conditions. The transformation

$$t = \frac{r^2}{2}, \qquad u(t) = w(r) \tag{3.5}$$

leads to the equation

$$u'' = \frac{u^2}{8t^2} + \frac{\lambda}{2} \qquad \text{for } t \in (0, 1/2] \tag{3.6}$$

with the conditions

$$\lim_{t \to 0_+} \sqrt{t}\, u'(t) = 0, \tag{3.7}$$

$$u(1/2) = 0, \tag{3.8}$$

and respectively

$$u(1/2) = u'(1/2). \tag{3.9}$$

Obviously, using the transformation (3.5) one can easily check that problems (3.1)–(3.3) (resp. (3.1), (3.2), (3.4)) and (3.6)–(3.8) (resp. (3.6), (3.7), (3.9)) are equivalent. By a solution to (3.6) we understand a function u belonging to the space $C^2_{loc}\big((0,1/2];\mathbb{R}\big)$ of functions $u : (0,1/2] \to \mathbb{R}$ such that $u \in C^2\big([a,b];\mathbb{R}\big)$ for every compact interval $[a,b] \subset (0,1/2]$, and satisfying (3.6).

4 Main Results

We start with the case of the Dirichlet conditions.

Theorem 4.1. *There exists a real number $\lambda_0 > 0$ such that problem (3.1)–(3.3) is solvable for every $\lambda \in [0,\lambda_0)$ and there is no solution to problem (3.1)–(3.3) if $\lambda > \lambda_0$. Furthermore, every solution w to (3.1)–(3.3) satisfies*

$$w(r) \le 0 \qquad \textit{for } r \in (0,1], \qquad \lim_{r\to 0_+} w(r) = 0. \tag{4.1}$$

The information contained in this Theorem is complementary to the existence and multiplicity results obtained in section 2 by means of variational arguments. On the other hand, the following Proposition gives us a localization of the critical value of λ for problem (3.1)–(3.3).

Proposition 4.1. *The number λ_0 from Theorem 4.1 admits the estimates*

$$144 \le \lambda_0 \le 307.$$

The situation is analogous for the Navier conditions.

Theorem 4.2. *There exists a real number $\lambda_0 > 0$ such that problem (3.1), (3.2), (3.4) is solvable for every $\lambda \in [0,\lambda_0)$ and there is no solution to problem (3.1), (3.2), (3.4) if $\lambda > \lambda_0$. Furthermore, every solution w to (3.1), (3.2), (3.4) satisfies (4.1).*

The following Proposition gives us a localization of the critical value of λ for problem (3.1), (3.2), (3.4).

Proposition 4.2. *The number λ_0 from Theorem 4.2 admits the estimates*

$$9 \le \lambda_0 \le \frac{128}{11} = 11.\overline{63}.$$

By comparing these estimates with the numerical results, one observes that in both cases the rigorous bounds capture the order of magnitude of the critical value of the parameter. Estimates for the Navier problem are more accurate, and in this case the upper bound is rather precise.

The rest of the paper is devoted to prove the latter results by means of a combination of techniques for differential inequalities and the method of upper and lower functions.

5 Auxiliary Propositions

In what follows, we establish some basic properties of a function $u \in C^2_{loc}\big((0,1/2];\mathbb{R}\big)$ satisfying the inequality

$$u''(t) \geq \frac{u^2(t)}{8t^2} + \frac{\lambda}{2} \qquad \text{for } t \in (0,1/2]. \tag{5.1}$$

Lemma 5.1. *Let $u \in C^2_{loc}\big((0,1/2];\mathbb{R}\big)$ satisfy (3.7) and (5.1). Then*

$$\lim_{t\to 0_+} u(t) = 0. \tag{5.2}$$

Proof. From (5.1) it follows that u' is a non-decreasing function. Therefore, there exists $t_0 \in (0,1/2]$ such that either

$$u'(t) \geq 0 \qquad \text{for } t \in (0,t_0]$$

or

$$u'(t) < 0 \qquad \text{for } t \in (0,t_0].$$

In both cases, the function u is monotone on the interval $(0,t_0]$. Therefore, there exists the (finite or infinite) limit $u(0_+)$. Assume that (5.2) does not hold. Then there exist $t_1 \in (0,1/2]$ and $\delta > 0$ such that

$$u^2(t) \geq \delta \qquad \text{for } t \in (0,t_1].$$

Thus the integration of (5.1) from t to t_1 yields

$$u'(t_1) - u'(t) \geq \frac{\delta}{8}\left(\frac{1}{t} - \frac{1}{t_1}\right) \qquad \text{for } t \in (0,t_1]. \tag{5.3}$$

Multiplying both sides of (5.3) by $\sqrt{t}$ and applying a limit as t tends to zero we obtain

$$-\lim_{t\to 0_+} \sqrt{t}\, u'(t) \geq \frac{\delta}{8} \lim_{t\to 0_+} \frac{1}{\sqrt{t}} = +\infty,$$

which contradicts (3.7). □

Remark 5.1. Note that every function $u \in C^2_{loc}\big((0,1/2];\mathbb{R}\big)$ satisfying (3.7) and (5.1) satisfies also

$$\lim_{t\to 0_+} \frac{u(t)}{\sqrt{t}} = 0. \tag{5.4}$$

Indeed, the equality (5.4) follows from Lemma 5.1 and de l'Hospital's rule. On the other hand, if (5.4) holds then (5.2) is fulfilled.

