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Moment analysis of paraxial propagation in a nonlinear graded index fibre

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Abstract. We study analytically the evolution of the beam parameters in a system of layered nonlinear graded index fibres within the framework of the paraxial wave equation. It is shown that the fibre parameters can be chosen to induce resonant behaviour, a phenomenon which could be used to achieve pulse compression or to decouple the beam from the fibre.

Keywords: Nonlinear optics, moment techniques

1. Introduction

Since the pioneering works in the early 1970s on soliton propagation in nonlinear optical waveguides [1], fundamental and applied research in this important area of nonlinear optics has experienced a tremendous and continuous growth, mainly due to its impact in telecommunication and photonic technologies.

From the purely theoretical point of view, the partial differential equations involved in nonlinear optical propagation problems represent an interesting challenge for physicists and mathematicians. Exact solutions are rare to find and the common option is to resort to computer simulations and approximate analytical models to describe the complex dynamics of light propagating through these kinds of media.

The most common starting point to study the dynamics of laser beams in nonlinear waveguides is the paraxial approximation, which takes into account only those rays close to the propagation direction, neglecting the effect of highly diffracted light. This approach is questionable when dealing with some ‘wild’ problems such as light collapse [2], where fields become so strong that even marginal rays can play an important role in beam dynamics. However, for a smooth behaviour, the ability to find accurate analytical solutions and high performance computational methods makes the paraxial approximation the ideal choice. This will be the case in the present paper, where we will start from the paraxial equation for light beams propagating through a periodic nonlinear waveguide. This will allow us to exploit the advantages of the so-called moment method [3], in which the explicit evolution of the electric field is not studied but only the evolution of a set of integral parameters characterizing the solution globally.

Although a powerful exact technique, its application has been limited because of the technical difficulties which appear when the nonlinearity is not cubic or new terms are taken into account. However, the potential of the method in its exact form is far from being exhausted, as this paper shows. On the other hand, when complemented by appropriate approximations such as the divergenceless phase ansatz [4], the moment method may be extended to deal with all problems which are usually handled by means of the variational method. The variational technique is extensively used in nonlinear optics [5] and other fields where nonlinear wave equations (e.g. nonlinear Schrödinger equations) arise, such as condensed matter [6] or recent problems in Bose–Einstein condensation [7].

We would like to emphasize that our treatment is exact and this is a remarkable property of moment equations: they allow us to make a rigorous analysis of nonlinear problems which cannot be explicitly solved. In a previous work [8] we have extended the technique to the case where some of the parameters are time-dependent. In this paper we further extend the analysis to particular forms of that dependence which are relevant to nonlinear optics.

2. The model

The model we will use is the paraxial wave equation for monochromatic beam propagation in a nonlinear Kerr-type

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media:

$$\triangle\Psi - 2ik\frac{\partial\Psi}{\partial z} + k^2\beta|\Psi|^2\Psi - k^2\alpha^2(z)(x^2+y^2)\Psi = 0 \quad (1)$$

where Ψ is the slowly varying envelope of the spatial part of the electric field, i.e. $E = \Psi\exp(-ikz)$, and it is assumed to depend only on the transverse coordinates x, y and the propagation direction z. $k = 2\pi/\lambda$ is the propagation constant in the medium and β is a nonlinear coefficient. Although we deal with a general Kerr medium the phenomena we will study here do not depend on nonlinear properties of the medium: in fact, the treatment is valid for the linear case. In this paper our interest will be restricted to particular forms of the guiding coefficient $\alpha(z)$. However, we must stress that for the general formalism it is not necessary to make any assumptions about it and we will keep it as an arbitrary function until section 4.

