Physica D 261 (2013) 81–84

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Periodic motions of fluid particles induced by a prescribed vortex path in a circular domain

Alberto Boscaggin [a], Pedro J. Torres [b,*]

[a] SISSA - International School for Advanced Studies, via Bonomea 265, 34136 Trieste, Italy
[b] Departamento de Matemática Aplicada, Universidad de Granada, 18071 Granada, Spain

HIGHLIGHTS

- A two-dimensional ideal fluid inside a circular domain under the action of a prescribed stirring protocol.
- The motion of advected particles follows a Hamiltonian system.
- The vortex induces a singularity on the angular variable.
- An infinite number of periodic solutions are found.

ARTICLE INFO

Article history:
Received 14 December 2012
Received in revised form
25 June 2013
Accepted 2 July 2013
Available online 11 July 2013
Communicated by I. Melbourne

Keywords:
Particle transport
Ideal fluid
Vortex
Periodic orbit
Poincaré–Birkhoff theorem

ABSTRACT

By means of a generalized version of the Poincaré–Birkhoff theorem, we prove the existence and multiplicity of periodic solutions for a Hamiltonian system modeling the evolution of advected particles in a two-dimensional ideal fluid inside a circular domain and under the action of a point vortex with prescribed periodic trajectory.

1. Introduction and main result

We consider the motion of a two-dimensional ideal fluid in a circular domain of radius $R > 0$ subjected to the action of a moving point vortex whose position, denoted as $z(t)$, is a prescribed T-periodic function of time. This model plays an important role in Fluid Mechanics as an idealized model of the stirring of a fluid inside a cylindrical tank by an agitator. A fundamental reference for this problem is the seminal paper [1], where the concept of *chaotic advection* was coined. Following the classical Lagrangian representation, the mathematical model under consideration is the planar system

$$\dot{\bar{\zeta}} = \frac{\Gamma}{2\pi i}\left(\frac{|z(t)|^2 - R^2}{(\zeta - z(t))(\zeta\bar{z}(t) - R^2)}\right), \qquad (1)$$

where the complex variable ζ represents the evolution on the position of a fluid particle induced by the so-called *stirring protocol* $z(t)$. System (1) is a T-periodically forced planar system with Hamiltonian structure, where the stream function

$$\Psi(t,\zeta) = \frac{\Gamma}{2\pi}\ln\left|\frac{\zeta - z(t)}{\bar{z}(t)\zeta - R^2}\right|$$

plays the role of the Hamiltonian.

The main contribution of Aref in [1] was to show that the flow may experience regular or chaotic regimes depending on the particular stirring protocol. For instance, system (1) is integrable if $z(t)$ is constant or $z(t) = z_0\exp(i\Omega t)$ but it is chaotic if $z(t)$ is piecewise constant (blinking protocol in the related literature). A naive way to measure the influence of the ideas presented in [1] is to note the more than a thousand citations of this inspiring paper to date. Aref's blinking protocol is piecewise integrable and the theory of linked twist maps permits a good analytical study of the underlying dynamics (see for instance [2,3]). More recently, other

* Corresponding author. Tel.: +34 958242941.
E-mail addresses: boscaggi@sissa.it (A. Boscaggin), ptorres@ugr.es (P.J. Torres).

0167-2789/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.physd.2013.07.004

strategies of stirring have been studied, for instance the figure-eight or the epitrochoidal protocol [4], but only from a numerical point of view. Our contribution in this paper is to prove that both regular and chaotic regimes share a common dynamical feature, namely the existence of an infinite number of periodic solutions labeled by the number of revolutions around the vortex in the course of a period.

To be precise, let us fix $z : \mathbb{R} \to \mathbb{C}$ a T-periodic function such that $|z(t)| < R$ for all t. For a periodic solution ζ of (1) with period kT, the *winding number* of ζ is defined as

$$\mathrm{rot}_{kT}(\zeta) = \frac{1}{2\pi i}\int_0^{kT} \frac{d(\zeta(t) - z(t))}{\zeta(t) - z(t)}$$

and provides the number of revolutions of $\zeta(t)$ around the vortex point $z(t)$ in the time interval $[0, kT]$. We proceed to state our main result.