Lemma 5.2. *Let* $u \in C^2_{loc}\big((0,1/2];\mathbb{R}\big)$ *satisfy* (3.8), (5.2), *and*

$$u''(t) \geq 0 \qquad \text{for } t \in (0,1/2]. \tag{5.5}$$

Then

$$u(t) \leq 0 \qquad \text{for } t \in (0,1/2]. \tag{5.6}$$

Proof. It is enough to observe that a convex function defined on a compact interval always attains its global maximum in one of the extremes. □

Lemma 5.3. *Let* $u \in C^2_{loc}\big((0,1/2];\mathbb{R}\big)$ *satisfy* (3.9), (5.2), *and* (5.5). *Then* (5.6) *holds.*

Proof. First we will show that

$$u(1/2) \leq 0. \tag{5.7}$$

Assume on the contrary that (5.7) does not hold. Then, according to (5.2), there exists $t_0 \in (0,1/2)$ such that

$$u(t_0) < u(1/2)(1/2 + t_0). \tag{5.8}$$

Obviously, (5.5) yields

$$u'(1/2) \geq \frac{u(1/2) - u(t_0)}{1/2 - t_0},$$

whence, in view of (3.9), we obtain

$$u(1/2) \geq \frac{u(1/2) - u(t_0)}{1/2 - t_0},$$

However, the latter inequality contradicts (5.8). Therefore, the inequality (5.7) holds.

Finally, (3.9), (5.5), and (5.7) imply

$$u'(t) \leq 0 \qquad \text{for } t \in (0,1/2],$$

which together with (5.2) results in (5.6). □

Lemma 5.4. *Let* $u \in C^2_{loc}\big((0,1/2];\mathbb{R}\big)$ *satisfy* (5.4)–(5.6). *Then* (3.7) *is fulfilled.*

Proof. In view of Remark 5.1 the equality (5.2) holds. Therefore we have

$$u(t) = \int_0^t u'(s)ds \qquad \text{for } t \in (0,1/2]. \tag{5.9}$$

According to (5.5), from (5.9) it follows that

$$u(t) \leq tu'(t) \qquad \text{for } t \in (0,1/2]. \tag{5.10}$$

Moreover, (5.2), (5.5), and (5.6) imply the existence of $t_0 \in (0,1/2]$ such that $u'(t) \leq 0$ for $t \in (0,t_0]$ which, together with (5.10), results in

$$\frac{u(t)}{\sqrt{t}} \leq \sqrt{t}\, u'(t) \leq 0 \qquad \text{for } t \in (0,t_0]. \tag{5.11}$$

Now we get (3.7) from (5.11) using (5.4). □

Lemma 5.5. *The problems* (3.6)–(3.8) *(resp.* (3.6), (3.7), (3.9)*) and* (3.6), (3.8), (5.4) *(resp.* (3.6), (3.9), (5.4)*) are equivalent.*

Proof. Let u be a solution to (3.6)–(3.8) (resp. (3.6), (3.7), (3.9)). Then, according to Remark 5.1, u is also a solution to (3.6), (3.8), (5.4) (resp. (3.6), (3.9), (5.4)).

On the other hand, let u be a solution to (3.6), (3.8), (5.4) (resp. (3.6), (3.9), (5.4)). Then, in view of (3.6), Remark 5.1 and Lemma 5.2 (resp. Lemma 5.3), the inequalities (5.5) and (5.6) hold. Thus we get that (3.7) is fulfilled from Lemma 5.4. □

Remark 5.2. It follows from Lemmas 5.1–5.3 that every solution u to (3.6)–(3.8) (resp. (3.6), (3.7), (3.9)) satisfies (5.2) and (5.6).

Lemma 5.6. *Let* $u \in C^2_{loc}\big((0,1/2];\mathbb{R}\big)$ *satisfy* (5.4)*. Then for every* $\mu \in [0,1)$ *we have*

$$\lim_{t\to 0_+} t^{1-\mu}\int_t^{1/2}\frac{u^2(s)}{s^2}ds = 0. \tag{5.12}$$

Proof. Put

$$f(t) = \frac{u^2(t)}{t} \qquad \text{for } t\in(0,1/2], \qquad g(t) = \frac{1}{t^\mu} \qquad \text{for } t\in(0,1/2].$$

Then, obviously, $g \in L\big([0,1/2];\mathbb{R}\big)$ and in view of (5.4), $f \in L^{+\infty}\big([0,1/2];\mathbb{R}\big)$. Consequently, $fg \in L\big([0,1/2];\mathbb{R}\big)$. Therefore, for every $n\in\mathbb{N}$ there exists $t_n \in (0,1/2]$ such that

$$\int_0^{t_n}\frac{u^2(s)}{s^{1+\mu}}ds \le \frac{1}{n}. \tag{5.13}$$

On the other hand, for any $n\in\mathbb{N}$, we have

$$\begin{aligned} 0 \le t^{1-\mu}\int_t^{1/2}\frac{u^2(s)}{s^2}ds &\le t^{1-\mu}\int_{t_n}^{1/2}\frac{u^2(s)}{s^2}ds + t^{1-\mu}\int_t^{t_n}\frac{u^2(s)}{s^2}ds \le \\ &\le t^{1-\mu}\int_{t_n}^{1/2}\frac{u^2(s)}{s^2}ds + \int_t^{t_n}\frac{s^{1-\mu}u^2(s)}{s^2}ds \le \\ &\le t^{1-\mu}\int_{t_n}^{1/2}\frac{u^2(s)}{s^2}ds + \int_t^{t_n}\frac{u^2(s)}{s^{1+\mu}}ds \qquad \text{for } t\in(0,t_n], \end{aligned} \tag{5.14}$$

and so, in view of (5.13),

$$0 \le \limsup_{t\to 0_+} t^{1-\mu}\int_t^{1/2}\frac{u^2(s)}{s^2}ds \le \int_0^{t_n}\frac{u^2(s)}{s^{1+\mu}}ds \le \frac{1}{n} \qquad \text{for } n\in\mathbb{N}.$$

