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Periodic Solutions and Chaotic Dynamics in Forced Impact Oscillators*

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Abstract. It is shown that a periodically forced impact oscillator may exhibit chaotic dynamics on two symbols, as well as an infinity of periodic solutions. Two cases are considered, depending on if the impact velocity is finite or infinite. In the second case, the Poincaré map is well defined by continuation of the energy. The proof combines the study of phase-plane curves together with the "stretching-along-paths" notion.

Key words. impact oscillator, singular potential, chaotic dynamics, periodic solution

AMS subject classifications. 34C28, 54H20

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1. Introduction.

The presence of impacts in a physical process is a classical source of nonlinearity and complex behavior. Partially elastic impacts are important in mechanical engineering, for instance, in the modeling of pneumatic hammers, drilling machines, and other industrial devices, whereas elastic impacts play a key role in billiard dynamics and other related models appearing in different branches of theoretical physics like the Fermi–Ulam oscillator and its many variants. The importance of this topic is reflected in thousands of analytical, numerical, and experimental papers that can be consulted in the bibliographies of the monographs [4, 19, 43].

For our purposes, systems with elastic impacts can be classified into two big families:

- Systems exhibiting impacts at finite velocity: most of the examples considered in the cited monographs belong to this type, for instance, the offset impact oscillator or the bouncing ball on a massive oscillating table. In general, we consider the motion of a particle following the Newtonian equation

$$u'' + f(t,u) = 0,$$

where f is smooth and T-periodic with respect to t. Besides, it is assumed that the particle experiences impacts against a T-periodically oscillating wall or barrier $q(t)$. If the impacts are elastic, the restitution rule is

$$u(t_0) = q(t_0) \implies u'(t_0^+) = -u'(t_0^-) + 2q'(t_0).$$

Summing up, if the relative distance $x(t) = u(t) - q(t)$ between the particle and the

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wall is taken as the relevant coordinate, the impact system reads

$$\begin{cases} x'' + g(t,x) = 0 & \text{for } x(t) > 0, \\ x(t_0) = 0 \implies x'(t_0^+) = -x'(t_0^-), \end{cases}$$

where $g(t,x) = f(t, x + q(t)) + q''(t)$.

- Systems exhibiting impacts at infinite velocity: these types of impacts are characterized by the presence of a singularity on the potential that rules the motion of the particle. It is said that a potential has a singularity if it becomes infinite at a given point. If a particle is moving under a singular potential, an eventual collision with the singularity will have infinite velocity. Since the gravitational potential is singular at the origin, the n-body problem and related models from celestial mechanics are the most prominent examples of this family. To continue a collision orbit after the impact, the usual restitution rule by reflection of the impact velocity is not effective and an energy-based continuation is required, as we will see later in more detail.

In short, the families described above will be called *regular systems* and *singular systems*, respectively. In the available bibliography, both families have been studied in distinct ways. Our objective is to show a unified method of study that can be implemented in both contexts. Such a method provides a way to construct concrete examples of periodic forces such that the associated Poincaré section induces chaotic dynamics on two symbols. In an informal way, we say that a map induces chaotic dynamics on two symbols if there exists an invariant set Λ being semiconjugate to the Bernoulli shift, topologically transitive, and having infinitely many periodic points (see Definition 5.1 and Theorem 5.1 in the appendix). This definition of chaos has been used before by several authors in [1, 7, 8, 27, 42].

In the related literature, a variety of methods have been developed to detect analytically the presence of chaos in a concrete dynamical system such as the Mel'nikov or the Sil'nikov methods, the Conley-Wazewki theory, estimates of Lyapunov exponents, ergodicity, or mixing associated to some invariant measures, etc. (see [5, 7, 8, 17, 21, 32, 37, 42, 45]). In this paper we study the notion of chaos under the perspective of topological horseshoes; see [21] and [47]. This topological point of view enables us to give explicit conditions in our models without using small/large parameters or hyperbolicity conditions. It is important to observe that a topological approach has been used in a broad variety of problems with physical and biological meaning (see [15, 16, 27, 33, 38]). However, to our knowledge, this is the first time that such an approach has been applied in the context of problems with impacts.

The arguments contained in this paper will describe a general strategy that can be applied to a variety of models. Rather than formulate an abstract setting, we have preferred to focus on some concrete and basic examples to better illustrate the underlying ideas. A paradigmatic regular system is

$$\begin{cases} x'' + a^2 x = p(t) & \text{for } x(t) > 0, \\ x(t_0) = 0 \implies x'(t_0^+) = -x'(t_0^-), \end{cases}$$

where $a \in \mathbb{R}$. Section 2 is devoted to the construction of periodic forcing terms $p(t)$ leading to chaotic dynamics in the sense exposed above. Note that this model comprises two important models in the field of impact dynamics, namely, the impact harmonic oscillator (if $a \neq 0$) and the bouncing ball over an oscillating table (if $a = 0$). Both models are touchstones and have been studied in many papers; see, for instance, [3, 13, 14, 39, 20, 24, 29, 30, 46] and the references therein (this is just a personal selection of the authors among the huge number of papers treating this topic).

On the other hand, as an example of a singular system we will analyze the model equation

$$x'' = -\frac{1}{x^2} + p(t), \tag{1.1}$$

with $p(t)$ a continuous and T-periodic function. Equation (1.1) is a forced second-order equation with an attractive singularity at the origin and can be regarded as a forced one-dimensional Kepler problem. It was first considered by Lazer and Solimini in the pioneering work [23], where it is proved that (1.1) has a positive T-periodic solution if and only if the mean value of the forcing term p is positive. Later, it was proved in [6, 26] that such a T-periodic solution is unique and a global dynamics of saddle type is organized around it. Hence, the dynamical behavior of classical solutions is extremely simple and reduced to a dichotomy: a solution lying in the stable manifold tends to a periodic one and the rest of the solutions tend to the infinite or collide with the singularity.

In an eventual collision, the velocity of the solution becomes infinite, but the total energy remains finite. In this way, collisions can be regularized by continuation of the energy, as presented by Sperling in [40]. In a recent paper [31], Ortega has combined this argument of regularization with the Poincaré–Birkhoff theorem in order to prove the existence and multiplicity of periodic solutions with a prescribed number of impacts. Let us mention that in the related bibliography, many regularization techniques have been developed (see [10, 11]), as well as other notions of generalized solutions (see for instance, [2, Definitions 3.1 and 3.23] or also used in [9], [6, Definition 4.1], or [18, section 3]).

In this paper, we will continue the study of the impact dynamics of (1.1) initiated in [31]. Specifically, our objective is to construct concrete examples of periodic forces $p(t)$ in such a way that the associated Poincaré section (considering the regularization of collisions) induces chaotic dynamics on two symbols.

The rest of the paper is organized as follows. In section 2, the regular case is studied by means of an analysis of phase-plane curves combined with the linked twist map technique and the "stretching-along-paths" definition. In section 3, first the regularization process is formalized and the Poincaré map is rigorously defined for the singular equation (1.1). After that, the main results for this model are stated and proved. Finally, in section 4 some extensions and further comments are presented. For the reader's convenience, the explicit definition of chaotic dynamics and some necessary background is included in a separate appendix.

2. Regular systems.

2.1. Bouncing solutions and the Poincaré map.

Given a general system

$$\begin{cases} x'' + g(t,x) = 0 & \text{for } x(t) > 0, \\ \text{if } \; x(t_0) = 0 \implies x'(t_0^+) = -x'(t_0^-), \end{cases} \tag{2.1}$$

where g is T-periodic in the first variable and of class $\mathcal{C}^1$, we say that $u : \mathbb{R} \longrightarrow [0, +\infty[=: \mathbb{R}^+$ is a *bouncing solution* of (2.1) if the following conditions hold:

- $Z = \{t \in \mathbb{R} : u(t) = 0\}$ is discrete;

- for any interval $I \subset \mathbb{R} \backslash Z$, the function u is of class $\mathcal{C}^2(I)$ and satisfies the differential equation $x'' + g(t,x) = 0$;
- for each $t_0 \in Z$, there is a constant $v \geq 0$ so that

$$\lim_{t \to t_0^-} u'(t) = -v,$$
$$\lim_{t \to t_0^+} u'(t) = v.$$

Along this section we assume, without further mention, that the force $p(t)$ in our regular model

$$\begin{cases} x'' + a^2 x = p(t) & \text{for } x(t) > 0, \\ x(t_0) = 0 \implies x'(t_0^+) = -x'(t_0^-), \end{cases} \tag{2.2}$$

is continuous, T-periodic, and strictly negative. Under these conditions we can check that for each initial condition $(x_0, v_0) \in X :=]0, \infty[\times \mathbb{R}$, there is a unique bouncing solution (defined on $\mathbb{R}$) associated with (2.2), namely, $x(t; x_0, v_0)$, satisfying that

$$x(0; x_0, v_0) = x_0,$$
$$x'(0; x_0, v_0) = v_0.$$

Note that as $p(t) < 0$, the set Z is always nonempty.

Given $x(t; x_0, v_0)$ a bouncing solution of (2.2), there is a time $t_1 := t_1(x_0, v_0) > 0$ and a constant $v_1 := v_1(x_0, v_0) \geq 0$, such that

$$x(t; x_0, v_0) > 0 \quad \text{for all } \; t \in [0, t_1[,$$
$$x(t_1; x_0, v_0) = 0,$$
$$\lim_{t \longrightarrow t_1^-} x'(t; x_0, v_0) = -v_1.$$

In this framework, an elementary argument of continuous dependence enables us to conclude that the map

$$\begin{aligned} X &\longrightarrow \mathbb{R}^2, \\ (x_0, v_0) &\mapsto (t_1(x_0, v_0), v_1(x_0, v_0)), \end{aligned} \tag{2.3}$$

is continuous (the same conclusion holds working with the previous collision). Following [35], it will be useful to introduce a successor map for (2.2). Specifically, given $\tau \in \mathbb{R}$ and $v \in \mathbb{R}^+$, let us denote $u(t; \tau, v)$ the unique bouncing solution of (2.2) satisfying the initial conditions

$$u(\tau; (\tau, v)) = 0,$$
$$u'(\tau^+; (\tau, v)) = v \geq 0.$$

If we assume that $\tau_1 > 0$ is the time of the next collision and v_1 is the corresponding velocity, then it can be proved as in [35] that the successor map

$$\begin{aligned} \mathcal{S} : \mathbb{R} \times \mathbb{R}^+ &\longrightarrow \mathbb{R} \times \mathbb{R}^+, \\ \mathcal{S}(\tau, v) := (\mathcal{S}_1(\tau, v), \mathcal{S}_2(\tau, v)) &= (\tau_1, v_1), \end{aligned} \tag{2.4}$$

is continuous, injective, and $\mathcal{S}_1(\tau, v)$ is strictly increasing with respect to the velocity. On the other hand, the map

$$(t, \tau, v) \in \mathcal{H} \mapsto (u(t; \tau, v), u'(t; \tau, v))$$

is continuous, where $\mathcal{H} = \{(t, \tau, v) : \tau < t < \tau_1\}$. Consequently, using the continuity of (2.3) and (2.4) we deduce that the function

$$(t; x_0, v_0) \in \mathcal{G} \longrightarrow (x(t; x_0, v_0), x'(t; x_0, v_0))$$

is continuous with $\mathcal{G} = \{(t, x_0, v_0) \in \mathbb{R} \times X : \ x(t; x_0, v_0) \neq 0\}$. Consider the topological space $(\Delta, \Im)$ where

$$\Delta = X \cup \{(0, v) : v \in \mathbb{R}^+\}.$$

The definition of the topology in Δ is as follows. A sequence $(x_n, y_n) \in \Delta$ converges to $(x_0, v_0) \in \Delta$ with $x_0 > 0$ if each coordinate converges in the classical sense. In the case $x_0 = 0$, (x_n, y_n) converges to $(0, v_0)$ if

$$x_n \longrightarrow 0,$$

$$|y_n| \longrightarrow v_0.$$

Clearly, $X \subset \Delta$ is an open set in $(\Delta, \Im)$. This topological space is natural under the notion of a bouncing solution. Indeed, take a solution, namely, $x(t; x_0, v_0)$, with a first impact at time T (assume that $t_1(x_0, v_0) = T$). In a neighborhood of (x_0, v_0) (under the Euclidean distance), some bouncing solutions have velocity close to $v_1(x_0, v_0)$ whereas other ones have velocity close to $-v_1(x_0, v_0)$ at T (it depends if the impact is before/after T). The key property in our topological space is that we "identify" these two behaviors so this situation does not produce any discontinuity.