Theorem 1.1. *Let $z : \mathbb{R} \to \mathbb{C}$ be a T-periodic function of class C^1, such that $|z(t)| < R$ for all t. Then, for every integer $k \geq 1$, system (1) has infinitely many kT-periodic solutions lying in the disk $\mathcal{B}_R(0)$. More precisely, for every integer $k \geq 1$, there exists an integer j_k^* such that, for every integer $j \geq j_k^*$, system (1) has two kT-periodic solutions $\zeta_{k,j}^{(1)}(t)$, $\zeta_{k,j}^{(2)}(t)$ such that, for $i = 1, 2$,*

$$\|\zeta_{k,j}^{(i)}\|_\infty \leq R \quad and \quad \mathrm{rot}_{kT}(\zeta_{k,j}^{(i)}) = j. \tag{2}$$

Moreover, for every $k \geq 1$, $j \geq j_k^*$ and $i = 1, 2$,

$$\lim_{j\to+\infty} |\zeta_{k,j}^{(i)}(t) - z(t)| = 0, \quad uniformly\ in\ t \in [0, kT]. \tag{3}$$

In particular, for $k = 1$, we find that (1) has infinitely many T-periodic solutions. For $k > 1$, we find subharmonic solutions of order k (i.e., kT-periodic solutions which are not lT-periodic for any $l = 1, \dots, k-1$) provided that j and k are relatively prime integers; we remark that in this case it is also possible to show that $\zeta_{k,j}^{(1)}(t)$, $\zeta_{k,j}^{(2)}(t)$ are not in the same periodicity class (namely, $\zeta_{k,j}^{(1)}(\cdot) \not\equiv \zeta_{k,j}^{(2)}(\cdot + lT)$ for every integer $l = 1, \dots, k-1$).

It is worth pointing out that the regularity condition on the stirring protocol plays an important role. In fact, Theorem 1.1 is not true for a discontinuous $z(t)$ (e.g. the blinking protocol), because condition (3) would imply unphysical discontinuous particle trajectories. The existence and multiplicity of periodic solutions for a general protocol, as well as their stability properties, remains as an open problem. We will come back to this issue in the final section.

The rest of the paper is divided in three section. In Section 2 the Poincaré section is defined, whereas Section 3 contains the proof of Theorem 1.1 by an application of a generalized version of the Poincaré–Birkhoff Theorem. The paper is concluded by Section 4 with a discussion on the physical meaning of the presented results and some other remarks.

2. Definition of the Poincaré section

For our purposes, it is convenient to write system (1) as

$$\dot{\bar{\zeta}} = \frac{\Gamma}{2\pi i}\left(\frac{1}{\zeta - z(t)} - \frac{1}{\zeta - \frac{R^2}{|z(t)|^2}z(t)}\right). \tag{4}$$

In this form, the first term on the right models the action of the vortex whereas the second term corresponds to the wall influence on the flow. Identifying $\mathbb{C}$ with $\mathbb{R}^2$ and setting $\zeta = (x, y)$, $z(t) = (a(t), b(t))$, we can rewrite system (4) in real notation as

$$\begin{cases} \dot{x} = \dfrac{\Gamma}{2\pi}\left(-\dfrac{y - b(t)}{|\zeta - z(t)|^2} + \dfrac{y - \frac{R^2}{|z(t)|^2}b(t)}{\left|\zeta - \frac{R^2}{|z(t)|^2}z(t)\right|^2}\right) \\ \dot{y} = \dfrac{\Gamma}{2\pi}\left(\dfrac{x - a(t)}{|\zeta - z(t)|^2} - \dfrac{x - \frac{R^2}{|z(t)|^2}a(t)}{\left|\zeta - \frac{R^2}{|z(t)|^2}z(t)\right|^2}\right), \end{cases}$$
$$\zeta = (x, y) \in \mathbb{R}^2. \tag{5}$$

Let $\mathcal{B}_R \subset \mathbb{R}^2$ be the closed disk centered at the origin with radius R. First, we recall a well known property of system (5).