Consequently, (5.12) holds. □

Lemma 5.7. *Let $u \in C^2_{loc}\big((0,1/2];\mathbb{R}\big)$ satisfy* (3.6) *and* (5.4). *Then*

$$u(t) = -\Bigg[(1/2-t)\int_0^t \frac{u^2(s)}{4s}ds + t\int_t^{1/2} \frac{u^2(s)}{4s^2}(1/2-s)ds \\ + \frac{\lambda}{4}t(1/2-t) - 2tu(1/2)\Bigg] \quad \text{for } t \in (0,1/2], \tag{5.15}$$

$$\lim_{t\to 0_+} \frac{u(t)}{t^\mu} = 0 \qquad \text{for } \mu \in [0,1), \tag{5.16}$$

and there exists the finite limit

$$\lim_{t\to 0_+} \frac{|u(t)|}{t} < +\infty. \tag{5.17}$$

Proof. For any $\tau \in (0,1/2)$ we have a representation

$$u(t) = -\frac{1}{1/2-\tau}\Bigg[(1/2-t)\int_\tau^t \frac{u^2(s)}{8s^2}(s-\tau)ds + (t-\tau)\int_t^{1/2} \frac{u^2(s)}{8s^2}(1/2-s)ds \\ + \frac{\lambda}{4}(t-\tau)(1/2-t)(1/2-\tau) - u(\tau)(1/2-t) - u(1/2)(t-\tau)\Bigg] \quad \text{for } t \in [\tau,1/2]. \tag{5.18}$$

According to (5.4) and Lemma 5.6 we have

$$\lim_{\tau\to 0_+}\int_\tau^t \frac{u^2(s)}{s^2}(s-\tau)ds = \lim_{\tau\to 0_+}\int_\tau^t \frac{u^2(s)}{s}ds - \lim_{\tau\to 0_+}\tau\int_\tau^t \frac{u^2(s)}{s^2}ds = \int_0^t \frac{u^2(s)}{s}ds \tag{5.19}$$

and, in view of Remark 5.1, (5.2) holds. Thus, on account of (5.19) and (5.2), from (5.18) it follows that (5.15) holds.

Now if we multiply both sides of (5.15) by $t^{-\mu}$ and apply the limit as t tends to zero, with respect to Lemma 5.6 and (5.4), we obtain that (5.16) holds true.

Finally, put

$$f(t) = \frac{u^2(t)}{t^{1+\mu}} \qquad \text{for } t \in (0,1/2], \qquad g(t) = \frac{1}{t^{1-\mu}} \qquad \text{for } t \in (0,1/2].$$

Then, in view of (5.16), we have $f \in L^{+\infty}\big([0,1/2];\mathbb{R}\big)$ and $g \in L\big([0,1/2];\mathbb{R}\big)$ provided $\mu \in (0,1)$. Consequently, $fg \in L\big([0,1/2];\mathbb{R}\big)$, i.e.,

$$\int_0^{1/2} \frac{u^2(s)}{s^2}ds < +\infty. \tag{5.20}$$

Now from (5.15), in view of (5.4) and (5.20), we get

$$\lim_{t\to 0_+} \frac{|u(t)|}{t} = \left|\int_0^{1/2} \frac{u^2(s)}{4s^2}(1/2-s)ds + \frac{\lambda}{8} - 2u(1/2)\right|,$$

i.e., (5.17) holds. □

6 Upper and Lower Functions

First we will recall the notion of lower and upper functions to the general equation

$$u'' = h(t, u, u'), \tag{6.1}$$

where $h \in \mathrm{Car}\big([a,b] \times \mathbb{R}^2; \mathbb{R}\big)$ is a Carathéodory function.

Definition 6.1. A continuous function $\gamma : [a,b] \to \mathbb{R}$ is said to be a lower (upper) function to (6.1) if $\gamma \in AC^1_{loc}\big([a,b] \setminus \{t_1, \dots, t_n\}; \mathbb{R}\big)$, where $a < t_1 < \cdots < t_n < b$, there exist finite limits $\gamma'(t_i+)$, $\gamma'(t_i-)$ $(i = 1, \dots, n)$,

$$\gamma'(t_i-) < \gamma'(t_i+) \qquad \Big(\gamma'(t_i-) > \gamma'(t_i+)\Big) \qquad (i = 1, \dots, n),$$

and the inequality

$$\gamma''(t) \geq h(t, \gamma(t), \gamma'(t)) \qquad \Big(\gamma''(t) \leq h(t, \gamma(t), \gamma'(t))\Big) \qquad \text{for a. e. } t \in [a,b]$$

holds.

The following two lemmas deal with the existence of a solution to (6.1) satisfying the boundary conditions

$$u(a) = c_1, \qquad u(b) = c_2, \tag{6.2}$$

and

$$u(a) = c_1, \qquad u(b) = u'(b). \tag{6.3}$$

The first one is a simple modification of the Scorza Dragoni theorem and its proof can be found in [6] (see also the more recent monograph [2]).

Lemma 6.1. *Let α and β be, respectively, lower and upper functions to* (6.1) *such that*

$$\alpha(t) \leq \beta(t) \qquad \text{for } t \in [a,b] \tag{6.4}$$

and

$$|h(t, x, y)| \leq q(t) \qquad \text{for a. e. } t \in [a,b], \quad \alpha(t) \leq x \leq \beta(t), \quad y \in \mathbb{R}, \tag{6.5}$$

where $q \in L\big([a,b]; \mathbb{R}_+\big)$. Then, for every $c_1 \in [\alpha(a), \beta(a)]$ and $c_2 \in [\alpha(b), \beta(b)]$, the problem (6.1), (6.2) *has a solution $u \in AC^1\big([a,b]; \mathbb{R}\big)$ satisfying the condition*

$$\alpha(t) \leq u(t) \leq \beta(t) \qquad \text{for } t \in [a,b]. \tag{6.6}$$

Lemma 6.2. *Let α and β be, respectively, lower and upper functions to* (6.1) *satisfying* (6.4) *and*

$$\alpha(b) \geq \alpha'(b), \qquad \beta(b) \leq \beta'(b). \tag{6.7}$$

Let, moreover, (6.5) *is fulfilled where $q \in L\big([a,b]; \mathbb{R}_+\big)$. Then, for every $c_1 \in [\alpha(a), \beta(a)]$, the problem* (6.1), (6.3) *has a solution $u \in AC^1\big([a,b]; \mathbb{R}\big)$ satisfying* (6.6).