Putting all the information together, we can prove that the map

$$\mathcal{P} : X \longrightarrow \Delta$$

given by

$$\mathcal{P}(x_0, v_0) = \begin{cases} (0, v_1(x_0, v_0)) & \text{if } t_1(x_0, v_0) = T, \\ (0, \mathcal{S}_2^j(t_1(x_0, v_0), v_1(x_0, v_0))) & \text{if } \mathcal{S}_1^j(t_1(x_0, v_0), v_1(x_0, v_0)) = T, \\ (x(T; x_0, v_0), x'(T; x_0, v_0)) & \text{otherwise,} \end{cases} \tag{2.5}$$

is continuous and injective. In the previous expression, $\mathcal{S}^j = (\mathcal{S}_1^j, \mathcal{S}_2^j)$ denotes the jth iterate of the successor map. Notice that $\Omega = \mathcal{P}^{-1}(X)$ is an open set and given an initial condition (x_0, v_0) in Ω, $x(T; x_0, v_0) \neq 0$. Throughout this section, we refer to this map as the Poincaré map of (2.2). Note that this is not the classical notion of Poincaré map commonly used in the literature of dynamical systems because the classical flux is extended by considering bouncing solutions.

2.2. Chaotic dynamics of (2.2). In this subsection we focus our attention on the construction of T-periodic forces producing chaotic dynamics in (2.2). By chaotic dynamics in (2.2) we understand that the map

$$\begin{aligned} \mathcal{P} : \Omega &\longrightarrow X, \\ (x_0, v_0) &\mapsto (x(T; x_0, v_0), x'(T; x_0, v_0)), \end{aligned} \tag{2.6}$$

has chaotic dynamics on two symbols; see Definition 5.1. In view of our definition of chaos, our results provide us criteria ensuring the existence of infinitely many subharmonics. Our method of proof consists of applying the results in the appendix to a geometrical configuration similar to the linked twists maps (LTM) (see [41, 25, 34, 5, 12, 45] for abstract results concerning LTM and [28, 44, 33] for different contexts of application).

The plan of this subsection is as follows. First, the class of forces is constructed and then we state and prove the main theorem. For convenience, we assume that $a^2 = 1$. The rest of the cases, including the bouncing ball $a = 0$, can be studied in a similar way.

We begin with a brief analysis of the geometry of orbits. Fix two constants $p_2 < p_1 < 0$ and consider the equations

$$x'' + x = p_1, \tag{2.7}$$

$$x'' + x = p_2. \tag{2.8}$$

Fix two nontrivial intervals $[H_1, h_1]$ and $[H_2, h_2]$ satisfying that, for all $h_1^* \in [H_1, h_1]$ and $h_2^* \in [H_2, h_2]$,

(C1) $0 < h_1^* < h_2^*$,

(C2) $p_2 + \sqrt{p_2^2 + 2h_2^*} < p_1 + \sqrt{p_1^2 + 2h_1^*}$.

Such properties imply some geometrical properties of the energy curves

$$\gamma_1 = \{(x, y) \in X : \mathcal{E}_1(x, y) = h_1^*\},$$
$$\gamma_2 = \{(x, y) \in X : \mathcal{E}_2(x, y) = h_2^*\},$$

where $\mathcal{E}_i(x, y) = \frac{y^2}{2} + \frac{x^2}{2} - p_i x$. Specifically, (C1) implies that for $\delta > 0$ small enough, $|y_2| > |y_1|$ provided $(x, y_1) \in \gamma_1$, $(x, y_2) \in \gamma_2$, and $0 < x < \delta$. On the other hand, condition (C2) implies that $x_1 > x_2$, where $(x_1, 0) = \gamma_1 \cap \{(x, 0) : x \in \mathbb{R}\}$ and $(x_2, 0) = \gamma_2 \cap \{(x, 0) : x \in \mathbb{R}\}$. See Figure 1. Now we fix four constants

$$\widetilde{H}_1 < \widetilde{h}_1 < \widetilde{H}_2 < \widetilde{h}_2$$

with $\widetilde{H}_1, \widetilde{h}_1 \in]H_1, h_1[$ and $\widetilde{H}_2, \widetilde{h}_2 \in]H_2, h_2[$. Define the topological rectangles

$$\mathcal{D}_1 := \{(x, y) \in X : \widetilde{H}_1 \le \mathcal{E}_1(x, y) \le \widetilde{h}_1, H_2 \le \mathcal{E}_2(x, y) \le h_2, y > 0\},$$
$$\mathcal{D}_2 := \{(x, y) \in X : H_1 \le \mathcal{E}_1(x, y) \le h_1, \widetilde{H}_2 \le \mathcal{E}_2(x, y) \le \widetilde{h}_2, y > 0\}.$$

See Figure 2.

Take $d_1 > 0$ such that the d_1-neighborhood of $\mathcal{D}_1^l \cup \mathcal{D}_1^r$ (see Figure 2, right), with

$$\mathcal{D}_1^l := \{(x, y) \in \mathcal{D}_1 : H_2 = \mathcal{E}_2(x, y)\},$$
$$\mathcal{D}_1^r := \{(x, y) \in \mathcal{D}_1 : h_2 = \mathcal{E}_2(x, y)\},$$

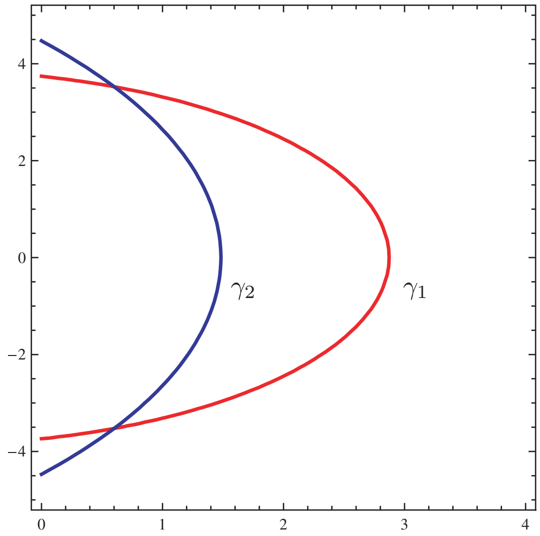

Figure 1. *The configuration of orbits* γ_1, γ_2.

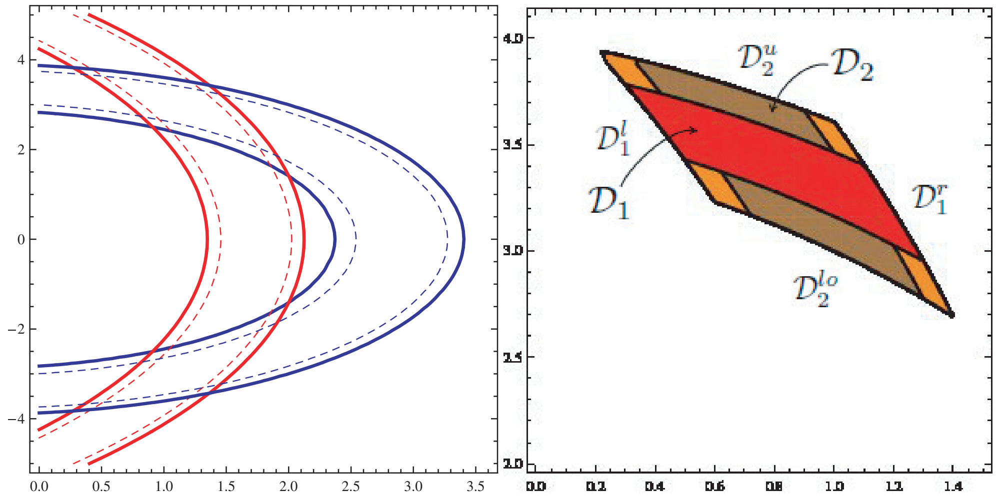

Figure 2. *Left: The "linked rings." The dashed lines denote the energy levels* $\widetilde{H}_1, \widetilde{h}_1, \widetilde{H}_2, \widetilde{h}_2$ *and the continuous lines the energy levels* H_1, h_1, H_2, h_2. *Right:* $\mathcal{D}_1, \mathcal{D}_2$, *and the distinguished boundaries. Note that this figure is a "zoom" of the upper intersection of the "rings."*

does not intersect the set $\mathcal{D}_2$ and the d_1-neighborhood of $\mathcal{D}_1$ does not intersect the sets

$$\{(x, y) \in X : h_1 = \mathcal{E}_1(x, y)\},$$
$$\{(x, y) \in X : H_1 = \mathcal{E}_1(x, y)\}.$$

Analogously, we pick $d_2 > 0$ such that the d_2-neighborhood of $\mathcal{D}_2^u \cup \mathcal{D}_2^{lo}$(see Figure 2, right), with

$$\mathcal{D}_2^u := \{(x,y) \in \mathcal{D}_2 : h_1 = \mathcal{E}_1(x,y)\},$$
$$\mathcal{D}_2^{lo} := \{(x,y) \in \mathcal{D}_2 : H_1 = \mathcal{E}_1(x,y)\},$$

does not intersect the set $\mathcal{D}_1$ and the d_2-neighborhood of $\mathcal{D}_2$ does not intersect the sets

$$\{(x,y) \in X : h_2 = \mathcal{E}_2(x,y)\},$$
$$\{(x,y) \in X : H_2 = \mathcal{E}_2(x,y)\}.$$

In the definition of the forcing $p(t)$, the following elementary estimate will be useful.

Lemma 2.1. *Fix two positive numbers d, M, and a compact set $K \subset X$. Then there exists a constant τ, depending on d, M, and K, such that for every continuous function satisfying $|p(t)| \le M$ and for all $(x_0, y_0) \in K$,*

$$|x(t;(s_0,x_0,y_0)) - x_0|^2 + |x'(t;(s_0,x_0,y_0)) - y_0|^2 < d \tag{2.9}$$

for all $t \in [s_0 - \tau, s_0 + \tau]$ ($x(t;(s_0,x_0,y_0))$ is the bouncing solution of (2.2) with $(x(s_0;(s_0,x_0,y_0)), x'(s_0;(s_0,x_0,y_0))) = (x_0,y_0)$).

Now, we have all the ingredients for the definition of $p(t)$. At a first stage, the previous lemma is applied twice. Specifically, by taking $d = d_1$, $M = |p_2|$, $K = \mathcal{D}_1$, we obtain a parameter $\tau_1 > 0$ satisfying condition (2.9); then, pick $d = d_2$, $M = |p_2|$, and $K = \mathcal{D}_2$ and obtain a parameter $\tau_2 > 0$ satisfying condition (2.9).