Lemma 2.1. *Let $\zeta : J \to \mathbb{R}^2$ be a solution of (5), with $J \subset \mathbb{R}$ its maximal interval of definition. If $|\zeta(t_0)| \leq R$ for some $t_0 \in J$, then $|\zeta(t)| \leq R$ for every $t \in J$, that is to say, the disk $\mathcal{B}_R$ is invariant for the flow associated to (5).*

Proof. Since $\mathcal{B}_R = \{(x, y) \in \mathbb{R}^2 \mid V(x, y) \leq R^2\}$ for $V(x, y) = x^2 + y^2$, by the standard result of flow-invariant sets, it is enough to prove that

$$\langle Z(t, x, y)|\nabla V(x, y)\rangle = 0, \quad \text{for every } t \in [0, T],\ x^2 + y^2 = R^2,$$

where $Z(t, x, y)$ denotes the vector field of the differential system (5). With simple computations, we find indeed

$$\begin{aligned} \langle Z(t, x, y)|\nabla V(x, y)\rangle &= \frac{1}{2}\Big(X(t, x, y)x + Y(t, x, y)y\Big) \\ &= \frac{\Gamma}{\pi}\Big(b(t)x - a(t)y\Big)\left(\frac{\left|\zeta - \frac{R^2}{|z(t)|^2}z(t)\right|^2 - \frac{R^2}{|z(t)|^2}|\zeta - z(t)|^2}{|\zeta - z(t)|^2\left|\zeta - \frac{R^2}{|z(t)|^2}z(t)\right|^2}\right) \\ &= \frac{\Gamma}{\pi}\Big(b(t)x - a(t)y\Big)\left(\frac{\left(1 - \frac{R^2}{|z(t)|^2}\right)\left(|\zeta|^2 - \frac{R^2}{|z(t)|^2}|z(t)|^2\right)}{|\zeta - z(t)|^2\left|\zeta - \frac{R^2}{|z(t)|^2}z(t)\right|^2}\right) \\ &= 0. \quad \square \end{aligned}$$

From now on, we will study solutions to system (5) belonging to the invariant disk $\mathcal{B}_R$; accordingly, the singularity of the vector field at $\zeta = \frac{R^2}{|z(t)|^2}z(t)$ (for which $|\zeta| > R$) will not play any role. On the contrary, we will take advantage of the singularity at $\zeta = z(t)$. To this aim, it is useful to introduce the change of variable

$$\eta = \zeta - z(t)$$

and set $\eta = (u, v)$, so that system (5) is transformed into

$$\begin{cases} \dot{u} = \dfrac{\Gamma}{2\pi}\left(-\dfrac{v}{|\eta|^2} + \dfrac{v + b(t)\left(1 - \frac{R^2}{|z(t)|^2}\right)}{\left|\eta + z(t)\left(1 - \frac{R^2}{|z(t)|^2}\right)\right|^2}\right) - \dot{a}(t) \\ \dot{v} = \dfrac{\Gamma}{2\pi}\left(\dfrac{u}{|\eta|^2} - \dfrac{u + a(t)\left(1 - \frac{R^2}{|z(t)|^2}\right)}{\left|\eta + z(t)\left(1 - \frac{R^2}{|z(t)|^2}\right)\right|^2}\right) - \dot{b}(t), \end{cases}$$
$$\eta = (u, v) \in \mathbb{R}^2. \tag{6}$$

In the following, given $\eta_0 \neq 0$, we will denote by $\eta(\cdot; \eta_0)$ the unique solution of (6) satisfying the initial condition $\eta(0) = \eta_0$.