Proof. Let $c_1 \in [\alpha(a), \beta(a)]$ be arbitrary but fixed. According to Lemma 6.1 there exists a solution u_1 to the equation (6.1) satisfying

$$u_1(a) = c_1, \qquad u_1(b) = \alpha(b), \tag{6.8}$$
$$\alpha(t) \le u_1(t) \le \beta(t) \qquad \text{for } t \in [a, b]. \tag{6.9}$$

On account of (6.7)–(6.9) we have

$$u_1(b) \ge \alpha(b) \ge \alpha'(b) \ge u_1'(b).$$

Furthermore, u_1 can be considered as a lower function to (6.1) and so, according to Lemma 6.1, there exists a solution v_1 to the equation (6.1) satisfying

$$v_1(a) = c_1, \qquad v_1(b) = \beta(b), \tag{6.10}$$
$$u_1(t) \le v_1(t) \le \beta(t) \qquad \text{for } t \in [a, b]. \tag{6.11}$$

On account of (6.7), (6.10), and (6.11) we have

$$v_1(b) \le \beta(b)\beta'(b) \le v_1'(b).$$

Now we will construct a sequences of solutions to (6.1) $(u_n)_{n=1}^{+\infty}$ and $(v_n)_{n=1}^{+\infty}$ in the following way: Having defined solutions u_n and v_n for some $n \in \mathbb{N}$ with

$$u_n(a) = c_1, \qquad v_n(a) = c_1, \qquad u_n(b) \ge u_n'(b), \qquad v_n(b) \le v_n'(b), \tag{6.12}$$
$$\alpha(t) \le u_n(t) \le v_n(t) \le \beta(t) \qquad \text{for } t \in [a, b], \tag{6.13}$$

we can consider them as a lower and an upper function, respectively, to (6.1). According to Lemma 6.1, there exists a solution u to (6.1) satisfying

$$u(a) = c_1, \qquad u(b) = \frac{u_n(b) + v_n(b)}{2}, \tag{6.14}$$
$$u_n(t) \le u(t) \le v_n(t) \qquad \text{for } t \in [a, b]. \tag{6.15}$$

Obviously, either

$$u(b) \le u'(b) \tag{6.16}$$

or

$$u(b) > u'(b). \tag{6.17}$$

If (6.16) holds we put

$$u_{n+1}(t) = u_n(t) \qquad \text{for } t \in [a, b], \qquad v_{n+1}(t) = u(t) \qquad \text{for } t \in [a, b]. \tag{6.18}$$

If (6.17) holds we put

$$u_{n+1}(t) = u(t) \qquad \text{for } t \in [a, b], \qquad v_{n+1}(t) = v_n(t) \qquad \text{for } t \in [a, b]. \tag{6.19}$$

Consequently, in view of (6.12)–(6.19), u_{n+1} and v_{n+1} are solutions to (6.1) satisfying

$$u_{n+1}(a) = c_1, \qquad v_{n+1}(a) = c_1, \qquad u_{n+1}(b) \geq u'_{n+1}(b), \qquad v_{n+1}(b) \leq v'_{n+1}(b),$$
$$\alpha(t) \leq u_n(t) \leq u_{n+1}(t) \leq v_{n+1}(t) \leq v_n(t) \leq \beta(t) \qquad \text{for } t \in [a,b]. \tag{6.20}$$

Obviously, in view of (6.14) and (6.18), resp. (6.19), and (6.20)

$$\lim_{n\to+\infty} u_n(b) = \lim_{n\to+\infty} v_n(b). \tag{6.21}$$

Moreover, in view of (6.12), for any $n \in \mathbb{N}$, there exist $t_n \in [a,b]$ and $s_n \in [a,b]$ such that

$$u'_n(t_n) = \frac{u_n(b) - c_1}{b-a}, \qquad v'_n(s_n) = \frac{v_n(b) - c_1}{b-a}.$$

Consequently, on account of (6.20), we have

$$|u'_n(t_n)| \leq M, \qquad |v'_n(s_n)| \leq M \qquad \text{for } n \in \mathbb{N}, \tag{6.22}$$

where

$$M = \frac{|\alpha(b)| + |\beta(b)| + |c_1|}{b-a}.$$

Since u_n and v_n are solutions to (6.1), with respect to (6.5), (6.20), and (6.22) we obtain

$$|u'_n(t)| \leq M + \int_a^b q(s)ds \qquad \text{for } t \in [a,b], \quad n \in \mathbb{N}, \tag{6.23}$$
$$|v'_n(t)| \leq M + \int_a^b q(s)ds \qquad \text{for } t \in [a,b], \quad n \in \mathbb{N}. \tag{6.24}$$

Thus, on account of (6.1), (6.20), (6.23), and (6.24), it follows that the sequences $(u_n)_{n=1}^{+\infty}$, $(u'_n)_{n=1}^{+\infty}$ and $(v_n)_{n=1}^{+\infty}$, $(v'_n)_{n=1}^{+\infty}$ are uniformly bounded and equicontinuous. Therefore, without loss of generality we can assume that there exists functions $u_0, v_0 \in C^2\big([a,b];\mathbb{R}\big)$ such that

$$u_0^{(j)}(t) = \lim_{n\to+\infty} u_n^{(j)}(t), \qquad v_0^{(j)}(t) = \lim_{n\to+\infty} v_n^{(j)}(t) \qquad \text{uniformly on } [a,b] \quad (j = 0,1).$$