Next, given two constants T_1 and T_2 we define a $(T_1+T_2+\tau_1+\tau_2)$-periodic function given by

$$p(t) = \begin{cases} p_1 & \text{if } 0 \le t < T_1, \\ \widetilde{p}_1(t) & \text{if } T_1 \le t < T_1 + \tau_1, \\ p_2 & \text{if } T_1 + \tau_1 \le t < T_1 + \tau_1 + T_2, \\ \widetilde{p}_2(t) & \text{if } T_1 + \tau_1 + T_2 \le t < T_1 + \tau_1 + T_2 + \tau_2, \end{cases} \tag{2.10}$$

so that $|\widetilde{p}_i(t)| \le |p_2|$ and the function $p(t)$ is continuous. Our purpose is to prove that, for suitable choices of the parameters T_1 and T_2, (2.2) with the previous force induces chaotic dynamics. Before studying this result, we introduce some notation. Consider ϕ_1 the Poincaré map associated with (2.7) at time T_1 (see (2.5)). For $\Omega_1 = \phi_1^{-1}(X)$ we have that

$$\begin{gathered} \phi_1 : \Omega_1 \longrightarrow X, \\ (x_0, v_0) \mapsto (x(T_1; x_0, v_0), x'(T_1; x_0, v_0)), \end{gathered} \tag{2.11}$$

where, for convenience, $(x(T_1;x_0,v_0), x'(T_1;x_0,v_0))$ is the bouncing solution of (2.7) with initial data at (x_0, v_0). Define ϕ_2 and Ω_2 in a similar way with (2.8) and T_2. On the other hand, in $\mathcal{U}$, a neighborhood of $\mathcal{D}_1 \cup \mathcal{D}_2$, by Lemma 2.1 and the definition of τ_1 and τ_2, we can define two maps,

$$h_1 : \mathcal{U} \longrightarrow X,$$
$$h_1(x_0,v_0) := (x(T_1+\tau_1;(T_1,x_0,v_0)), x'(T_1+\tau_1;(T_1,x_0,v_0))),$$
$$h_2 : \mathcal{U} \longrightarrow X,$$
$$\begin{gathered} h_2(x_0,v_0) := (x(T_1+\tau_1+T_2+\tau_2;(T_1+\tau_1+T_2,x_0,v_0)), \\ x'(T_1+\tau_1+T_2+\tau_2;(T_1+\tau_1+T_2,x_0,v_0))), \end{gathered}$$

where $(x(t;(t_0,x_0,v_0)),x'(t;(t_0,x_0,v_0)))$ refers to the bouncing solution of (2.2) with force (2.10). Note that, by expression (2.10), if $(x,v)\in\Omega_1$, $\phi_1(x,v)\in\mathcal{U}$, $h_1(\phi_1(x,v))\in\Omega_2$, and $\phi_2(h_1(\phi_1(x,v)))\in\mathcal{U}$, then

$$\mathcal{P}(x,v)=h_2\circ\phi_2\circ h_1\circ\phi_1(x,v), \tag{2.12}$$

where $\mathcal{P}$ is the Poincaré map of (2.2) with force (2.10).

Theorem 2.1. *There exist $T_1^*,T_2^*>0$ with the following property: for every function $p(t)$ given in (2.10) with parameters $T_1>T_1^*$ and $T_2>T_2^*$, (2.2) has chaotic dynamics.*

Proof. The proof of this theorem is divided into five steps.

Step 1. *Stretching property for* $[0,T_1]$.

In this step we study system (2.2) when $p(t)=p_1$ (as we pointed out before, we always assume that $a^2=1$). First we observe that given a point

$$(x_0,v_0)\in\{(x,y)\in X:\mathcal{E}_1(x,y)=h_1\},$$

the bouncing solution associated with this initial condition is periodic with period $\mathcal{S}_1(0,\sqrt{2h_1})$ ($\mathcal{S}=(\mathcal{S}_1,\mathcal{S}_2)$ is the successor map associated with (2.7); see (2.4)). We recall that $\mathcal{S}_1(x,v)$ is strictly increasing with respect to the velocity. Therefore,

$$P_1=\mathcal{S}_1\left(0,\sqrt{2\widetilde{H}_1}\right)<P_2=\mathcal{S}_1\left(0,\sqrt{2\widetilde{h}_1}\right). \tag{2.13}$$

Now we define

$$T_1^*=\frac{5P_1P_2}{P_2-P_1},$$

and take a constant T_1 such that $T_1>T_1^*$. Property (2.13) implies the dynamical behavior for ϕ_1 illustrated in Figure 3.

Our aim is to find two disjoint compact sets $\mathcal{K}_1,\mathcal{K}_0\subset\mathcal{D}_1$ such that (see Definition 5.3)

$$(\mathcal{K}_i,\phi_1):\widetilde{\mathcal{D}_1^0}\Leftrightarrow\!\!\!\!\!\!\!\!\longrightarrow\widetilde{\mathcal{D}_1}\qquad\text{for } i=0,1,$$

where $\phi_1:\Omega_1\longrightarrow X$ and $\widetilde{\mathcal{D}_1^0}=(\mathcal{D}_1,(\mathcal{D}_1^0)^-)$ with

$$(\mathcal{D}_1^0)^-=\{(x,y)\in\mathcal{D}_1:\widetilde{H}_1=\mathcal{E}_1(x,y)\}\cup\{(x,y)\in\mathcal{D}_1:\widetilde{h}_1=\mathcal{E}_1(x,y)\},$$

and $\widetilde{\mathcal{D}}_1=(\mathcal{D}_1,(\mathcal{D}_1)^-)$, where $(\mathcal{D}_1)^-=(\mathcal{D}_1^l\cup\mathcal{D}_1^r)$. Indeed, first consider m_* the smallest integer satisfying

$$\frac{T_1}{P_2}<m_* \tag{2.14}$$

and m^* the largest integer satisfying

$$\frac{T_1}{P_1}>m^*. \tag{2.15}$$

From the choice of T_1 we deduce that $m^*-m_*>5$. Observe that property (2.14) implies that given any initial condition $(x_0,y_0)\in\{(x,y)\in X:\mathcal{E}_1(x,y)=\widetilde{h}_1\}$, solution

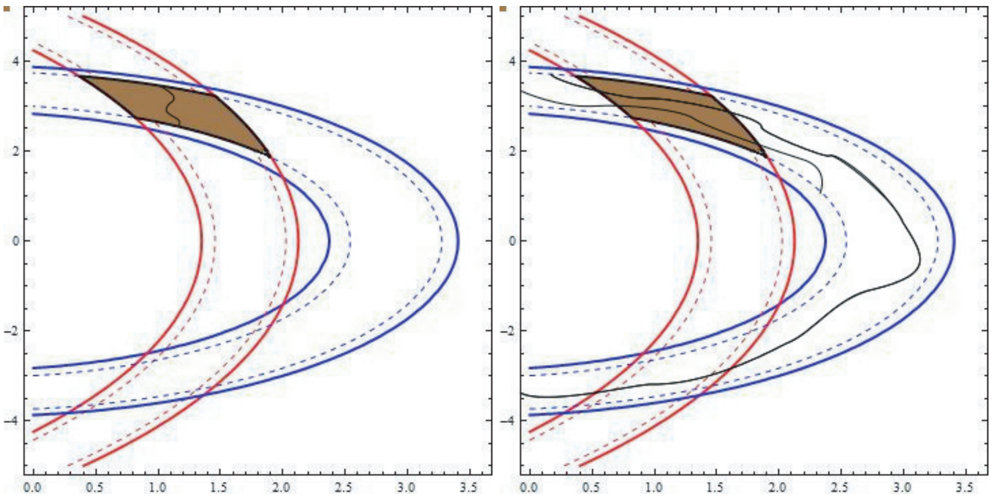

Figure 3. *Left: A path* $\gamma : [0,1] \longrightarrow \mathcal{D}_1$ *joining the sides* $\{(x,y) \in \mathcal{D}_1 : \mathcal{E}_1(x,y) = \widetilde{h}_1\}$ *and* $\{(x,y) \in \mathcal{D}_1 : \mathcal{E}_1(x,y) = \widetilde{H}_1\}$*. Right: Illustration of* $\phi_1(\Omega_1 \cap \gamma(t))$*. Note that this effect is caused by the "twist" property* (2.13).

$(x(t;x_0,y_0), x'(t;x_0,y_0))$ finds the point (x_0,y_0) at most m_* times for the interval $[0,T_1]$. Similarly, property (2.15) says that for all initial conditions $(x_0,y_0) \in \{(x,y) \in X : \mathcal{E}_1(x,y) = \widetilde{H}_1\}$, solution $(x(t;x_0,y_0), x'(t;x_0,y_0))$ finds the point (x_0,y_0) at least m^* times for the interval $[0,T_1]$. After that, we define the compact sets

$$\mathcal{K}_1 = \{(x,y) \in \mathcal{D}_1 : \frac{T_1}{\mathcal{S}_1(0,\sqrt{2\mathcal{E}_1(x,y)})} \in [m_*, m_*+2]\},$$
$$\mathcal{K}_0 = \{(x,y) \in \mathcal{D}_1 : \frac{T_1}{\mathcal{S}_1(0,\sqrt{2\mathcal{E}_1(x,y)})} \in [m^*-2, m^*]\}.$$

Now we focus our attention on the stretching property (see Definition 5.3). Take

$$\gamma : [0,1] \longrightarrow \mathcal{D}_1,$$

a path with

$$\gamma(0) \in \{(x,y) \in \mathcal{D}_1 : \mathcal{E}_1(x,y) = \widetilde{H}_1\},$$
$$\gamma(1) \in \{(x,y) \in \mathcal{D}_1 : \mathcal{E}_1(x,y) = \widetilde{h}_1\}.$$

Using that $\mathcal{S}_1(0,\sqrt{2\mathcal{E}_1(\gamma(t))})$ is continuous, we deduce that there exist two disjoint subintervals $[A_0, A_0']$ and $[A_1, A_1']$ such that

$$\frac{T_1}{\mathcal{S}_1(0,\sqrt{2\mathcal{E}_1(\gamma(s))})} \in [m_*, m_*+2] \qquad \text{for } s \in [A_0, A_0'],$$

with $\frac{T_1}{\mathcal{S}_1(0,\sqrt{2\mathcal{E}_1(\gamma(A_0))})} = m_*$ and $\frac{T_1}{\mathcal{S}_1(0,\sqrt{2\mathcal{E}_1(\gamma(A_0'))})} = m_*+2$ and

$$\frac{T_1}{\mathcal{S}_1(0,\sqrt{2\mathcal{E}_1(\gamma(s))})} \in [m^*-2, m^*] \qquad \text{for } s \in [A_1, A_1'],$$

with $\frac{T_1}{\mathcal{S}_1(0,\sqrt{2\mathcal{E}_1(\gamma(A_1))})} = m^*$ and $\frac{T_1}{\mathcal{S}_1(0,\sqrt{2\mathcal{E}_1(\gamma(A_1'))})} = m^*-2$. Now we concentrate on the interval $[A_0, A_0']$. Clearly, the solutions with initial conditions at $\gamma(A_0)$ and $\gamma(A_0')$ have exactly m_* and (m_*+2) collisions, respectively, on the interval $[0, T_1]$. This property implies that there exists a subinterval $[\widetilde{S}_0, \widetilde{S}_0'] \subset [A_0, A_0']$ such that

$$\begin{aligned}\Big(x(T_1,\gamma(\widetilde{S}_0)), x'(T_1,\gamma(\widetilde{S}_0))\Big) &\in \{(x,y) : x < \min\{x_0 : (x_0,y) \in \mathcal{D}_1\}\},\\ \Big(x(T_1,\gamma(\widetilde{S}_0')), x'(T_1,\gamma(\widetilde{S}_0'))\Big) &\in \{(x,y) : y = 0\},\\ (x(T_1,\gamma(s), x(T_1,\gamma(s))) &\in \{(x,y) : y \geq 0\}\end{aligned}$$

for all $s \in [\widetilde{S}_0, \widetilde{S}_0']$. Finally, we easily obtain the desired subinterval $[S_0, S_0']$. Observe that previously we have used that

$$\widetilde{H}_1 \leq \mathcal{E}_1(x(T_1;\gamma(t)), x'(T_1;\gamma(t))) \leq \widetilde{h}_1$$

for all $t \in [0,1]$.

Step 2. *Behavior in the interval* $[T_1, T_1+\tau_1]$.

By the definition of $p(t)$, the following property holds as a direct consequence of Lemma 2.1: For all continuous path $\gamma(t) : [0,1] \longrightarrow \mathcal{D}_1$ with $\gamma(0) \in \mathcal{D}_1^l$, $\gamma(1) \in \mathcal{D}_1^r$, there exists a subinterval $[R_0, R_0'] \subset [0,1]$ so that the curve

$$\beta(t) = (x(T_1+\tau_1;(T_1,\gamma(t))), x'(T_1+\tau_1;(T_1,\gamma(t))))$$

satisfies that

$$\begin{aligned}\beta([R_0, R_0']) &\subset \mathcal{D}_2,\\ \beta(R_0) &\in \{(x,y) \in \mathcal{D}_2 : \mathcal{E}_2(x,y) = \widetilde{H}_2\},\\ \beta(R_0') &\in \{(x,y) \in \mathcal{D}_2 : \mathcal{E}_2(x,y) = \widetilde{h}_2\}.\end{aligned}$$

This property is illustrated in Figure 4.