Lemma 2.2. *There exists $r > 0$ such that, if $0 < |\eta_0| \leq r$, then the solution $\eta(\cdot; \eta_0)$ exists on $\mathbb{R}$ and satisfies $|\eta(t; \eta_0) + z(t)| \leq R$, for every $t \in \mathbb{R}$.*

Proof. Define

$$r = R - |z(0)| > 0.$$

Then, for $0 < |\eta_0| \le r$, the function $\zeta(t) = \eta(t;\eta_0) + z(t)$ solves (5) and

$$|\zeta(0)| \le |\eta_0| + |z(0)| \le r + |z(0)| = R.$$

From Lemma 2.1, we have the *a priori* bound

$$|\eta(t;\eta_0) + z(t)| \le R, \quad \text{for every } t \in J, \tag{7}$$

where $J \subset \mathbb{R}$ denotes the maximal interval of definition of $\eta(t;\eta_0)$. Our objective is to show that actually $J = \mathbb{R}$, completing the proof of the lemma. Notice that, in view of the *a priori* bound (7), we just have to show that $\eta(t;\eta_0)$ cannot reach the singularity $\eta = 0$ in finite time. First, we are going to consider the particular case of $z(t) = (a(t), b(t))$ belonging to the C^2 class, then the general case is proved by a standard limiting argument.

Define the function (to simplify the notation, we take advantage here of both real and complex notation)

$$K(t,\eta) = \frac{\Gamma}{2\pi}\left(\ln|\eta| - \ln\left|\bar{z}(t)(\eta + z(t)) - R^2\right|\right) + \dot{a}(t)v - \dot{b}(t)u$$

and set $k(t) = K(t, \eta(t;\eta_0))$ for $t \in J$. Since $K(t,\eta)$ is a Hamiltonian function for (6), we have

$$\langle \nabla_\eta K(t,\eta(t;\eta_0)) | \eta'(t,\eta_0)\rangle = 0,$$

so that (writing for simplicity $\eta(t;\eta_0) = \eta(t)$),

$$\begin{aligned} |k'(t)| &= \left|\frac{\partial K}{\partial t}(t,\eta(t;\eta_0))\right| \\ &= \left| -\frac{\Gamma}{2\pi} \frac{\langle \bar{z}(t)(\eta+z(t)) - R^2 | \gamma(t)\rangle}{\left|\bar{z}(t)(\eta+z(t)) - R^2\right|^2} \right. \\ &\qquad \left. + \ddot{a}(t)v(t) - \ddot{b}(t)u(t) \right| \\ &\le \frac{\Gamma}{2\pi} \frac{|\gamma(t)|}{\left|\bar{z}(t)(\eta+z(t)) - R^2\right|} + |\ddot{a}(t)v(t) - \ddot{b}(t)u(t)|, \end{aligned}$$

being $\gamma(t) = \bar{z}'(t)\eta(t) + 2\langle z(t)|z'(t)\rangle$. From the *a priori* bound (7) one gets

$$\begin{aligned} \left|\bar{z}(t)(\eta+z(t)) - R^2\right| &\ge R^2 - \left|\bar{z}(t)(\eta(t)+z(t))\right| \\ &\ge R\Big(R - |\bar{z}(t)|\Big) > 0, \end{aligned} \tag{8}$$

so there exists $M > 0$ (independent on η_0) such that $|k'(t)| \le M$ for every $t \in J$. Hence,

$$|K(t,\eta(t)) - K(0,\eta_0)| \le M|t|, \quad \text{for every } t \in J. \tag{9}$$

Since $K(t,\eta)$ is unbounded near $\eta = 0$, this shows that $\eta(t)$ cannot reach the singularity in finite time, thus concluding the proof. For the general C^1 case, one can approach uniformly $z(t)$ by C^2 functions, and the result follows from the continuous dependence of the solutions of the initial value problem with respect to parameters. □

Fix now an integer $k \ge 1$. We can then define the Poincaré map Ψ_k at time kT as

$$\mathcal{B}_r \setminus \{0\} \ni \eta_0 \mapsto \Psi_k(\eta_0) = \eta(kT;\eta_0).$$

By the fundamental theory of ODEs, it turns out that Ψ_k is a global homeomorphism of $\mathcal{B}_r \setminus \{0\}$ onto $\Psi_k(\mathcal{B}_r \setminus \{0\})$, preserving area and orientation; moreover, from (9) we see that Ψ_k can be extended (as an area and orientation preserving homeomorphism) to the whole disk $\mathcal{B}_r$ by setting $\Psi_k(0) = 0$.