It is convenient to deal with equation (1) in nondimensional form. To do so, let us define the new variables $\tau = z\cdot k, r = k\sqrt{x^2+y^2}, u = \sqrt{\beta/2}\Psi^*, \Omega = \alpha/k$. It is also convenient to consider the case of radial symmetry, which simplifies the complexity of the problem. However, as will be discussed later, this is not an essential assumption. Mathematically we will search for solutions of the form $\psi(r,\theta,z) = u(r,z)\mathrm{e}^{im\theta}$, which includes both the typical radially symmetric problem corresponding to $m = 0$, and vortex solutions, with $m \neq 0$. The simplified equation for u is

$$\mathrm{i}\frac{\partial u}{\partial\tau} = -\frac{1}{2r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \left(\frac{m^2}{2r} + s|u|^2 + \frac{1}{2}\Omega^2(\tau)r^2\right)u. \quad (2)$$

As usual this is an equation of nonlinear Schrödinger type, which is known to be related to many different phenomena in nonlinear optics and other fields. Equation (2) is non-integrable and has no exact solutions even in the constant Ω case. In this form of the equation $s = -1$ for self-focusing nonlinear media and $s = +1$ in the self-defocusing case. Throughout this paper we will restrict ourselves (though this is not an essential point) to the consideration of self-defocusing media.

3. Moment theory

Let us define the following integral quantities [3]:

$$I_1(\tau) = \int |u|^2\,\mathrm{d}^2x, \quad (3a)$$

$$I_2(\tau) = \int |u|^2 r^2\,\mathrm{d}^2x, \quad (3b)$$

$$I_3(\tau) = \mathrm{i}\int\left[u\frac{\partial u^*}{\partial\tau} - u^*\frac{\partial u}{\partial\tau}\right]r\,\mathrm{d}^2x, \quad (3c)$$

$$I_4(\tau) = \frac{1}{2}\int\left(|\nabla u|^2 + \frac{m^2}{r^2}|u|^2 + s|u|^4\right)\mathrm{d}^2x. \quad (3d)$$

These parameters are related physically to the nondimensional intensity (I_1), width (I_2), curvature radius (I_3) and divergence (I_4), which are related to the beam moments as defined in the framework of moment theory [3]. Their evolution laws are

$$\frac{\mathrm{d}I_1}{\mathrm{d}\tau} = 0, \quad (4a)$$

$$\frac{\mathrm{d}I_2}{\mathrm{d}\tau} = I_3, \quad (4b)$$

$$\frac{\mathrm{d}I_3}{\mathrm{d}\tau} = -2\Omega^2(\tau)I_2 + 4I_4, \quad (4c)$$

$$\frac{\mathrm{d}I_4}{\mathrm{d}\tau} = -\tfrac{1}{2}\Omega^2(\tau)I_3. \quad (4d)$$

Equations (4) form a system for the unknowns, $I_j, j = 1,\ldots,4$. An invariant of this system under propagation is $Q = 2I_4I_2 - I_3^2/4 > 0$.

The system (4) can be reduced to a single equation for the most relevant parameter $I_2(\tau)$, which is

$$\frac{\mathrm{d}^2I_2}{\mathrm{d}\tau^2} - \frac{1}{2I_2}\left(\frac{\mathrm{d}I_2}{\mathrm{d}\tau}\right)^2 + 2\Omega^2(\tau)I_2 = \frac{Q}{I_2}. \quad (5)$$

If we were able to solve equation (5) then the use of equations (4) would allow us to track the evolution of the remaining momenta. We can do so by defining $w(\tau) = \sqrt{I_2}$, whose physical meaning is the (nondimensional) beam width, and substituting it into (5). This procedure gives us

$$\frac{\mathrm{d}^2w}{\mathrm{d}\tau^2} + \Omega^2(\tau)w = \frac{Q}{w^3}. \quad (6)$$

The resulting equation is a singular (nonlinear) Hill equation of Ermakov type whose general solution is [11]

$$w(\tau) = \sqrt{u^2(\tau) + \frac{Q}{W^2}v^2(\tau)}, \quad (7)$$

where $u(\tau)$ and $v(\tau)$ are the two linearly independent solutions of the equation

$$\ddot{x} + \Omega^2(\tau)x = 0, \quad (8)$$

which satisfies $u(\tau_0) = w(\tau_0)$, $\dot{u}(\tau_0) = w'(\tau_0)$, $v(\tau_0) = 0$, $v'(\tau_0) \neq 0$, and W is the Wronskian $W = u\dot{v} - \dot{u}v =$ constant $\neq 0$.

The conclusion is that the evolution of the beam width is closely related to the solutions of the Hill equation (8). This is a well studied problem [10] which is explicitly solvable only for particular choices of $\Omega(\tau)$, but whose solutions are well characterized and many of its properties are known. In [8,9] we used this fact to prove the existence of resonances in extended systems with harmonic $\Omega(\tau)$. Here we will focus our attention on functional forms of Ω corresponding to nonlinear optical systems.