To prove this property, we use that

$$\beta(t) \in \{(x,y) \in X : H_1 \leq \mathcal{E}_1(x,y) \leq h_1, y > 0\}$$

and $\beta(0)$ and $\beta(1)$ belong to different connected components of

$$X \backslash \{(x,y) : \widetilde{H}_2 \leq \mathcal{E}_2(x,y) \leq \widetilde{h}_2\}.$$

Step 3. *Stretching property in the interval* $[T_1+\tau_1, T_1+\tau_1+T_2]$.

In this step we study (2.2) when $p(t) = p_2$. Consider the corresponding successor map $\widetilde{\mathcal{S}}$ associated with this equation. As in the first step, we define

$$T_2^* = \frac{5\widetilde{P}_1\widetilde{P}_2}{\widetilde{P}_2 - \widetilde{P}_1},$$

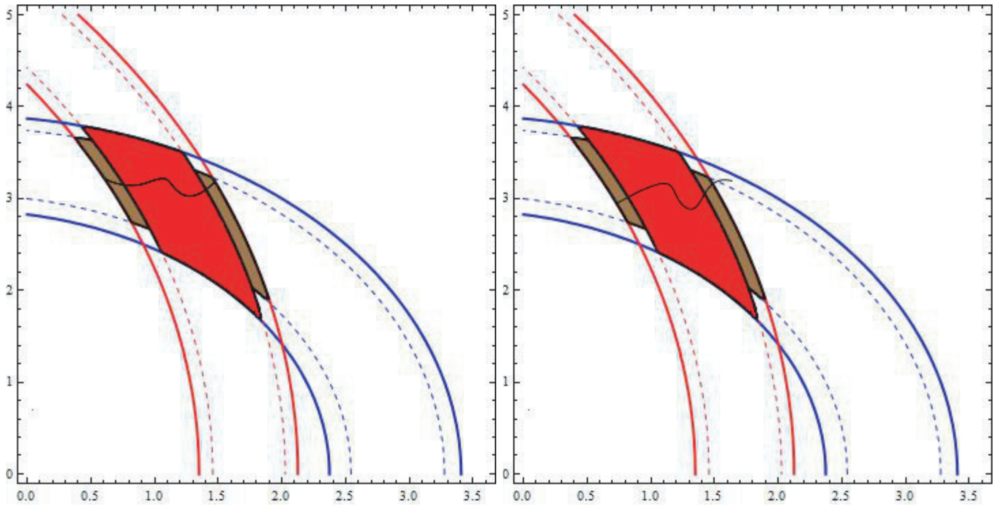

Figure 4. *Left: A path* $\gamma : [0,1] \longrightarrow \mathcal{D}_1$ *joining the sides* $\{(x,y) \in \mathcal{D}_1 : H_2 = \mathcal{E}_2(x,y)\}$ *and* $\{(x,y) \in \mathcal{D}_1 : h_2 = \mathcal{E}_2(x,y)\}$ *. Right: Illustration of* $\beta(t)$*. Note that by Lemma 2.1,* $\beta(0)$*,* $\beta(1)$ *do not touch* $\mathcal{D}_2$ *and* $\beta(t)$ *does not touch* $\{(x,y) \in X : H_1 = \mathcal{E}_1(x,y)\}$ *and* $\{(x,y) \in X : h_1 = \mathcal{E}_1(x,y)\}$*.*

where $\widetilde{P}_1 = \widetilde{\mathcal{S}}_1(0, \sqrt{2\widetilde{H}_2}) < \widetilde{P}_2 = \widetilde{\mathcal{S}}_1(0, \sqrt{2\widetilde{h}_2})$. After that, fix a constant $T_2 > T_2^*$ and consider $\phi_2 : \Omega_2 \longrightarrow X$, where ϕ_2 is the Poincaré map associated with (2.2) (for $p(t) = p_2$) at time T_2 and $\Omega_2 = \phi_2^{-1}(X)$. Reasoning as in Step 1, we can prove that for every continuous path

$$\gamma : [0,1] \longrightarrow \mathcal{D}_2$$

with

$$\gamma(0) \in \{(x,y) \in \mathcal{D}_2 : \mathcal{E}_2(x,y) = \widetilde{H}_2\}$$

and

$$\gamma(1) \in \{(x,y) \in \mathcal{D}_2 : \mathcal{E}_2(x,y) = \widetilde{h}_2\},$$

there exists a subinterval $[M_0, M_0']$ satisfying that

$$\phi_2(\gamma([M_0, M_0'])) \subset \mathcal{D}_2$$

with $\phi_2(\gamma(M_0)) \in \mathcal{D}_2^u$ and $\phi_2(\gamma(M_0')) \in \mathcal{D}_2^{lo}$.

Step 4. *Behavior in the interval* $[T_1 + T_2 + \tau_1, T_1 + T_2 + \tau_1 + \tau_2]$.

Reasoning as in Step 2, we obtain that for all continuous path $\gamma(t) : [0,1] \longrightarrow \mathcal{D}_2$ with $\gamma(0) \in \mathcal{D}_2^u$, $\gamma(1) \in \mathcal{D}_2^{lo}$, there exists a subinterval $[C_0, C_0'] \subset [0,1]$ so that the curve

$$\beta(t) = (x(T_1 + \tau_1 + T_2 + \tau_2; (T_1 + \tau_1 + T_2, \gamma(t))), x'(T_1 + \tau_1 + \tau_2 + T_2; (T_1 + \tau_1 + T_2, \gamma(t))))$$

satisfies that

$$\beta([C_0, C_0']) \subset \mathcal{D}_1,$$
$$\beta(C_0) \in \{(x,y) \in \mathcal{D}_1 : \mathcal{E}_1(x,y) = \widetilde{H}_1\},$$
$$\beta(C_0') \in \{(x,y) \in \mathcal{D}_1 : \mathcal{E}_1(x,y) = \widetilde{h}_1\}.$$

Step 5. *Conclusion*

Putting all the information together we deduce that $\mathcal{P}$ with $\mathcal{P}$ the Poincaré map associated with (2.2) with function $p(t)$ given by (2.10) has the properties

$$(\mathcal{K}_i, \mathcal{P}) : \widetilde{\mathcal{D}_1^0} \Leftrightarrow\!\!\!\rightarrow \widetilde{\mathcal{D}_1^0}$$

for $i = 0, 1$. To prove this claim, we use the previous steps and (2.12). Finally we apply Theorem 5.2. ■

3. Singular systems.

3.1. Bouncing solutions and the Poincaré map.

Consider the equation

$$x'' = -\frac{1}{x^2} + p(t), \tag{3.1}$$

with $p(t)$ a periodic function of class $\mathcal{C}^1$. Let us denote by $(x_c(t; x_0, v_0),\ x'_c(t; x_0, v_0))$ the maximal solution of (3.1) satisfying the initial condition

$$(x_c(0; x_0, v_0), x'_c(0; x_0, v_0)) = (x_0, v_0) \in\ X :=]0, +\infty[\times\mathbb{R}.$$

As was observed in the introduction, many solutions are not defined on $\mathbb{R}$. However, in such a case we have a nice property (see section 2 in [31]). Specifically, if the maximal interval of definition of $(x_c(t; x_0, v_0),\ x'_c(t; x_0, v_0))$ is $I =]t_0, t_1[$ with $t_0 > -\infty$ (resp., $t_1 < \infty$), then

(P1) $\lim_{t\to t_0^+} x_c(t; x_0, v_0) = 0$ (resp., $\lim_{t\to t_1^-} x_c(t; x_0, v_0) = 0$),

(P2) $\lim_{t\to t_0^+} x'_c(t; x_0, v_0) = +\infty$ (resp., $\lim_{t\to t_1^-} x'_c(t; x_0, v_0) = -\infty$),

(P3) by defining

$$h(t; x_0, v_0) := \frac{x'_c(t; x_0, v_0)^2}{2} - \frac{1}{x_c(t; x_0, v_0)},$$

there exists a constant h_0 such that $\lim_{t\to t_0^+} h(t; x_0, v_0) = h_0$ (resp., there exists a constant h_1 such that $\lim_{t\to t_1^-} h(t; x_0, v_0) = h_1$).

From a physical point of view, the previous identities can be interpreted in the following way. If the solution associated with the particle motion is not defined for all time then the particle has a collision at a finite time and the energy function at that instant is finite and well defined. On the other hand, from a mathematical point of view, the previous properties allow us to define a natural notion of a generalized or bouncing solution in (3.1).

Definition 3.1. *A bouncing solution of* (3.1) *is a continuous function* $u : \mathbb{R} \longrightarrow [0, \infty[$ *satisfying*

- $Z = \{t \in \mathbb{R} : u(t) = 0\}$ *is discrete;*
- *for any interval* $I \subset \mathbb{R}\backslash Z$*, the function* u *is of class* $\mathcal{C}^2(I)$ *and satisfies* (3.1) *on* I*;*
- *for each* $t_0 \in Z$*, the limit*

$$\lim_{t\to t_0} \frac{1}{2}u'(t)^2 - \frac{1}{u(t)} \tag{3.2}$$

exists.

There are some remarks to be made concerning the previous definition. Of course, a classical solution defined in $\mathbb{R}$ is a bouncing solution with $Z = \emptyset$. Limit (3.2) is taken from both sides of t_0. Therefore, the energy function must be preserved at the collision. As was mentioned in the introduction, this notion of collision regularization by continuation of the energy is due to Sperling [40]. For other techniques of extensions after collision in celestial dynamics, one can consult [9] and the references in this paper.

As proved in Proposition 3.1 in [31], an advantage of the notion of a bouncing solution is that for all initial conditions $(x_0, v_0) \in X$, there exists a unique bouncing solution denoted by $(x(t; x_0, v_0),\, x'(t; x_0, v_0))$ so that

$$(x(0; x_0, v_0), x'(0; x_0, v_0)) = (x_0, v_0).$$

Observe that $x(t; x_0, v_0)$ is defined for all $t \in \mathbb{R}$, $x'(t; x_0, v_0)$ is defined for all $t \in \mathbb{R}\backslash Z$, and given $t_0 \in Z$,

$$\lim_{t \to t_0^-} x'(t; x_0, v_0) = -\infty,$$
$$\lim_{t \to t_0^+} x'(t; x_0, v_0) = +\infty.$$

These last properties are a direct consequence of (P2) and the second condition in Definition 3.1.

For illustrative purposes, we study this notion of solution in (3.1) when $p(t)$ is a negative constant. Indeed, if $p < 0$, a simple phase portrait analysis shows that each classical solution has a bounded maximal interval I and stays in the curve

$$\beta_h = \left\{ (x, y) \in X : \frac{y^2}{2} - \frac{1}{x} - px = h \right\}$$

with $h \in \mathbb{R}$. In this scenario, if $I =]t_0, t_1[$ is the maximal interval of the classical solution $(x_c(t; x_0, v_0),\, x_c'(t; x_0, v_0))$, then $(x(t; x_0, v_0), x'(t; x_0, v_0)) := (x_c(t - n(t_1 - t_0); x_0, v_0),\, x_c'(t - n(t_1 - t_0); x_0, v_0))$, where $t \in]t_0 + n(t_1 - t_0), t_1 + n(t_1 - t_0)[$.

After this discussion, we define the Poincaré map associated with (3.1) by using the notion of a bouncing solution. For it, we need the following result.

Lemma 3.1. *Consider the set*

$$\mathcal{B} := \{(x_0, v_0) \in]0, \infty[\times \mathbb{R} : (x_c(t; x_0, v_0), x_c'(t; x_0, v_0)) \text{ has}$$
$$\text{a maximal interval }]t_0, t_1[\text{ with } t_1 < \infty\}.$$

Then $\mathcal{B}$ is open and the map

$$\begin{aligned} \mathcal{B} &\longrightarrow \mathbb{R}^2, \\ (x_0, v_0) &\mapsto (t_1, h_1), \end{aligned} \tag{3.3}$$

is continuous, where t_1 is the time of the first collision and h_1 is the energy at that instant of the solution $(x_c(t; x_0, v_0), x_c'(t; x_0, v_0))$ (see (P1) *and* (P3)*).*

Proof. We split the proof into two steps.