3. Proof of the main result

By Section 2, for any integer $k \ge 1$ there exists a well-defined homeomorphism $\Psi_k : \mathcal{B}_r \to \Psi_k(\mathcal{B}_r)$ preserving area and orientation. Moreover, $\Psi_k(0) = 0$. For the reader's convenience, we recall here the generalized version of the Poincaré–Birkhoff theorem which we are going to apply (see [5,6]).

Generalized Poincaré–Birkhoff theorem. Let $0 < r_1 < r_2$ and set $\mathcal{A} = \{(x,y) \in \mathbb{R}^2 \mid r_1^2 \le x^2 + y^2 \le r_2^2\}$. Let $\Psi : \mathcal{B}_{r_2} \to \Psi(\mathcal{B}_{r_2})$ be an area-preserving homeomorphism with $\Psi(0) = 0$. Assume that, on the universal covering space $\{(\rho,\theta) \in \mathbb{R}^2 \mid \rho > 0\}$ with covering projection $\Pi(\rho,\theta) = (\rho\cos\theta, \rho\sin\theta)$, $\Psi|_{\mathcal{A}}$ has a lifting of the form

$$\widetilde{\Psi}(\rho,\theta) = (R(\rho,\theta), \theta + \gamma(\rho,\theta)),$$

$R(\rho,\theta), \gamma(\rho,\theta)$ being continuous functions 2π-periodic in the second variable. Finally, suppose that, for a suitable $j \in \mathbb{Z}$, the twist condition

$$\gamma(r_1,\theta) > 2\pi j \quad \text{and} \quad \gamma(r_2,\theta) < 2\pi j, \quad \text{for every } \theta \in \mathbb{R},$$

is fulfilled. Then there exist two distinct points $(\rho^{(1)},\theta^{(1)})$, $(\rho^{(2)},\theta^{(2)}) \in]r_1,r_2[\times[0,2\pi[$ such that (for $i = 1,2$) $\widetilde{\Psi}(\rho^{(i)},\theta^{(i)}) = (\rho^{(i)},\theta^{(i)} + 2\pi j)$.

To apply this theorem, we therefore write

$$\eta(t) = (\rho(t)\cos\theta(t), \rho(t)\sin\theta(t)), \quad \rho(t) > 0,$$

transforming system (6) into

$$\begin{cases} \dot{\rho} = I(t,\rho,\theta) \\ \dot{\theta} = \Theta(t,\rho,\theta), \end{cases} \tag{10}$$

being

$$\begin{aligned} I(t,\rho,\theta) &= \frac{\Gamma}{2\pi}\left(\frac{(b(t)\cos\theta - a(t)\sin\theta)\left(1 - \frac{R^2}{|z(t)|^2}\right)}{\left|(\rho\cos\theta,\rho\sin\theta) + z(t)\left(1 - \frac{R^2}{|z(t)|^2}\right)\right|^2}\right) \\ &\quad - \dot{a}(t)\cos\theta - \dot{b}(t)\sin\theta \end{aligned}$$

$$\begin{aligned} &\Theta(t,\rho,\theta) \\ &= \frac{\Gamma}{2\pi}\left(\frac{1}{\rho^2} - \frac{\rho + (a(t)\cos\theta + b(t)\sin\theta)\left(1 - \frac{R^2}{|z(t)|^2}\right)}{\rho\left|(\rho\cos\theta,\rho\sin\theta) + z(t)\left(1 - \frac{R^2}{|z(t)|^2}\right)\right|^2}\right) \\ &\quad + \frac{\dot{a}(t)\sin\theta - \dot{b}(t)\cos\theta}{\rho}. \end{aligned}$$

We denote by $(\rho(\cdot;\rho_0,\theta_0), \theta(\cdot;\rho,\theta_0))$ the unique solution to (10) satisfying the initial condition $(\rho(0),\theta(0)) = (\rho_0,\theta_0)$. In view of Lemma 2.2, such solutions globally exist (and $\rho(t) \ne 0$) if $\rho_0 \in]0,r]$.