4. Beam width evolution in layered graded index media

4.1. Model

Our goal now will be to solve equation (8) in the realistic case where Ω is a piecewise constant function depending on τ. In practical situations this can be easily achieved by just

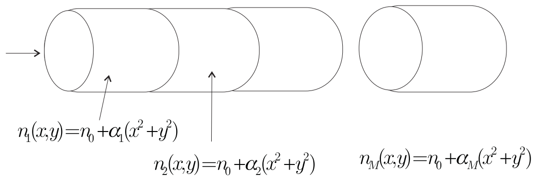

Figure 1. The system considered: the fibres are arranged to form a layered structure.

joining different fibres with different parabolic profiles in the refraction index.

It is well known that equation (8) may describe parametric resonances when Ω is periodic. This opens the possibility of inducing a strong response of the pulse width to (may be small) variations of the guiding coefficient. This phenomenon could be useful for two purposes: (i) to decouple the beam from the fibre and allow switching, or (ii) to compress transversally the pulse, as will be discussed later.

Let us consider a fibre made up of different graded index sections as described in figure 1. Although this general case can be studied with our formalism, we will consider a simplified case which is interesting enough to deserve detailed study and allows us to keep the physical insight. Specifically, we will choose $\Omega(\tau)$ as a piecewise constant T-periodic function of the form

$$\Omega(\tau) = \begin{cases} a^2 & \tau \in [0, T_1] \\ b^2 & \tau \in [T_1, T_2 = T] \end{cases} \tag{9}$$

which corresponds to the case where $\alpha_{2j+1} = \alpha_{2j-1} \propto a^2$ and $\alpha_{2j} = \alpha_{2j+2} \propto b^2$ for integer j values.

4.2. Formal solution

Equation (8) with $\Omega(\tau)$ given by equation (9) is a particular type of Hill equation known as the Meissner equation. It can be integrated explicitly once the initial data are known. Let us first consider the interval $[0, T_1]$. It is evident that the solution is given by

$$x(\tau) = x_0 \cos(a\tau) + \frac{v_0}{a}\sin(a\tau), \qquad \tau \in [0, T_1]. \tag{10}$$

Using as initial conditions $x(T_1)$, $x'(T_1)$ we can compute $x(\tau)$ in the interval $[T_1, T]$ as

$$x(\tau) = x(T_1)\cos(b(\tau - T_1)) + \frac{x'(T_1)}{b}\sin(b(\tau - T_1)). \tag{11}$$

Recursively we can compute $x(\tau)$ in a general interval $[nT, (n+1)T]$ with integer n, the result being

$$x(\tau) = x(nT)\cos(a(\tau - nT)) + \frac{x'(nT)}{a}\sin(a(\tau - nT)), \qquad \tau \in [nT, nT + T_1]. \tag{12}$$

Although we have an explicit solution, which could be of interest in some cases, we will concentrate in what follows on the analysis of the resonance conditions, which can be made without the explicit knowledge of the solution.

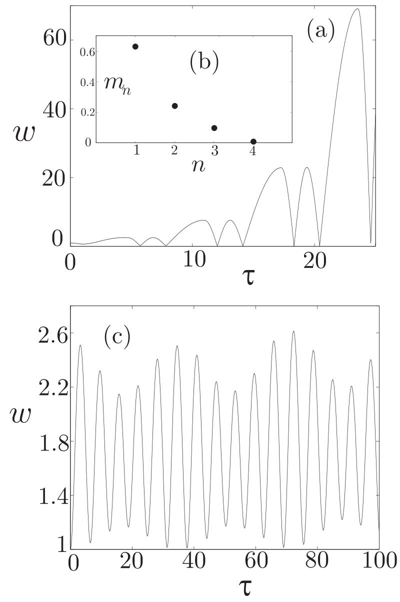

Figure 2. Solutions of equation (8) with $\Omega(\tau)$ given by equation (9). (*a*) Resonant solution for parameter values $T_1 = 10\pi$, $T = 20\pi$, $a = 0.05$, $b = 0.15$ and (*b*) amplitude of the absolute minimum for period n of the perturbation m_n. A decrease of the minimum width is observed at the same time as the maximum amplitude increases. (*c*) Bounded solution for $T_1 = 10\pi$, $T = 20\pi$, $a = 0.05$, $b = 0.02$.