- *Step* 1. *Continuity in the first component.*
  Take $(x_0, v_0) \in \mathcal{B}$ and $\varepsilon > 0$. By (P1) and (P2), there exists $\tau < t_1(x_0, v_0) =: t_1$ satisfying

$$x_c^2(\tau; x_0, v_0)\|p\|_\infty < 1, \tag{3.4}$$

$$x_c(\tau; x_0, v_0) > 0, \tag{3.5}$$

$$x_c'(\tau; x_0, v_0) < 0, \tag{3.6}$$

$$0 < -\frac{x_c(\tau; x_0, v_0)}{x_c'(\tau; x_0, v_0)} < \varepsilon. \tag{3.7}$$

  Observe that these properties imply

$$x_c''(t; x_0, v_0) < 0 \quad \text{for all } t \in]\tau, t_1[.$$

  Therefore,

$$x_c'(t; x_0, v_0) < x_c'(\tau; x_0, v_0) \quad \text{for all } t \in]\tau, t_1[$$

  and so,

$$0 < x_c(t; x_0, v_0) < x_c(\tau; x_0, v_0) + x_c'(\tau; x_0, v_0)(t - \tau). \tag{3.8}$$

  Clearly, by (3.8) we deduce that $\tau < t_1(x_0, v_0) < \tau + \varepsilon$. To conclude that $\mathcal{B}$ is an open set and the continuity in the first component we use an argument of continuous dependence to guarantee (3.4)–(3.7).
- *Step* 2. *Continuity in the second component.*
  Take $\{(x_0^n, v_0^n)\} \longrightarrow (x_0, v_0)$. We know, by continuous dependence, that

$$x_c(t; x_0^n, v_0^n) \longrightarrow x_c(t; x_0, v_0)$$

  pointwise provided $0 \le t < t_1(x_0, v_0)$. On the other hand, by the previous step, we also deduce that

$$\chi_{[0, t_1(x_0^n, v_0^n)]} \longrightarrow \chi_{[0, t_1(x_0, v_0)]}$$

  pointwise, where $\chi_{[0, t_1(x_0^n, v_0^n)]}(t) = 1$ if $t \in [0, t_1(x_0^n, v_0^n)]$ and 0 otherwise. Apart from these arguments, an easy computation shows that

$$h'(t; x_0^n, v_0^n) = p(t) x_c'(t; x_0^n, v_0^n) \quad \text{for all } t \in [0, t_1(x_0^n, v_0^n)].$$

  Collecting all the information, we obtain that

$$h(t_1(x_0^n, v_0^n); x_0^n, v_0^n) - h_1(0; x_0^n, v_0^n) = -p(0) x_0^n - \int_0^{t_1(x_0^n, v_0^n)} p'(s) x_c(s; x_0^n, v_0^n) ds.$$

The proof is now complete after using Lebesgue's dominated convergence theorem. ■

Remark 3.1. If we replace, in the previous lemma, $\mathcal{B}$ by

$$\mathcal{C} := \{(x_0, v_0) \in]0, \infty[\times\mathbb{R} : (x_c(t; , x_0, v_0), x_c'(t; x_0, v_0)) \text{ has}$$
$$\text{a maximal interval }]t_0, t_1[\text{ with } t_0 > -\infty\}$$

and consider the map

$$\begin{gathered} \mathcal{C} \longrightarrow \mathbb{R}^2, \\ (x_0, v_0) \mapsto (t_0, h_0) \end{gathered} \tag{3.9}$$

with h_0 the energy at time t_0, then the same conclusions as Lemma 3.1 hold.

Lemma 3.1 allows us to pass, in a continuous way, from (position, velocity) at the initial time to (time, energy) of the first collision. Analogously, by Remark 3.1, we can repeat the same procedure in the past. On the other hand, given $(t_0, h_0) \in \mathbb{R}^2$, by Proposition 3.1 in [31] we know that there exists a unique bouncing solution, namely, $u(t; (t_0, h_0))$, satisfying

(i) $\lim_{t\to t_0} u(t, (t_0, h_0)) = 0$,
(ii) $\lim_{t\to t_0^+} u'(t, (t_0, h_0)) = -\infty$,
(iii) $\lim_{t\to t_0} \frac{u'(t,(t_0,h_0))^2}{2} - \frac{1}{u(t,(t_0,h_0))} = h_0$.

As in the regular case, if $t_1 > t_0$ is the next instant of collision and h_1 is the corresponding energy, then it was proved in [31, section 5] that the successor map

$$\begin{gathered} S : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \\ S(t_0, h_0) = (S_1(t_0, h_0), S_2(t_0, h_0)) = (t_1, h_1) \end{gathered} \tag{3.10}$$

is continuous and injective with $D := \{(t_0, h_0) \in \mathbb{R}^2 : t_1 < \infty\}$ and the map

$$(t, t_0, h_0) \in \mathcal{H} \mapsto (u(t; t_0, h_0), u'(t; t_0, h_0)) \in \mathbb{R}^2$$

is continuous, where $\mathcal{H} := \{(t; t_0, h_0) \in \mathbb{R}^3 : t_0 < t < t_1\}$ (observe that t_1 can be ∞). Consequently we deduce the continuity of the function

$$(t; x_0, v_0) \in \mathcal{G} \longrightarrow (x(t; x_0, v_0), x'(t; x_0, v_0)), \tag{3.11}$$

where $\mathcal{G} = \{(t; x_0, v_0) : x(t; x_0, v_0) \neq 0\}$. Putting all the information together and reasoning inductively as in the second step of Lemma 3.1 we obtain that the energy function

$$\begin{gathered} h : \mathbb{R}\times]0, \infty[\times\mathbb{R} \longrightarrow \mathbb{R}, \\ h(t, x_0, v_0) = \frac{x'(t;x_0,v_0)^2}{2} - \frac{1}{x(t;x_0,v_0)}, \end{gathered} \tag{3.12}$$

is continuous (see (P3) when $x(t; x_0, v_0) = 0$).

Once these remarks have been made, we need to introduce a topological space $(\mathcal{A}, \Im)$ where

$$\mathcal{A} = \left\{(x, y, z) \in X \times \mathbb{R} : z = \frac{y^2}{2} - \frac{1}{x}\right\} \cup \{(0, \infty, z) : z \in \mathbb{R}\}.$$

Using the convention $y < \infty$ for all $y \in \mathbb{R}$, the definition of the topology in $\mathcal{A}$ is as follows: a sequence (x_n, y_n, z_n) converges to $(x_0, y_0, z_0) \in \mathcal{A}$ with $x_0 > 0$ if each coordinate converges in the classical sense. In the case $x_0 = 0$, a sequence $(x_n, y_n, z_n) \subset \mathcal{A}$ converges to $(0, \infty, z_0)$ if

$$\begin{aligned} x_n &\longrightarrow 0, \\ |y_n| &\longrightarrow \infty, \\ z_n &\longrightarrow z_0. \end{aligned}$$

In this topological space, it is useful to introduce the maps

$$\begin{aligned} i : X &\longrightarrow \mathcal{A}, \\ (x, v) &\mapsto (x, v, \tfrac{v^2}{2} - \tfrac{1}{x}) \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} \pi : i(X) \subset \mathcal{A} &\longrightarrow X, \\ (x, v, h) &\mapsto (x, v). \end{aligned} \tag{3.14}$$

Clearly these two maps are continuous and injective.

Next we properly define the Poincaré map associated with (3.1). Specifically,

$$\mathcal{P} : X \longrightarrow \mathcal{A}$$

given by

$$\mathcal{P}(x_0, v_0) = \begin{cases} (0, \infty, h(T; x_0, v_0)) & \text{if}\, t_1(x_0, v_0) = T, \\ (0, \infty, h(T; x_0, v_0)) & \text{if } S_1^j(t_1(x_0, v_0), h_1(x_0, v_0)) = T, \\ (x(T; x_0, v_0), x'(T; x_0, v_0), h(T; x_0, v_0)) & \text{otherwise}, \end{cases} \tag{3.15}$$

where $S^j = (S_1^j, S_2^j)$ denotes the jth iterate of the successor map.

Proposition 3.1. *The Poincaré map* (3.15) *is injective and continuous.*

Proof. . By uniqueness of the bouncing solution, we know that $\mathcal{P}$ is injective. On the other hand, from the comments above on the definition of $(\mathcal{A}, \Im)$ (see (3.10)–(3.12)), $\mathcal{P}$ is continuous at the points (x_0, v_0) such that $\mathcal{P}(x_0, v_0) \neq (0, \infty, z)$. After that, we concentrate on points (x_0, v_0) with $\mathcal{P}(x_0, v_0) = (0, \infty, z)$. Indeed, pick a point (x_0, v_0) such that $S_1^j(t_1(x_0, v_0), h_1(x_0, v_0)) = T$ (the proof is completely analogous if $t_1(x_0, v_0) = T$), and a sequence $\{(x_0^n, v_0^n)\} \longrightarrow (x_0, v_0)$. We split the proof into three cases.

Case 1. *The sequence* $S_1^j(t_1(x_0^n, v_0^n), h_1(x_0^n, v_0^n)) = T$ *for all* $n \in \mathbb{N}$.

The assertion of the proposition is clear by the continuity of the energy function (see (3.12)).

Case 2. *The sequence* $S_1^j(t_1(x_0^n, v_0^n), h_1(x_0^n, v_0^n)) < T$ *for all* $n \in \mathbb{N}$.

Fix ϵ, $k > 0$. After that, notice that $h(T; x_0^n, y_0^n) \longrightarrow h(T; x_0, v_0)$. By properties (P1), (P2), we can take $s < T$ such that

- $x^2(s; x_0, v_0)\|p\|_\infty < 1$,
- $0 < x(s; x_0, v_0) < \epsilon$,
- $-x'(s; x_0, v_0) > k$.

By continuous dependence, we deduce that, for n large enough, $(x(s;x_0^n,v_0^n),\ x'(s;x_0^n, v_0^n))$ also enjoys these properties. At this moment, we reason as in Step 1 of Lemma 3.1, to obtain that

$$\begin{aligned} 0 < x(T;x_0^n,v_0^n) < \epsilon, \\ -x'(T;x_0^n,v_0^n) > k. \end{aligned}$$

Case 3. *The sequence* $S_1^j(t_1(x_0^n,v_0^n),h_1(x_0^n,v_0^n)) > T$ *for all* $n \in \mathbb{N}$.
Again, fix ϵ, $k > 0$. First observe that $h(T;x_0^n,y_0^n) \longrightarrow h(T;x_0,v_0)$. Now take $s > T$ such that

- $x^2(s;x_0,v_0)\|p\|_\infty < 1$,
- $0 < x(s;x_0,v_0) < \epsilon$,
- $x'(s;x_0,v_0) > k$.

Reasoning as in the previous case, we obtain, for n large enough,

$$\begin{aligned} 0 < x(T;x_0^n,v_0^n) < \epsilon, \\ x'(T;x_0^n,v_0^n) > k. \end{aligned} \qquad \blacksquare$$

As a direct consequence of this lemma we deduce that $\Omega := \mathcal{P}^{-1}(i(X)) \subset X$ is an open set and

$$\begin{aligned} \phi = \pi \circ \mathcal{P} : \Omega \longrightarrow X, \\ (x_0,v_0) \mapsto (x(T;x_0,v_0), x'(T;x_0,v_0)), \end{aligned} \tag{3.16}$$

is continuous. Notice that given $(x_0,v_0) \in \Omega$, we know that the solution with this initial condition does not have a collision at T. It is also important to observe that the key point for the continuity of $\mathcal{P}$ is the continuity of the energy function at the collisions.

3.2. Chaotic dynamics of (3.1). In this section we concentrate on the construction of T-periodic forces with alternating sign inducing chaotic dynamics in (3.1). By chaotic dynamics in (3.1) we understand that the map

$$\begin{aligned} \pi \circ \mathcal{P} : \Omega \longrightarrow X, \\ (x_0,v_0) \mapsto (x(T;x_0,v_0), x'(T;x_0,v_0)), \end{aligned} \tag{3.17}$$

induces chaotic dynamics on two symbols (see Definition 5.1). In our results the notion of a bouncing solution is essential. To appreciate this fact, we recall that, in the classical sense, either there are no T-periodic solutions (this happens if $\int_0^T p(t)dt \le 0$, see [23]) or (3.1) has a dynamics of "saddle-type" (see [23, 26]). In both cases, no chaos is present. As in the regular case, first we construct the class of functions and then we give the main results.