By a standard way it can be shown that $u_0, v_0 \in AC^1\big([a,b];\mathbb{R}\big)$, and they are also solutions to (6.1). Moreover, from (6.12), (6.20), and (6.21) we have

$$\alpha(t) \leq u_0(t) \leq v_0(t) \leq \beta(t) \qquad \text{for } t \in [a,b], \tag{6.25}$$
$$u_0(a) = c_1, \qquad v_0(a) = c_1, \qquad u_0(b) \geq u'_0(b), \qquad v_0(b) \leq v'_0(b), \tag{6.26}$$
$$u_0(b) = v_0(b). \tag{6.27}$$

On the other hand, from (6.25) and (6.27) it follows that

$$u'_0(b) \geq v'_0(b). \tag{6.28}$$

Now (6.25)–(6.28) imply that $u \overset{def}{\equiv} u_0$ is a solution to (6.1), (6.3) satisfying (6.6). □

Lemma 6.3. *Let there exists a function $\alpha \in C^2_{loc}\big((0,1/2];\mathbb{R}\big)$ such that its restriction to any closed interval $[a,b] \subset (0,1/2]$ is a lower function to* (3.6) *on $[a,b]$. Let, moreover,*

$$\alpha(t) \le 0 \qquad \textit{for } t \in (0,1/2], \tag{6.29}$$

$$\lim_{t\to 0_+} \frac{|\alpha(t)|}{t} < +\infty. \tag{6.30}$$

Then there exists a solution u to (3.6), (3.8), (5.4) *with*

$$\alpha(t) \le u(t) \le 0 \qquad \textit{for } t \in (0,1/2]. \tag{6.31}$$

Proof. Note that from (6.30) it follows that

$$\lim_{t\to 0_+} \alpha(t) = 0, \qquad \lim_{t\to 0_+} \frac{\alpha(t)}{\sqrt{t}} = 0, \tag{6.32}$$

$$\int_0^{1/2} \frac{\alpha^2(s)}{s^2} ds < +\infty. \tag{6.33}$$

Further, from (6.32) it follows that

$$\alpha^* = \sup\big\{|\alpha(t)| : t \in (0,1/2]\big\} < +\infty. \tag{6.34}$$

Let $t_n \in (0,1/2)$ for $n \in \mathbb{N}$ be such that

$$t_{n+1} < t_n \qquad \text{for } n \in \mathbb{N}, \qquad \lim_{n\to+\infty} t_n = 0. \tag{6.35}$$

Obviously, for every $n \in \mathbb{N}$, $\beta \equiv 0$ is an upper function to (3.6) on the interval $[t_n, 1/2]$ satisfying

$$\beta(t_n) = 0, \qquad \beta(1/2) = 0.$$

Therefore, according to Lemma 6.1, in view of (6.29), for every $n \in \mathbb{N}$ there exists a solution u_n to (3.6) on the interval $[t_n, 1/2]$ satisfying

$$u_n(t_n) = 0, \qquad u_n(1/2) = 0, \tag{6.36}$$

$$\alpha(t) \le u_n(t) \le 0 \qquad \text{for } t \in [t_n, 1/2]. \tag{6.37}$$

Moreover, for every $n \in \mathbb{N}$ there exists $s_n \in (t_n, 1/2)$ such that

$$u_n'(s_n) = 0. \tag{6.38}$$

Therefore, integrating (3.6) from s_n to t, on account of (6.33), (6.37), and (6.38), we obtain

$$|u_n'(t)| = \left| \int_{s_n}^t \frac{u_n^2(s)}{8s^2} ds + \frac{\lambda}{2}(t - s_n) \right| \le \int_0^{1/2} \frac{\alpha^2(s)}{8s^2} ds + \frac{\lambda}{4} \qquad \text{for } t \in [t_n, 1/2], \; n \in \mathbb{N}. \tag{6.39}$$

Moreover, from (3.6) and (6.37) we get

$$|u_n''(t)| \le \frac{\alpha^2(t)}{8t^2} + \frac{\lambda}{2} \qquad \text{for } t \in [t_n, 1/2], \quad n \in \mathbb{N}. \tag{6.40}$$

Thus, on account of (6.33)–(6.35), (6.37), (6.39), and (6.40), we have that the sequences $(u_n)_{n=1}^{+\infty}$, $(u_n')_{n=1}^{+\infty}$ are uniformly bounded and equicontinuous on every compact subinterval of $(0, 1/2]$. Therefore, according to the Arzelà–Ascoli theorem, there exists $u_0 \in C^1_{loc}\big((0, 1/2]; \mathbb{R}\big)$ such that

$$\lim_{n\to+\infty} u_n^{(j)}(t) = u_0^{(j)}(t) \quad \text{uniformly on every compact interval of } (0, 1/2] \quad (j = 0, 1).$$

Moreover, since u_n are solutions to (3.6), $u_0 \in C^2_{loc}\big((0, 1/2]; \mathbb{R}\big)$ and it is also a solution to (3.6). Furthermore, from (6.32), (6.36), and (6.37), it follows that

$$\alpha(t) \le u_0(t) \le 0 \qquad \text{for } t \in (0, 1/2], \qquad u_0(1/2) = 0, \qquad \lim_{t\to 0_+} \frac{u_0(t)}{\sqrt{t}} = 0.$$

□

Lemma 6.4. *Let the assumptions of Lemma 6.3 be fulfilled. Let, moreover,*

$$\alpha(1/2) \ge \alpha'(1/2). \tag{6.41}$$

Then there exists a solution u to (3.6), (3.9), (5.4) *satisfying* (6.31).