4.3. Resonances and beam compression

Solutions of the Meissner equation are of two different types. First, there are bounded solutions (periodic or quasi-periodic) corresponding to width oscillations. The new feature is that there are solutions which oscillate resonantly. In figure 2 we show the two different types of solutions for a particular choice of parameters. It is also qualitatively clear in figure 2(*a*) that the minimum beam width decreases and the maximum width increases as the beam passes through the layered structure. In figure 2(*b*) we quantify the decrease of the minimum width for each period of the structure.

This fact suggests the idea that one could achieve a significant compression at a given distance. Also, the increasing part is interesting since it can be used for several purposes and specifically to decouple a beam with a given wavelength from the fibre (T is related to the wavelength and is involved in the resonance condition). Finally, as stated in the introduction there is a fundamental interest in our result since this is a nonlinear optical system for which analytical information can be provided, since our treatment does not involve any approximations.

For practical applications it is necessary to analyse the stability regions in a precise way. This can be done by studying the discriminant of equation (8), which is defined as the trace of a monodromy matrix, mathematically

$$D(a, b, T_1, T) := \phi_1(T) + \phi_2'(T). \tag{13}$$

ϕ_1, ϕ_2 are the solutions of equation (8) which satisfy the initial data $\phi_1(0) = 1$, $\phi_1'(0) = 0$ and $\phi_2(0) = 0$, $\phi_2'(0) = 1$, respectively. In our case a simple computation leads to

$$\phi_1(T) = \cos(aT_1)\cos(b(T - T_1))$$

$$- \frac{a}{b} \sin(aT_1) \sin(b(T - T_1)), \tag{14a}$$

$$\phi_2'(T) = \cos(aT_1)\cos(b(T - T_1)) - \frac{b}{a} \sin(aT_1) \sin(b(T - T_1)). \tag{14b}$$

After some algebra the form of the discriminant is found:

$$D(a, b, T_1, T) = 2\cos(aT_1 + b(T - T_1)) - \frac{(a-b)^2}{ab} \sin(aT_1)\sin(b(T - T_1)). \tag{15}$$

It is known from Floquet theory for linear periodic coefficient differential equations that the stability of solutions of equation (8) for particular parameter values is related to the discriminant. Resonance regions correspond to those parameter regions for which $|D| > 2$, while if $|D| < 2$ all the solutions are bounded. The equations $D(a, b, T_1, T) = 2$ and $D(a, b, T_1, T) = -2$ are manifolds which limit stability regions in the four-dimensional parameter space. Those manifolds contain all the information related to resonances.

Since it is very difficult to fully study exhaustively the four-dimensional parameter space involved in this problem, we will concentrate in the following section on parameter regions of practical interest.

5. Examples

As stated before, in the general case we have four positive parameters in our model system: a, b, T_1 and T. In what follows we will study some particular examples which are either illustrative or interesting from the point of view of applications.

5.1. Equal length fibres: resonance regions

To obtain some insight into the shape of the resonance regions, let us consider the case where both sections are equal, $T_1 = T_2 = T$, so that the discriminant depends only on the parameters aT, bT and is found to be

$$D(a, b) = 2\cos\left(\frac{(a+b)T}{2}\right) - \frac{(aT - bT)^2}{abT^2} \sin\left(\frac{aT}{2}\right) \sin\left(\frac{bT}{2}\right). \tag{16}$$

Now the manifolds $|D| = 2$ are curves and it is possible to draw them. For this particular case we plot them together with the resonance regions in figure 3. It can be seen that there are large regions in the parameter space which correspond to resonant solutions. Also resonance regions have finite width which makes the system robust against uncontrollable parameter imprecisions.

For instance, taking typical parameters according to the usual fabrication techniques of nonlinear optical waveguides [12], $T_1 = 100\pi$, $a = 0.1$, we obtain $aT/2\pi \sim 5$. To induce resonances one should choose $bT/2\pi$ suitably, e.g. on the interval [0,3] there are wide resonance regions which should be robust enough to parameter imprecisions.