Fix two constants $p_1 < 0 < p_2$ and consider the equations

$$x'' = \frac{-1}{x^2} + p_1, \tag{3.18}$$

$$x'' = \frac{-1}{x^2} + p_2. \tag{3.19}$$

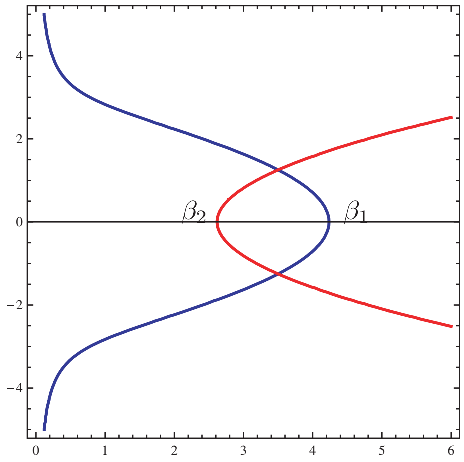

Figure 5. *The configuration of orbits β_1, β_2.*

Next, take $e_1, e_2 < 0$ with $e_2 < -2\sqrt{p_2}$ so that

$$\frac{-e_1 - \sqrt{e_1^2 - 4p_1}}{2p_1} > \frac{-e_2 - \sqrt{e_2^2 - 4p_2}}{2p_2}. \tag{3.20}$$

Define by $\mathcal{F}_i(x, y) = \frac{y^2}{2} - \frac{1}{x} - p_i x$ for $i = 1, 2$ the (conserved) energy for both autonomous equations (3.18) and (3.19). Let

$$\beta_1 = \{(x, y) : \mathcal{F}_1(x, y) = e_1\}$$

and

$$\beta_2 = \{(x, y) : \mathcal{F}_2(x, y) = e_2, x \geq \sigma\},$$

where $\sigma = \sqrt{1/p_2}$ is the unique equilibrium of (3.19). By (3.20), the intersection points between β_i and the x-axis, namely, $(z_1, 0)$ and $(z_2, 0)$, respectively, satisfy that

$$z_1 > z_2. \tag{3.21}$$

Using this property, a simple analysis implies that the intersection $\beta_1 \cap \beta_2$ is made by two points, specifically $\{(x_1, y_1), (x_2, y_2)\}$ with $y_1 < 0 < y_2$. See Figure 5.

Clearly, as $e_2 < -2\sqrt{p_2}$, we can take $T_2^* > 0$ such that

$$(x_2(T_2^*; (x_1, y_1)), x_2'(T_2^*; (x_1, y_1))) = (x_2, y_2),$$

where $(x_2(t; (x_1, y_1)), x_2'(t; (x_1, y_1)))$ is the solution of (3.19) with initial condition at (x_1, y_1). After that, pick a constant $\widetilde{e}_2$ such that

(i) $e_2 < \widetilde{e}_2 < -2\sqrt{p_2}$,

(ii) $\frac{-e_1 - \sqrt{e_1^2 - 4p_1}}{2p_1} > \frac{-\widetilde{e}_2 - \sqrt{\widetilde{e}_2^2 - 4p_2}}{2p_2}$.

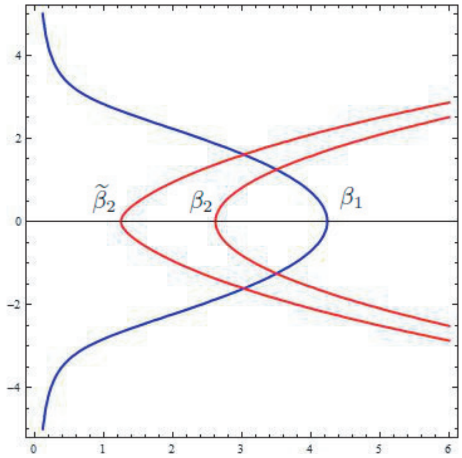

Figure 6. *The configuration of orbits* $\beta_1, \beta_2, \widetilde{\beta}_2$.

By an elementary phase portrait analysis, we know that

$$x_1 < \widetilde{x}_1,$$
$$y_1 < \widetilde{y}_2,$$

and

$$x_2 < \widetilde{x}_2,$$
$$\widetilde{y}_2 < y_2,$$

where $(\widetilde{x}_1, \widetilde{y}_1)$ and $(\widetilde{x}_2, \widetilde{y}_2)$ are the intersection points between β_1 and $\widetilde{\beta}_2$ with

$$\widetilde{\beta}_2 = \{(x, y) : \mathcal{F}_2(x, y) = \widetilde{e}_2, x \geq \sigma\}.$$

See Figure 6.

Equation (3.19) preserves the usual ordering in $\mathbb{R}^2$ (see Theorem 2.1 and its proof in [6]) and so we have that

$$x_2(T_2^*, (x_1, y_1)) < x_2(T_2^*, (\widetilde{x}_1, \widetilde{y}_1)),$$
$$x_2'(T_2^*, (x_1, y_1)) < x_2'(T_2^*, (\widetilde{x}_1, \widetilde{y}_1)).$$

By these inequalities, we can take a constant $\widetilde{e}_1$ close to e_1 satisfying the following:

1. $e_1 < \widetilde{e}_1$.
2. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the connected components of

$$\{(x, y) : e_2 \leq \mathcal{F}_2(x, y) \leq \widetilde{e}_2, e_1 \leq \mathcal{F}_1(x, y) \leq \widetilde{e}_1, x \geq \sigma\}$$

with $\mathcal{H}_1 \subset \{(x, y) : y < 0\}$. Then we can fix a time $T_2 < T_2^*$ such that for all $(x_0, y_0) \in \{(x, y) \in \mathcal{H}_1 : \mathcal{F}_2(x, y) = e_2\}$ we have that

(a1) $\{(x, y) \in \mathcal{H}_1 : \mathcal{F}_2(x, y) = \widetilde{e}_2\} \subset \{(x, y) : x_0 < x, y_0 < y\}$,

(a2) $\{(x, y) : x_2(T_2; (x_0, y_0)) < x, x_2'(T_2; (x_0, y_0)) < y\} \cap \{(x, y) \in \mathcal{H}_2 : \mathcal{F}_2(x, y) = \widetilde{e}_2\} = \emptyset$,

(a3) $\{(x, y) \in \mathcal{H}_2 : \mathcal{F}_2(x, y) = e_2\} \subset \{(x, y) : x_2(T_2; (x_0, y_0)) < x, x_2'(T_2; (x_0, y_0)) < y\}$.

See Figure 7.

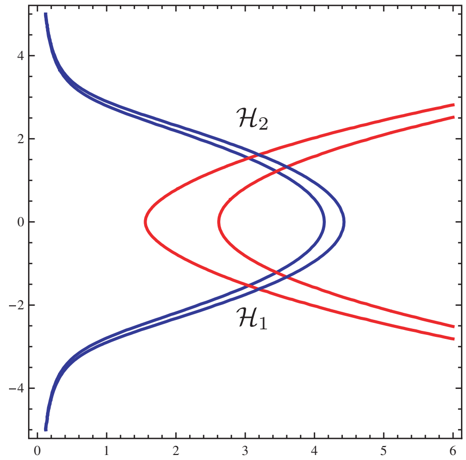

Figure 7. *Illustration of* $\mathcal{H}_1$ *and* $\mathcal{H}_2$.

Observe that these properties have deep consequences. Take $(x_0, y_0) \in \{(x, y) \in \mathcal{H}_1 : \mathcal{F}_2(x, y) = e_2\}$ and $(\widetilde{x}_0, \widetilde{y}_0) \in \{(x, y) \in \mathcal{H}_1 : \mathcal{F}_2(x, y) = \widetilde{e}_2\}$. Clearly,

$$x_0 < \widetilde{x}_0,$$
$$y_0 < \widetilde{y}_0.$$

Then, using that (3.19) preserves the usual ordering in $\mathbb{R}^2$,

$$x_2(T_2; (x_0, y_0)) < x_2(T_2; (\widetilde{x}_0, \widetilde{y}_0)),$$
$$x_2'(T_2; (x_0, y_0)) < x_2'(T_2; (\widetilde{x}_0, \widetilde{y}_0)).$$

Using 2 above, we deduce that

$$(x_2(T_2; (x_0, y_0)), x_2'(T_2; (x_0, y_0))) \notin \mathcal{H}_2 \text{ (see (a3))},$$
$$(x_2(T_2; (\widetilde{x}_0, \widetilde{y}_0)), x_2'(T_2; (\widetilde{x}_0, \widetilde{y}_0))) \notin \mathcal{H}_2 \text{ (see (a2))}.$$

In fact, these points belong to different connected components of

$$X \backslash \{(x, y) : e_1 \le \mathcal{F}_1(x, y) \le \widetilde{e}_1\}.$$

Finally, we fix four additional constants $E_1 < \widetilde{E}_1$ and $E_2 < \widetilde{E}_2$ so that

- $[E_1, \widetilde{E}_1] \subset]e_1, \widetilde{e}_1[$, $[e_2, \widetilde{e}_2] \subset]E_2, \widetilde{E}_2[$,
- for all $e_1^* \in [e_1, \widetilde{e}_1]$ and for all $e_2^* \in [E_2, \widetilde{E}_2]$

$$\frac{-e_1^* - \sqrt{(e_1^*)^2 - 4p_1}}{2p_1} > \frac{-e_2^* - \sqrt{(e_2^*)^2 - 4p_2}}{2p_2}$$

holds. See Figure 8.

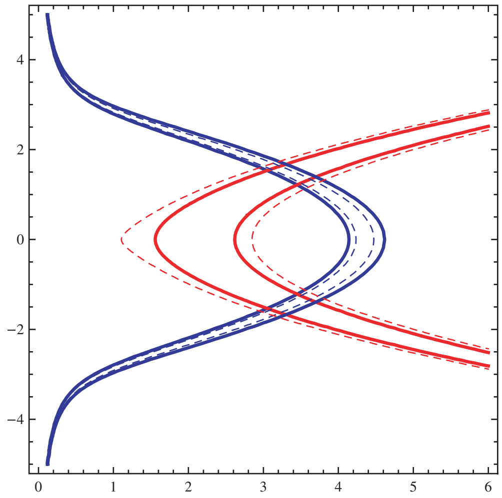

Figure 8. *Illustration of the configuration of the orbits. The dashed lines correspond to the orbits associated with the energy levels denoted by* $E_1, \widetilde{E}_1, E_2, \widetilde{E}_2$.