The proof of Lemma 6.4 is similar to that of Lemma 6.3, just Lemma 6.2 is used instead of Lemma 6.1 and the estimate of u_n' is produced using the fact that there exists $s_n \in (t_n, 1/2)$ such that

$$|u_n'(s_n)| = \frac{|u_n(1/2)|}{1/2 - t_n} \le \frac{|\alpha(1/2)|}{1/2 - t_1}.$$

7 Lemmas on Estimation of λ

Lemma 7.1. *The set of numbers $\lambda \ge 0$, for which there exists a solution to* (3.6) *satisfying* (5.4) *and* (5.6)*, is nonempty and bounded from above.*

Proof. Obviously, for $\lambda = 0$ there is a zero solution with the appropriate properties. Therefore the set is nonempty. If $\lambda = 0$ is the only element of the set, then, clearly, the set is bounded from above. Let, therefore, $\lambda > 0$ and let u be a solution to (3.6) satisfying (5.4) and (5.6). Then, according to Remark 5.1 and Lemma 5.4, we have that (3.7) and (5.2) hold.

On the other hand, from (3.6) it follows that

$$(tu'(t) - u(t))' = \frac{u^2(t)}{8t} + \frac{\lambda}{2}t \qquad \text{for } t \in (0, 1/2]. \tag{7.1}$$

Integration (7.1) from 0 to t, in view of (3.7), (5.2), and (5.4), yields

$$tu'(t) - u(t) = \int_0^t \frac{u^2(s)}{8s} ds + \frac{\lambda}{4} t^2 \qquad \text{for } t \in (0, 1/2]. \tag{7.2}$$

Put

$$v(t) = -\frac{u(t)}{t} \qquad \text{for } t \in (0, 1/2].$$

Then, on account of (5.6) and (7.2) we have

$$v'(t) = -\frac{1}{8t^2} \int_0^t v^2(s) s ds - \frac{\lambda}{4} \qquad \text{for } t \in (0, 1/2], \tag{7.3}$$

$$v(t) \geq 0 \qquad \text{for } t \in (0, 1/2]. \tag{7.4}$$

Moreover, from (7.3) it follows that v is a decreasing function and thus (7.3) and (7.4) result in

$$v'(t) \leq -\frac{v^2(t) + 4\lambda}{16} \qquad \text{for } t \in (0, 1/2]$$

whence we get

$$-\frac{v'(t)}{v^2(t) + 4\lambda} \geq \frac{1}{16} \qquad \text{for } t \in (0, 1/2]. \tag{7.5}$$

Now the integration of (7.5) from t to $1/2$ results in

$$\int_{v(1/2)}^{v(t)} \frac{dx}{x^2 + 4\lambda} = -\int_t^{1/2} \frac{v'(s) ds}{v^2(s) + 4\lambda} \geq \frac{1/2 - t}{16}.$$

Hence we get

$$\frac{\pi}{4\sqrt{\lambda}} = \int_0^{+\infty} \frac{dx}{x^2 + 4\lambda} \geq \lim_{t \to 0_+} \int_{v(1/2)}^{v(t)} \frac{dx}{x^2 + 4\lambda} \geq \frac{1}{32}. \tag{7.6}$$

Consequently, (7.6) implies $\lambda \leq 64\pi^2$. □

Lemma 7.2. *If the problem* (3.6), (3.8), (5.4) *(resp.* (3.6), (3.9), (5.4)*) is solvable for some* $\lambda_0 \geq 0$ *then it is solvable also for every* $\lambda \in [0, \lambda_0]$.

Proof. Let u be a solution to the problem (3.6), (3.8), (5.4) (resp. (3.6), (3.9), (5.4)) with $\lambda = \lambda_0$. Put $\alpha(t) = u(t)$. Then according to Remark 5.1, Lemma 5.2 (resp. Lemma 5.3), and Lemma 5.7, the function α satisfies the assumptions of Lemma 6.3 (resp. Lemma 6.4), where the equation (3.6) is considered with $\lambda \in [0, \lambda_0]$. Consequently, according to Lemma 6.3 (resp. Lemma 6.4), the problem (3.6), (3.8), (5.4) (resp. (3.6), (3.9), (5.4)) is also solvable. □

Lemma 7.3. *Let*

$$\lambda \leq 144. \tag{7.7}$$

Then there exists $\alpha \in C^2_{loc}\big((0, 1/2]; \mathbb{R}\big)$ *satisfying the assumptions of Lemma 6.3.*

Proof. Put
$$\alpha(t) = -48t\left(1-\sqrt{2t}\right) \qquad \text{for } t \in (0,1/2]. \tag{7.8}$$
Obviously, (6.29) and (6.30) hold. We will show that
$$\alpha''(t) \geq \frac{\alpha^2(t)}{8t^2} + \frac{\lambda}{2} \qquad \text{for } t \in (0,1/2]. \tag{7.9}$$
In view of (7.8) we have
$$\alpha''(t) - \frac{\alpha^2(t)}{8t^2} - 72 = \frac{72}{\sqrt{2t}}\left(1-\sqrt{2t}\right)\left(1-2\sqrt{2t}\right)^2 \geq 0 \qquad \text{for } t \in (0,1/2]$$
and thus, on account of (7.7), the inequality (7.9) is fulfilled. □

Lemma 7.4. *Let*
$$\lambda \leq 9. \tag{7.10}$$
Then there exists $\alpha \in C^2_{loc}\big((0,1/2];\mathbb{R}\big)$ satisfying the assumptions of Lemma 6.4.