Define $j_k^* \ge 1$ as the smallest integer such that

$$\theta(kT;r,\theta_0) - \theta(0;r,\theta_0) < 2\pi j_k^*, \quad \text{for every } \theta_0 \in [0,2\pi[. \tag{11}$$

Fix now an integer $j \ge j_k^*$; we claim that there exists $r_j \in]0,r[$ such that

$$\theta(kT;r_j,\theta_0) - \theta(0;r_j,\theta_0) > 2\pi j, \quad \text{for every } \theta_0 \in [0,2\pi[. \tag{12}$$

Indeed, arguing similarly as in (8) we see that

$$\left|(\rho\cos\theta,\rho\sin\theta) + z(t)\left(1 - \frac{R^2}{|z(t)|^2}\right)\right|^2$$

is bounded away from zero for $\rho \in]0, r]$; accordingly, we can find $\hat{r}_j \in]0, r[$ such that

$$\Theta(t, \rho, \theta) > \frac{2\pi j}{kT}, \quad \text{for every } t \in \mathbb{R},\ \rho \in]0, \hat{r}_j],\ \theta \in \mathbb{R}. \tag{13}$$

Now, a well-known argument (usually referred to as the "elastic property"), relying on the continuous dependence of the solutions from the initial conditions and on the fact that $\Psi_k(0) = 0$, yields the existence of $r_j \in]0, \hat{r}_j[$ such that

$$\rho_0 = r_j \Longrightarrow \rho(t; \rho_0, \theta_0) \le \hat{r}_j, \quad \text{for every } t \in [0, kT],\ \theta_0 \in [0, 2\pi[.$$

Hence (12) follows from (13), after integrating the second equation in (10).

In view of (11) and (12), the Poincaré–Birkhoff fixed point theorem implies the existence of at least two distinct points $(\rho_{k,j}^{(1)}, \theta_{k,j}^{(1)}), (\rho_{k,j}^{(2)}, \theta_{k,j}^{(2)}) \in]r_j, r[\times[0, 2\pi[$ such that, for $i = 1, 2$,

$$\begin{aligned} &\rho(kT; \rho_{k,j}^{(i)}, \theta_{k,j}^{(i)}) = \rho(0; \rho_{k,j}^{(i)}, \theta_{k,j}^{(i)}), \\ &\theta(kT; \rho_{k,j}^{(i)}, \theta_{k,j}^{(i)}) = \theta(0; \rho_{k,j}^{(i)}, \theta_{k,j}^{(i)}) + 2\pi j. \end{aligned} \tag{14}$$

Accordingly,

$$\zeta_{k,j}^{(i)}(t) = \eta(t; (\rho_{k,j}^{(i)} \cos\theta_{k,j}^{(i)}, \rho_{k,j}^{(i)} \sin\theta_{k,j}^{(i)})) + z(t)$$

is a kT-periodic solution to (5) such that, in view of Lemma 2.2, $\|\zeta_{k,j}^{(i)}\|_\infty \le R$.

The second relation in (2) is just a consequence of (14), using complex notation. Indeed, $\zeta_{k,j}^{(i)}(t) - z(t) = \rho(t; \rho_{k,j}^{(i)}, \theta_{k,j}^{(i)}) e^{i\theta(t; \rho_{k,j}^{(i)}, \theta_{k,j}^{(i)})}$ so that, with easy computations,

$$\begin{aligned} \mathrm{rot}_{kT}(\zeta_{k,j}^{(i)}) &= \frac{1}{2\pi i} \int_0^{kT} \frac{d(\zeta_{k,j}^{(i)}(t) - z(t))}{\zeta_{k,j}^{(i)}(t) - z(t)} \\ &= \frac{1}{2\pi i} \int_0^{kT} \left(\frac{d}{dt}\Big(\log(\rho(t; \rho_{k,j}^{(i)}, \theta_{k,j}^{(i)}))\Big) \right. \\ &\qquad \left. + i\theta'(t; \rho_{k,j}^{(i)}, \theta_{k,j}^{(i)}) \right) dt = j. \end{aligned}$$

From this information, we can conclude that (3) holds true. Indeed, the continuity of the winding number as a function $\zeta \mapsto \mathrm{rot}_{kT}(\zeta)$ implies that an upper bound exists for the winding numbers of any family of solutions uniformly bounded away from zero. Hence, the solutions $\{\zeta_{k,j}^{(i)}(t)\}_j$ $(i = 1, 2)$ cannot be bounded away from zero and the previous "elastic property" implies that they converge to zero uniformly.