Denote

$$Q_1 = \{(x, y) : E_1 \leq \mathcal{F}_1(x, y) \leq \widetilde{E}_1, E_2 \leq \mathcal{F}_2(x, y) \leq \widetilde{E}_2, y > 0, x \geq \sigma\},$$
$$Q_2 = \{(x, y) : e_1 \leq \mathcal{F}_1(x, y) \leq \widetilde{e}_1, e_2 \leq \mathcal{F}_2(x, y) \leq \widetilde{e}_2, y < 0, x \geq \sigma\}.$$

Clearly we can choose $r_2 > 0$ such that the r_2-neighborhood of

$$\{(x, y) : e_1 = \mathcal{F}_1(x, y), e_2 \leq \mathcal{F}_2(x, y) \leq \widetilde{e}_2, y > 0, x \geq \sigma\}$$

and

$$\{(x, y) : \widetilde{e}_1 = \mathcal{F}_1(x, y), e_2 \leq \mathcal{F}_2(x, y) \leq \widetilde{e}_2, y > 0, x \geq \sigma\}$$

does not touch Q_1 and the r_2-neighborhood of

$$\mathcal{J} = \{(x, y) : \widetilde{e}_2 \leq \mathcal{F}_2(x, y) \leq e_2, \widetilde{e}_1 \leq \mathcal{F}_1(x, y) \leq e_1, y > 0, x \geq \sigma\}$$

does not touch the sets

$$\{(x, y) : \mathcal{F}_2(x, y) = E_2, x \geq \sigma\},$$
$$\{(x, y) : \mathcal{F}_2(x, y) = \widetilde{E}_2, x \geq \sigma\}.$$

Analogously, we can choose $r_1 > 0$ such that the r_1-neighborhood of

$$\{(x, y) : E_2 = \mathcal{F}_2(x, y), E_1 \leq \mathcal{F}_1(x, y) \leq \widetilde{E}_1, y < 0, x \geq \sigma\}$$

and

$$\{(x, y) : \widetilde{E}_2 = \mathcal{F}_2(x, y), E_1 \leq \mathcal{F}_1(x, y) \leq \widetilde{E}_1, y < 0, x \geq \sigma\}$$

does not touch Q_2 and the r_1-neighborhood of

$$R_2 = \{(x, y) : E_1 \leq \mathcal{F}_1(x, y) \leq \widetilde{E}_1, E_2 \leq \mathcal{F}_2(x, y) \leq \widetilde{E}_2, y < 0, x \geq \sigma\}$$

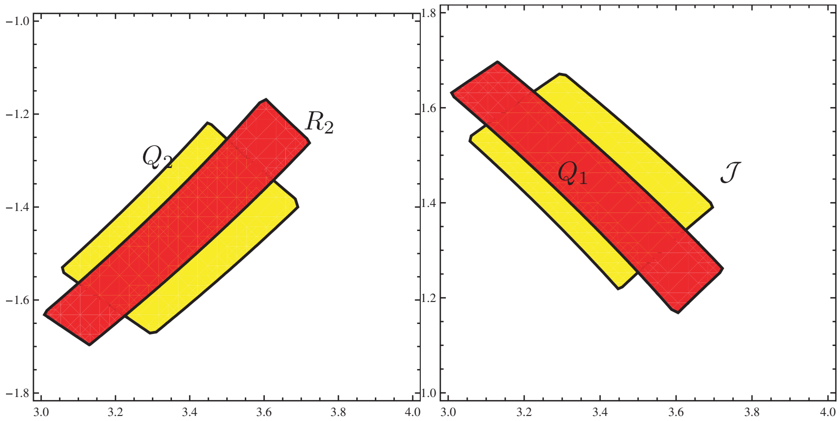

Figure 9. *Detail of the intersections between the linked orbits. Left: Lower intersection. Right: Upper intersection.*

does not touch the sets

$$\{(x, y) : \mathcal{F}_1(x, y) = e_1\},$$

$$\{(x, y) : \mathcal{F}_1(x, y) = \widetilde{e}_1\}.$$

See Figure 9 for an illustration of these sets.

The next step is to point out that Lemma 2.1 holds exactly the same for (3.1). Now, we are in condition to construct the forcing term $p(t)$. First, apply Lemma 2.1 with $d = r_1$, $M = \max\{|p_1|, |p_2|\}$, and $K = R_2$ to obtain a parameter $\tau_1 > 0$ satisfying condition (2.9). After that, apply Lemma 2.1 with $d = r_2$, $M = \max\{|p_1|, |p_2|\}$, and $K = \mathcal{J}$ to obtain a parameter $\tau_2 > 0$ satisfying condition (2.9). Now, for a parameter $T_1 > 0$ we define

$$p(t) = \begin{cases} p_1 & \text{if } 0 \le t < T_1, \\ \widetilde{p}_1(t) & \text{if } T_1 \le t < T_1 + \tau_1, \\ p_2 & \text{if } T_1 + \tau_1 \le t < T_1 + \tau_1 + T_2, \\ \widetilde{p}_2(t) & \text{if } T_1 + \tau_1 + T_2 \le t < T_1 + \tau_1 + T_2 + \tau_2, \end{cases} \tag{3.22}$$

so that $|\widetilde{p}_i(t)| \le \max\{|p_1|, |p_2|\}$ and function $p(t)$ is of class $\mathcal{C}^1$. Recall that the parameter T_2 is fixed above (see (a1)–(a3)). As in the regular case, we need to introduce some notation. Consider Φ_1 the Poincaré map associated with (3.18) at time T_1 (see (3.15)). For $\Omega_1 = \Phi_1^{-1}(i(X))$ we have that

$$\begin{gathered} \pi \circ \Phi_1 : \Omega_1 \longrightarrow X, \\ (x_0, v_0) \mapsto (x(T_1; x_0, v_0), x'(T_1; x_0, v_0)), \end{gathered} \tag{3.23}$$

where, for convenience, $(x(T_1; x_0, v_0), x'(T_1; x_0, v_0))$ is the bouncing solution of (3.18) with initial data at (x_0, v_0). Define Φ_2 and Ω_2 in a similar way with system (3.19) and T_2. On the other hand, in $\mathcal{U}$ a neighborhood of $Q_1 \cup Q_2$, by Lemma 2.1 and the definition of τ_1 and τ_2, we can define two maps,

$$h_1 : \mathcal{U} \longrightarrow X,$$
$$h_1(x_0, v_0) := (x(T_1 + \tau_1; (T_1, x_0, v_0), x'(T_1 + \tau_1; (T_1, x_0, v_0))),$$
$$h_2 : \mathcal{U} \longrightarrow X,$$
$$h_2(x_0, v_0) := (x(T_1 + \tau_1 + T_2 + \tau_2; (T_1 + \tau_1 + T_2, x_0, v_0), x'(T_1 + \tau_1 + T_2 + \tau_2; (T_1 + \tau_1 + T_2, x_0, v_0))),$$

where $(x(t; (t_0, x_0, v_0)), x'(t; (t_0, x_0, v_0)))$ refers to the bouncing solution of (3.1) with force (3.22). Note that, by the expression of (3.22), if $(x, v) \in \Omega_1$, $\pi \circ \Phi_1(x, v) \in \mathcal{U}$, $h_1(\pi \circ \Phi_1(x, v)) \in \Omega_2$, and $\pi \circ \Phi_2(h_1(\pi \circ \Phi_1(x, v))) \in \mathcal{U}$, then

$$\pi \circ \mathcal{P}(x, v) = h_2 \circ \pi \circ \Phi_2 \circ h_1 \circ \pi \circ \Phi_1(x, v), \tag{3.24}$$

where $\mathcal{P}$ is the Poincaré map of (3.1) with force (3.22).

Theorem 3.1. *There exists T_1^* with the following property: for every function $p(t)$ given in (3.22) with parameter $T_1 > T_1^*$, (3.1) has chaotic dynamics.*

Proof. The proof of this theorem is divided into five steps.

Step 1. *Stretching property for* $[0, T_1]$.

We follow a similar reasoning as Step 1 in Theorem 2.1. Consider the successor map $S = (S_1, S_2)$ associated with (3.18). Observe that given a point $(x_0, v_0) \in \{(x, y) : \mathcal{F}_1(x, y) = h_1\}$, the bouncing solution of (3.18) with this initial condition is periodic with period $S_1(0, h_1)$. We recall that by Proposition 5.1 in [31], $S_1(x, h)$ is a strictly increasing function in the second component. Now we define

$$T_1^* = \frac{5P_1P_2}{P_2 - P_1},$$

where $P_1 = S_1(0, E_1)$ and $P_2 = S_1(0, \widetilde{E}_1)$.

Fix $T_1 > T_1^*$. We have to find two compact sets $\mathcal{K}_0, \mathcal{K}_1 \subset Q_1$ such that (see Definition 5.3)

$$(\mathcal{K}_i, \pi \circ \Phi_1) : \widetilde{Q}_1 \Leftrightarrow\!\!\!\!\!\!\!\longrightarrow \widetilde{R}_2,$$

where Φ_1 is the Poincaré map of (3.18) at the instant T_1, $\widetilde{Q}_1 = (Q_1, (Q_1)^-)$ with

$$Q_1^- = \{(x, y) \in Q_1 : \mathcal{F}_1(x, y) = E_1\} \cup \{(x, y) \in Q_1 : \mathcal{F}_1(x, y) = \widetilde{E}_1\},$$

and $\widetilde{R}_2 = (R_2, R_2^-)$ with

$$R_2^- = \{(x, y) \in R_2 : \mathcal{F}_2(x, y) = E_2\} \cup \{(x, y) \in R_2 : \mathcal{F}_2(x, y) = \widetilde{E}_2\}.$$

In Figure 10, we illustrate this dynamical property.

Indeed, first consider m_* the smallest integer satisfying

$$\frac{T_1}{P_2} < m_* \tag{3.25}$$

and m^* the largest integer satisfying

$$\frac{T_1}{P_1} > m^*. \tag{3.26}$$

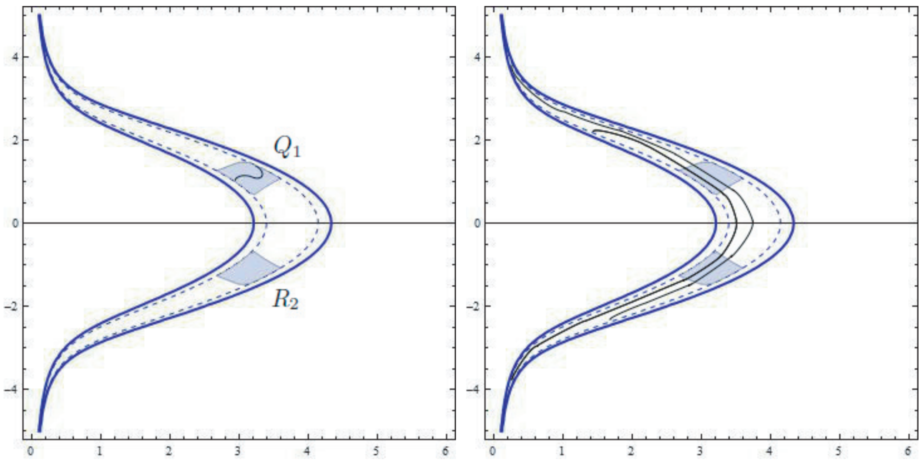

Figure 10. *Left: A path* $\gamma : [0,1] \longrightarrow Q_1$ *joining the sides* $\{(x,y) \in Q_1 : \mathcal{F}_1(x,y) = E_1\}$ *and* $\{(x,y) \in Q_1 : \mathcal{F}_1(x,y) = \widetilde{E}_1\}$. *Right: Illustration of* $\pi \circ \Phi_1(\Omega_1 \cap \gamma(t))$. *Note that this effect is caused by the "twist" property.*

By the definition of T_1 we know that $m^* - m_* > 5$. Observe that property (3.25) says that for all initial condition $(x_0, y_0) \in \{(x,y) : \mathcal{F}_1(x,y) = \widetilde{E}_1\}$, solution $(x(t; x_0, y_0), x'(t; x_0, y_0))$ finds the point (x_0, y_0) at most m_* times for the interval $[0, T_1]$. Similarly, property (3.26) says that for all initial condition $(x_0, y_0) \in \{(x,y) : \mathcal{F}_1(x,y) = E_1\}$, solution $(x(t; x_0, y_0), x'(t; x_0, y_0))$ finds the point (x_0, y_0) at least m^* times for the interval $[0, T_1]$. Next, we consider the compact sets

$$\mathcal{K}_1 = \left\{ (x,y) \in Q_1 : \frac{T_1}{S_1(0, \mathcal{F}_1(x,y))} \in [m_*, m_* + 2] \right\},$$

$$\mathcal{K}_0 = \left\{ (x,y) \in Q_1 : \frac{T_1}{S_1(0, \mathcal{F}_1(x,y))} \in [m^* - 2, m^*] \right\}.$$

Now we prove the stretching property (see Definition 5.3).