Proof. Put
$$\alpha(t) = -6t\left(2-\sqrt{2t}\right) \qquad \text{for } t \in (0,1/2]. \tag{7.11}$$
Obviously, (6.29), (6.30), and (6.41) hold. We will show that (7.9) is valid. In view of (7.11) we have
$$\alpha''(t) - \frac{\alpha^2(t)}{8t^2} - \frac{9}{2} = \frac{9}{2\sqrt{2t}}\left(2-\sqrt{2t}\right)\left(1-\sqrt{2t}\right)^2 \geq 0 \qquad \text{for } t \in (0,1/2]$$
and thus, on account of (7.10), the inequality (7.9) is fulfilled. □

Lemma 7.5. *Let there exist a function $v \in C^2_{loc}\big((0,1/2];\mathbb{R}\big) \cap L^{+\infty}\big([0,1/2];\mathbb{R}\big)$ satisfying*
$$v'(t) = -\frac{1}{8t^2}\int_0^t v^2(s)s\,ds - \frac{\lambda}{4} \qquad \textit{for } t \in (0,1/2], \tag{7.12}$$
$$v(t) \geq 0 \qquad \textit{for } t \in (0,1/2], \tag{7.13}$$
$$v(1/2) = 0. \tag{7.14}$$
Then
$$\lambda \leq 384 \tag{7.15}$$
and
$$v(t) \geq 192\left(1-\sqrt{1-\frac{\lambda}{384}}\right)(1/2-t) \qquad \textit{for } t \in (0,1/2]. \tag{7.16}$$

Proof. From (7.12) it follows that

$$v'(t) \le 0 \qquad \text{for } t \in (0, 1/2]. \tag{7.17}$$

Furthermore, $v' \in C^2_{loc}\big((0,1/2];\mathbb{R}\big)$ and

$$v''(t) = \frac{1}{4t^3}\int_0^t v^2(s)sds - \frac{v^2(t)}{8t} \qquad \text{for } t \in (0, 1/2]. \tag{7.18}$$

From (7.18), on account of (7.13) and (7.17) we obtain

$$v''(t) \ge 0 \qquad \text{for } t \in (0, 1/2]. \tag{7.19}$$

Therefore, (7.12) and (7.19) yield

$$v'(t) \le v'(1/2) = -\frac{1}{2}\int_0^{1/2} v^2(s)sds - \frac{\lambda}{4} \le -\frac{\lambda}{4} \qquad \text{for } t \in (0, 1/2]. \tag{7.20}$$

Now the integration of (7.20) from t to $1/2$, with respect to (7.14), results in

$$v(t) \ge \frac{\lambda}{4}(1/2 - t) \qquad \text{for } t \in (0, 1/2]. \tag{7.21}$$

Put

$$c_1 = \frac{\lambda}{4}, \qquad c_{n+1} = \frac{c_n^2}{384} + \frac{\lambda}{4} \qquad \text{for } n \in \mathbb{N}. \tag{7.22}$$

We will show that for every $n \in \mathbb{N}$ the inequality

$$v(t) \ge c_n(1/2 - t) \qquad \text{for } t \in (0, 1/2] \tag{7.23}$$

holds. Obviously, (7.21) and (7.22) yield the validity of (7.23) for $n = 1$. Assume therefore that (7.23) holds for some $n \in \mathbb{N}$. Then from (7.20) we obtain

$$v'(t) \le -\frac{1}{2}\int_0^{1/2} v^2(s)sds - \frac{\lambda}{4} \le -\left(\frac{c_n^2}{2}\int_0^{1/2}(1/2 - s)^2 sds + \frac{\lambda}{4}\right) = -c_{n+1} \qquad \text{for } t \in (0, 1/2]. \tag{7.24}$$

The integration of (7.24) from t to $1/2$, with respect to (7.14), results in

$$v(t) \ge c_{n+1}(1/2 - t) \qquad \text{for } t \in (0, 1/2].$$

Thus (7.23) holds for every $n \in \mathbb{N}$. Moreover,

$$c_2 = \frac{c_1^2}{384} + \frac{\lambda}{4} \ge \frac{\lambda}{4} = c_1 \ge 0,$$

and, assuming $c_n \ge c_{n-1} \ge 0$ for some $n \in \mathbb{N}$, we get

$$c_{n+1} = \frac{c_n^2}{384} + \frac{\lambda}{4} \ge \frac{c_{n-1}^2}{384} + \frac{\lambda}{4} = c_n.$$

Thus $(c_n)_{n=1}^{+\infty}$ is a non-decreasing sequence of numbers which are, on account of (7.23), bounded from above. Therefore, there exists $c_0 \in \mathbb{R}$ such that

$$c_0 = \lim_{n \to +\infty} c_n,$$

and from (7.22) and (7.23) we obtain

$$c_0 = \frac{c_0^2}{384} + \frac{\lambda}{4}, \tag{7.25}$$

$$v(t) \ge c_0(1/2 - t) \qquad \text{for } t \in (0, 1/2]. \tag{7.26}$$

Now (7.25) implies (7.15) and

$$c_0 \ge 192\left(1 - \sqrt{1 - \frac{\lambda}{384}}\right)$$

and, consequently, from (7.26) we get (7.16). □

Lemma 7.6. *Let there exist a function* $v \in C^2_{loc}\big((0,1/2];\mathbb{R}\big) \cap L^{+\infty}\big([0,1/2];\mathbb{R}\big)$ *satisfying* (7.12), (7.13)*, and*

$$v(1/2) = -v'(1/2). \tag{7.27}$$

Then

$$\lambda \le \frac{128}{11}\,. \tag{7.28}$$

Proof. From (7.12) it follows that (7.17) is fulfilled. Further, $v' \in C^2_{loc}\big((0,1/2];\mathbb{R}\big)$ and (7.18) holds. From (7.18), on account of (7.13) and (7.17) we obtain (7.19). Therefore, (7.12) and (7.19) yield

$$v'(t) \le v'(1/2) = -c \tag{7.29}$$

where

$$c = \frac{1}{2}\int_0^{1/2} v^2(s)s ds + \frac{\lambda}{4}. \tag{7.30}$$

Now the integration of (7.29) from t to $1/2$, results in

$$v(1/2) - v(t) \le -c(1/2 - t) \qquad \text{for } t \in (0, 1/2],$$

whence, with respect to (7.27) and (7.29), we get

$$v(t) \ge c(3/2 - t) \qquad \text{for } t \in (0, 1/2]. \tag{7.31}$$

Now using (7.31) in (7.30) we obtain

$$c \geq \frac{c^2}{2} \int_0^{1/2} (3/2 - s)^2 s ds + \frac{\lambda}{4},$$

i.e.,

$$\frac{11}{128} c^2 - c + \frac{\lambda}{4} \leq 0. \tag{7.32}$$

However, (7.32) implies (7.28). □

Lemma 7.7. *Let* $u \in C^2_{loc}\big((0, 1/2]; \mathbb{R}\big)$ *satisfy* (3.6), (3.8), *and* (5.4). *Then*