4. Discussion and final remarks

We have considered a simple 2D model of the stirring of a fluid inside a cylindrical tank by an agitator or rod following a smooth (regular) periodic protocol. The associated mathematical model is a non-autonomous planar Hamiltonian system with a moving singularity (vortex). By a non-standard application of the Poincaré–Birkhoff theorem, we prove the existence of an infinite number of periodic and subharmonic solutions orbiting around the moving vortex. Intuitively, a vortex induces a singularity on the angular variable, twisting the flux around it, so the Poincaré–Birkhoff Theorem becomes a natural tool of potential application here, and in fact in more general contexts like arbitrary boundary domains [7,8] or the presence of multiple vortices [9,10]. Such extensions will be the subject of future works.

The physical relevance of periodic motions of particles in models of fluid mixing is nicely described in [3, Section 2]. Typically, the periodic orbits obtained by means of the Poincaré–Birkhoff theorem are inscribed in a classical KAM scenario and conforms the "skeleton" of global mixing properties. Stable periodic orbits produce "stability islands" around them, whereas hyperbolic (unstable) orbits provide invariant (stable and unstable) manifolds acting as flux barriers and, on the other hand, may support Smale horseshoes. Therefore, it is natural to conjecture that our periodic orbits should be encapsulated by quasiperiodic orbits (KAM tori) conforming the ubiquitous KAM dynamics around the vortex. It should be easy to find numerical evidence of this conjecture, although a rigorous mathematical proof may be very difficult. Anyway, the mathematical results presented in this paper have a clear physical interpretation: for any smooth periodic stirring protocol, there are infinitely many fluid particles rotating around the agitator and following it all the time ("sticky particles"). Note that this fact is independent of how fast the agitator is moved. In this sense, one can say that a discontinuous protocol (like the classical Aref blinking protocol) would be more convenient for an efficient mixing.

Acknowledgments

The first and second authors were partially supported by project MTM2011-23652, Ministerio de Ciencia e Innovación, Spain. The first author wishes to thank all the members of the Department of Applied Mathematics of the University of Granada for the warm hospitality.

References

[1] H. Aref, Stirring by chaotic advection, J. Fluid Mech. 143 (1984) 1–21.
[2] T. Carletti, A. Margheri, Measuring the mixing efficiency in a simple model of stirring: some analytical results and a quantitative study via frequency map analysis, J. Phys. A 39 (2006) 299–312.
[3] S. Wiggins, J.M. Ottino, Foundations of chaotic mixing, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 362 (2004) 937–970.
[4] J.-L. Thiffeault, M.D. Finn, E. Gouillart, T. Hall, Topology of chaotic mixing patterns, Chaos 18 (2008) 033123. http://dx.doi.org/10.1063/1.2973815. p. 8.
[5] W.Y. Ding, A generalization of the Poincaré–Birkhoff theorem, Proc. Amer. Math. Soc. 88 (1983) 341–346.
[6] C. Rebelo, A note on the Poincaré–Birkhoff fixed point theorem and periodic solutions of planar systems, Nonlinear Anal. 29 (1997) 291–311.
[7] P. Franzese, L. Zannetti, Advection by a point vortex in closed domains, Eur. J. Mech. B-Fluids 12 (1993) 1–24.
[8] L. Zannetti, P. Franzese, The non-integrability of the restricted problem of two vortices in closed domains, Physica D 76 (1994) 99–109.
[9] G. Boffetta, A. Celani, P. Franzese, Trapping of passive tracers in a point vortex system, J. Phys. A: Math. Gen. 29 (1996) 3749–3759.
[10] P. Boyland, M. Stremler, H. Aref, Topological fluid mechanics of point vortex motions, Physica D 175 (2003) 69–95.