The details of the resonance regions can be discussed more precisely fixing the relative value of the coefficients.

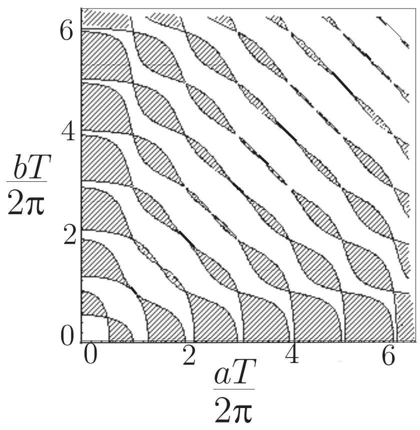

Figure 3. Resonance regions (shaded) for $T_1 = T_2 = T/2$.

To present the methodology with a particular case we choose the value $b = 2a$:

$$D(a, T) = 2\cos\left(\frac{3aT}{2}\right) - \frac{1}{2}\sin\left(\frac{aT}{2}\right)\sin(aT). \tag{17}$$

Since the discriminant depends only on the product aT we can define a new variable $x = aT/2$ and then the problem is simplified since the discriminant depends on only one variable $D(x)$ (figure 4). The characteristic curves are hyperbolas of the type $2x_{\pm}^{(n)} = aT$, $x_+^{(n)}, x_-^{(n)}$ being, respectively, the solutions of the algebraic equations

$$f_+(x) = 2\cos 3x - \tfrac{1}{2}\sin x \sin 2x - 2 = 0 \tag{18a}$$

$$f_-(x) = 2\cos 3x - \tfrac{1}{2}\sin x \sin 2x + 2 = 0. \tag{18b}$$

It is easy to prove that resonance regions are contained between two consecutive zeros of f_+ or f_-. Zeros of $f_\pm$ can be obtained numerically by using a standard method. When the plotted quantities are a and $1/T$ hyperbolas become straight lines, and resonance regions correspond to the shaded regions of figure 4(*a*). Evidently, the picture repeats due to the 2π periodicity of $D(x)$ and there are only four basic (and their 'harmonics') resonance regions contained in the intervals (roots of f_+ and f_-): $x \in [0.84, 1.23] \cup [1.91, 2.3] \cup [3.98, 4.37] \cup [5.05, 5.44]$.

Another case of interest corresponds to the situation where one of the fibres is not a graded index one, i.e. $b = 0$. Then the discriminant is given by the limit of equation (15) when $b \to 0$:

$$D(a, T) = -aT\sin\left(\frac{aT}{2}\right) + 2\cos\left(\frac{aT}{2}\right). \tag{19}$$

As in the previous case the only relevant parameter is $x = aT/2$, resonance regions in the a–T plane are hyperbolas and the relevant quantities are the zeros of f_+, f_-, which are now given by

$$f_+(x) = -x\sin x + 2\cos x - 2 = 0 \tag{20a}$$

$$f_-(x) = -x\sin x + 2\cos x + 2 = 0. \tag{20b}$$

Now there is no exact periodicity in the positions of the zeros but at least high-order zeros are easy to find. To do so

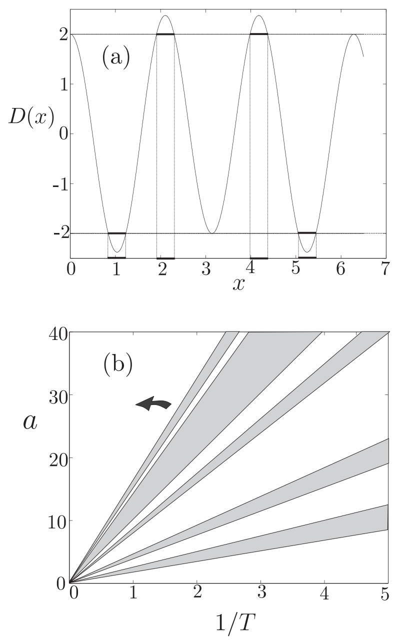

Figure 4. (*a*) First four resonance regions in x for $T_1 = T/2$, $b = 2a$ as a function of $x = aT/2$. (*b*) First five resonance regions (shaded) on the a–$1/T$ plane.

let us realize that for large x the dominant term in both cases is $f_{\pm}(x) \simeq -x \sin x$ which means that the zeros are given by $x = n\pi$. The rate of convergence is $O(1/n)$ as can be seen perturbatively. Writing $x_{\pm}^{(n)} = n\pi + \epsilon_{\pm}^{(n)}$ and substituting into equations (20) one finds that

$$\epsilon_{\pm}^{(n)} \simeq (-1)^{n+1} \frac{1 \pm 2}{n\pi}. \tag{21}$$

This type of analysis can be made to locate precisely resonance regions for any restriction on parameters.