$$\lambda < 307.$$

Proof. Assume that there exists $u \in C^2_{loc}\big((0, 1/2]; \mathbb{R}\big)$ satisfying (3.6), (3.8), (5.4). According to Remark 5.1 and Lemma 5.2, u satisfies (5.6). Moreover, according to Lemma 5.7, u satisfies (5.15) and (5.17). Put

$$v(t) = -\frac{u(t)}{t} \qquad \text{for } t \in (0, 1/2]. \tag{7.33}$$

Then, in view of (3.8), (5.6), (5.15), and (5.17) we have that v satisfies all the assumptions of Lemma 7.5. Consequently, (7.15) and (7.16) hold.

On the other hand, (3.8), (5.15), and (7.33) yield

$$v(t) = \left[\frac{(1/2 - t)}{4t} \int_0^t v^2(s) s ds + \int_t^{1/2} \frac{v^2(s)}{4} (1/2 - s) ds + \frac{\lambda}{4} (1/2 - t) \right] \qquad \text{for } t \in (0, 1/2]. \tag{7.34}$$

Obviously, (7.12) results in (7.17), and thus, in view of (7.13), we have

$$\int_0^t v^2(s) s ds \geq v^2(t) \frac{t^2}{2} \qquad \text{for } t \in (0, 1/2]. \tag{7.35}$$

Now using (7.16) and (7.35) in (7.34) we obtain

$$v(t) \geq \frac{(1/2 - t)t}{8} v^2(t) + \frac{c^2}{4} \int_t^{1/2} (1/2 - s)^3 ds + \frac{\lambda}{4} (1/2 - t) \quad \text{for } t \in (0, 1/2], \tag{7.36}$$

where

$$c = 192 \left(1 - \sqrt{1 - \frac{\lambda}{384}} \right). \tag{7.37}$$

The inequality (7.36) means that for every fixed $t \in (0, 1/2]$, the value of the function v at the point t is a solution to the quadratic inequality

$$\frac{(1/2 - t)t}{8} x^2 - x + \frac{(1/2 - t)}{4} \left[\frac{c^2 (1/2 - t)^3}{4} + \lambda \right] \leq 0. \tag{7.38}$$

Thus (7.38) results in

$$f(\lambda,t) \stackrel{def}{=} \frac{(1/2-t)^2 t}{8}\left[\frac{c^2(1/2-t)^3}{4}+\lambda\right] \le 1 \qquad \text{for } t\in(0,1/2],$$

with c given by (7.37). However, it can be verified by a direct calculation that

$$f(307,1/8) > 1.$$

Therefore, if $\lambda = 307$, there is no solution to (3.6), (3.8), (5.4), and, according to Lemma 7.2, there is no solution to (3.6), (3.8), (5.4) for $\lambda \ge 307$ either. □

Lemma 7.8. *Let* $u \in C^2_{loc}\big((0,1/2];\mathbb{R}\big)$ *satisfy* (3.6)*,* (3.9)*, and* (5.4)*. Then* (7.28) *is fulfilled.*

Proof. Assume that there exists $u \in C^2_{loc}\big((0,1/2];\mathbb{R}\big)$ satisfying (3.6), (3.9), (5.4). According to Remark 5.1 and Lemma 5.3, u satisfies (5.6). Moreover, according to Lemma 5.7, u satisfies (5.15) and (5.17). Define v by (7.33). Then, in view of (3.9), (5.6), (5.15), and (5.17) we have that v satisfies all the assumptions of Lemma 7.6. Consequently, (7.28) holds. □

8 Proofs of the Main Results

Theorem 4.1 (resp. 4.2) follows from Remark 5.1, Lemmas 5.2 (resp. 5.3), 5.5, 7.1, 7.2, and transformation (3.5). Proposition 4.1 follows from Lemmas 5.5, 6.3, 7.3, 7.7, and transformation (3.5). Proposition 4.2 follows from Lemmas 5.5, 6.4, 7.4, 7.8, and transformation (3.5).

References

[1] A.-L. Barabási and H. E. Stanley, *Fractal Concepts in Surface Growth* (Cambridge University Press, Cambridge, 1995).

[2] C. De Coster and P. Habets, *Two-point bondary value problems, lower and upper solutions*, Mathematics in Science and Engineering Vol. 205 (Elsevier 2006).

[3] C. Escudero, *Geometric principles of surface growth*, Phys. Rev. Lett. **101** (2008) 196102.

[4] C. Escudero and E. Korutcheva, *Origins of scaling relations in nonequilibrium growth*, J. Phys. A: Math. Theor. **45** (2012) 125005.

[5] C. Escudero, R. Hakl, I. Peral, and P. J. Torres,*On the radial stationary solutions to a model of nonequilibrium growth*, to appear in European Journal of Applied Mathematics .

[6] I. T. Kiguradze and B. L. Shekhter, *Singular boundary-value problems for second-order ordinary differential equations*, J. Sov. Math. **43** (1988), No. 2, 2340–2417.

[7] G. Lengel, R. J. Phaneuf, E. D. Williams, S. Das Sarma, W. Beard, and F. G. Johnson, *Nonuniversality in mound formation during semiconductor growth*, Phys. Rev. B **60** (1999) R8469–R8472.

[8] M. Marsili, A. Maritan, F. Toigo, and J. R. Banavar, *Stochastic growth equations and reparametrization invariance*, Rev. Mod. Phys. **68** (1996) 963–983.