5.2. Short graded index sections

Let us now consider the case when one of the graded index fibres is much shorter than the other one. To fix ideas let us consider the case when $T_1 = 9T/10$, $T_2 = T/10$ to obtain the discriminant

$$\begin{aligned} D(a, b, T) = 2\cos\left(\frac{9aT}{10} + \frac{bT}{10}\right) \\ - \frac{(a-b)^2}{ab} \sin\frac{9a}{10} \sin\frac{b}{10} \\ - \frac{(a-b)^2}{ab} \sin\frac{9aT}{10} \sin\frac{bT}{10}. \end{aligned} \tag{22}$$

The resonance regions are seen in figure 5.

Taking particular values for the parameters corresponding to real optical fibres we get large resonance regions in realistic regions of parameter space.

6. Conclusions and discussion

In this paper we have computed analytically the evolution of the width of a light beam ruled by the paraxial equation in a system of layered graded index (nonlinear) fibres. Throughout this paper we have made use of a special symmetry for

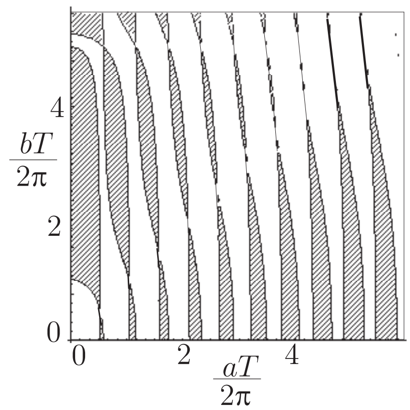

Figure 5. Resonance regions (shaded) for $T_1 = 9T/10$ and $T_2 = T/10$.

the confining potential and for the solution in order to make our theory exact. When these constraints are removed and we turn to a nonsymmetric two-dimensional model, a corresponding set of coupled Hill's equations may still be obtained by some kind of approximation, e.g. using a variational ansatz [8]. We think that our situation here is very similar, the only difference being the use of a piecewise constant periodic function instead of the harmonic function discussed there. Our method is exact, which is an improvement over the more conventional method of using multimode expansions and matching of the modes at the interfaces.

When losses are included in equation (2), e.g. by the addition of a new term of the type $i\sigma\psi$, it is not possible to obtain a set of closed equations for the momenta and thus one cannot solve exactly the problem. However, as discussed in [4], it is possible to study the system in the framework of the parabolic phase approximation to show that the effect of the damping is not essential.

One possible application of resonances could be to achieve temporal pulse compression induced by the transverse focusing of the beam. Although the detailed analysis of the problem exceeds the purposes of the present paper, the intuitive idea is very simple: as the beam is focused in the transverse plane, the peak power is increased. The rising of the field amplitude induces a self-modulated nonlinear phase which yields to pulse compression in the propagation direction. Thus, by suitably controlling the geometrical parameters of the waveguide, pulse compressors can be designed for specific purposes. A complete study of the problem involves propagation in $1+3$ dimensions that are currently in progress.

The applications of our results could involve pulse compression or optical switching. Independently of the particular application, the results presented here are of fundamental interest since they correspond to a complex system where beam parameter evolution can be computed analytically.

Finally let us comment that, although we include a possible nonlinear response of the medium, the phenomena presented here are not exclusive to nonlinear media. In fact, the only difference between linear and nonlinear media in our treatment is the particular value of Q. However, since nonlinear media correspond to $Q > 1$ they correspond to a 'harder' response of the system to a parametric perturbation which leads to the easier establishment of resonances. It is a remarkable fact that the moment method is able to handle in an unified way both situations and provide exact answers for the behaviour of the beam parameters, even in the complex case considered in this paper.

Acknowledgments

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