Take

$$\gamma : [0,1] \longrightarrow Q_1,$$

a path with

$$\gamma(0) \in \{(x,y) \in Q_1 : \mathcal{F}_1(x,y) = E_1\},$$
$$\gamma(1) \in \{(x,y) \in Q_1 : \mathcal{F}_1(x,y) = \widetilde{E}_1\}.$$

Using that $S_1(0, \mathcal{F}_1(\gamma(t)))$ is continuous, we deduce that there exist two disjoint subintervals $[A_0, A_0']$ and $[A_1, A_1']$ such that

$$\frac{T_1}{S_1(0, \mathcal{F}_1(\gamma(s)))} \in [m_*, m_* + 2] \qquad \text{for } s \in [A_0, A_0'],$$

with $\frac{T_1}{S_1(0, \mathcal{F}_1(\gamma(A_0)))} = m_*$ and $\frac{T_1}{S_1(0, \mathcal{F}_1(\gamma(A_0')))} = m_* + 2$ and

$$\frac{T_1}{S_1(0, \mathcal{F}_1(\gamma(s)))} \in [m^* - 2, m^*] \qquad \text{for } s \in [A_1, A_1'],$$

with $\frac{T_1}{S_1(0,\mathcal{F}_1(\gamma(A_1)))} = m^*$ and $\frac{T_1}{S_1(0,\mathcal{F}_1(\gamma(A_1')))} = m^* - 2$. Now we concentrate on the interval $[A_0, A_0']$. Clearly, the solutions with initial conditions at $\gamma(A_0)$ and $\gamma(A_0')$ have exactly m_* and $(m_* + 2)$ collisions, respectively, for the interval $[0, T_1]$. This property implies that there exists a subinterval $[\widetilde{S}_0, \widetilde{S}_0'] \subset [A_0, A_0']$ such that

$$(x(T_1;\gamma(\widetilde{S}_0)), x'(T_1;\gamma(\widetilde{S}_0))) \in \{(x,y) : x < \min\{x_0 : (x_0,y) \in R_2\}\},$$
$$(x(T_1;\gamma(\widetilde{S}_0')), x'(T_1;\gamma(\widetilde{S}_0'))) \in \{(x,y) : y = 0\},$$
$$(x(T_1;\gamma(s)), x(T_1;\gamma(s))) \in \{(x,y) : y \le 0\}$$

for all $s \in [\widetilde{S}_0, \widetilde{S}_0']$. Finally, we easily obtain the desired subinterval $[S_0, S_0']$. Observe that previously we have used that

$$E_1 \le \mathcal{F}_1(x(T_1;\gamma(t)), x'(T_1;\gamma(t))) \le \widetilde{E}_1$$

for all $t \in [0,1]$.

Step 2. *Behavior in the interval* $[T_1, T_1 + \tau_1]$.

By the definition of $p(t)$ and as a direct consequence of the variant of Lemma 2.1 for (3.1) we have the following property: For all continuous path

$$\gamma : [0,1] \longrightarrow R_2$$

with

$$\gamma(0) \in \{(x,y) \in R_2 : \mathcal{F}_2(x,y) = E_2\},$$
$$\gamma(1) \in \{(x,y) \in R_2 : \mathcal{F}_2(x,y) = \widetilde{E}_2\},$$

there is a subinterval $[R_0, R_0'] \subset [0,1]$ so that the curve

$$\beta(t) = (x(T_1 + \tau_1, (T_1, \gamma(t))), x'(T_1 + \tau_1, (T_1, \gamma(t))))$$

satisfies

$$\beta([R_0, R_0']) \subset Q_2$$

with $\beta(R_0) \in \{(x,y) \in Q_2 : \mathcal{F}_2(x,y) = e_2\}$ and $\beta(R_0') \in \{(x,y) \in Q_2 : \mathcal{F}_2(x,y) = \widetilde{e}_2\}$.

Step 3. *Stretching property in the interval* $[T_1 + \tau_1, T_1 + \tau_1 + T_2]$.

In this step we prove the following stretching property: Given a path

$$\gamma : [0,1] \longrightarrow Q_2$$

with

$$\gamma(0) \in \{(x,y) \in Q_2 : \mathcal{F}_2(x,y) = e_2\},$$
$$\gamma(1) \in \{(x,y) \in Q_2 : \mathcal{F}_2(x,y) = \widetilde{e}_2\},$$

there is a subinterval $[M_0, M_0'] \subset [0,1]$ such that

$$\pi \circ \Phi_2(\gamma(t)) \subset \mathcal{J}$$

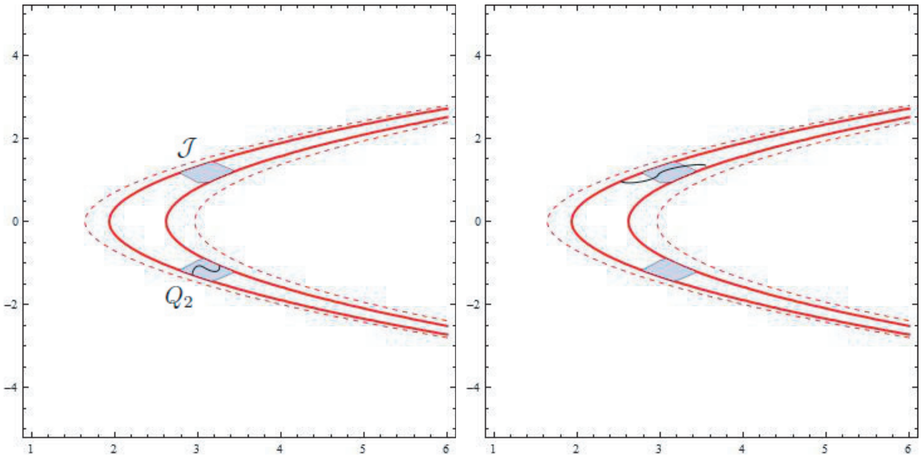

Figure 11. *Left: A path* $\gamma : [0,1] \longrightarrow Q_2$ *joining the sides* $\{(x,y) \in Q_2 : \mathcal{F}_2(x,y) = e_2\}$ *and* $\{(x,y) \in Q_2 : \mathcal{F}_2(x,y) = E_2\}$*. Right: Illustration of* $\pi \circ \Phi_2(\gamma(t))$*.*

with $\pi \circ \Phi_2(\gamma(M_0)) \in \{(x,y) \in \mathcal{J} : \mathcal{F}_1(x,y) = e_1\}$ and $\pi \circ \Phi_2(\gamma(M_0')) \in \{(x,y) \in \mathcal{J} : \mathcal{F}_1(x,y) = \widetilde{e}_1\}$. We denote by Φ_2 the Poincaré map associated with (3.19).

To see this claim, we recall that, by the definition of T_2 (see (a1)–(a3), $\pi \circ \Phi_2(\gamma(0))$ and $\pi \circ \Phi_2(\gamma(1))$ belong to different components of

$$X \backslash \{(x,y) : e_1 \leq \mathcal{F}_1(x,y) \leq \widetilde{e}_1\}.$$

In addition, clearly $\pi \circ \Phi_2(\gamma(t)) \in \{(x,y) : \widetilde{e}_2 \leq \mathcal{F}_2(x,y) \leq e_2, x \geq \sigma, y > 0\}$ for all $t \in [0,1]$. The situation is illustrated by Figure 11. From these facts, the proof is clear.

Step 4. *Behavior in the interval* $[T_1 + T_2 + \tau_1, T_1 + T_2 + \tau_1 + \tau_2]$.

Reasoning as in Step 2, we obtain that for all continuous path $\gamma(t) : [0,1] \longrightarrow \mathcal{J}$ with $\gamma(0) \in \{(x,y) \in \mathcal{J} : \mathcal{F}_1(x,y) = e_1\}$, $\gamma(1) \in \{(x,y) \in \mathcal{J} : \mathcal{F}_1(x,y) = \widetilde{e}_1\}$, there exists a subinterval $[C_0, C_0'] \subset [0,1]$ so that the curve

$$\beta(t) = (x(T_1 + \tau_1 + T_2 + \tau_2; (T_1 + T_2 + \tau_1, \gamma(t))), x'(T_1 + \tau_1 + T_2 + \tau_2; (T_1 + T_2 + \tau_1, \gamma(t))))$$

satisfies

$$\beta([C_0, C_0']) \subset Q_1,$$

with $\beta(C_0)$ and $\beta(C_0')$ lying on different connected components of Q_1^-.

Step 5. *Conclusion.*

Putting all the information together and using (3.24) we deduce that $\pi \circ \mathcal{P}$ with $\mathcal{P}$ the Poincaré map associated with (3.1) has the properties

$$(\mathcal{K}_i, \pi \circ \mathcal{P}) : \widetilde{Q_1} \Leftrightarrow\!\!\!\!\!\!\longrightarrow \widetilde{Q_1}$$

for $i = 0, 1$. Finally we apply Theorem 5.2. ∎

4. Further examples and remarks. In connection with our results on chaotic dynamics we point out the general method of LTM has the following advantages:

- All the parameters involved can be estimated.
- A quantitative estimation of regions of initial data where the chaotic behavior occurs can be performed.
- The method is robust under small perturbations. Since the method is of topological nature, any type of conservation of energy or Hamiltonian structure is not essential. This means that our main results hold if a small friction coefficient is added in the equation or, more importantly, if the impact rule includes a restitution coefficent $\lambda > 0$ such that
$$x(t_0) = 0 \implies x'(t_0^+) = -\lambda x'(t_0^-).$$
This introduces a dissipation in the model, but if λ is close to 1 our results are preserved.

Concerning the singular equation, the original equation considered by Lazer and Solimini in [23] is

$$x'' = \frac{-1}{x^\gamma} + p(t) \tag{4.1}$$

with $p(t)$ continuous and T-periodic. For (4.1) with $\gamma \geq 1$ and $p(t)$ of class $\mathcal{C}^1$ and T-periodic, the same regularization technique can be applied and our results are valid in order to obtain the analogous conclusion. Notice that we need to adapt (in a direct way) the results taken from [31]. On the order hand, the weak singularity case $0 < \gamma < 1$ is simpler and has been considered in [36]. In this case, the collision velocity is finite and the solution can be continued by a simple elastic rule, hence the regularization in not necessary and analogous results are verified. It is important to observe that a similar construction as in section 2 can be detected in (4.1).

5. Appendix: Background on chaotic dynamics. In this section we introduce the definition of chaos and the topological tools which are used throughout this paper. For the reader's convenience, we present all the notions in the framework of the applications.

Definition 5.1. *Consider $(\mathcal{J}, d)$ a metric space and take $\mathcal{D}$ an open set. We say that a continuous map $\psi : \mathcal{D} \to \mathcal{J}$ induces "chaotic dynamics on two symbols" if there exist two disjoint compact sets $\mathcal{K}_0, \mathcal{K}_1 \subset \mathcal{D}$ such that, for each two-sided sequence $(s_i)_{i\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$, there exists a corresponding sequence $(\omega_i)_{i\in\mathbb{Z}} \in (\mathcal{K}_0 \cup \mathcal{K}_1)^{\mathbb{Z}}$ such that*

$$\omega_i \in \mathcal{K}_{s_i} \quad \text{and} \quad \omega_{i+1} = \psi(\omega_i) \quad \text{for all } i \in \mathbb{Z}, \tag{5.1}$$

and, whenever $(s_i)_{i\in\mathbb{Z}}$ is a k-periodic sequence (that is, $s_{i+k} = s_i$ for all $i \in \mathbb{Z}$) for some $k \geq 1$, there exists a k-periodic sequence $(\omega_i)_{i\in\mathbb{Z}} \in (\mathcal{K}_0 \cup \mathcal{K}_1)^{\mathbb{Z}}$ satisfying (5.1).

Definition 5.1 guarantees natural properties of complex dynamics such as sensitive dependence on the initial conditions or the presence of an invariant set Λ being transitive and semiconjugate with the Bernoulli shift. In addition we must note that our definition of chaos ensures as well the existence of periodic points of any period $n \in \mathbb{N}$. In relation to other definitions of chaos, we notice that if a concrete map is chaotic according to our definition it is also chaotic in the sense of Block–Coppel and in the sense of coin tossing; see [1], [22]. Next we collect some important properties of Definition 5.1.

Theorem 5.1 ([27, Theorem 2.2]). *Assume that* ψ, $\mathcal{K}_0$, *and* $\mathcal{K}_1$ *are as in Definition* 5.1 *and set*

$$\mathcal{K} := \mathcal{K}_1 \cup \mathcal{K}_0.$$

Defining the nonempty compact set

$$\mathcal{I}_\infty = \bigcap_{n=0}^{\infty} \psi^{-n}(\mathcal{K}), \tag{5.2}$$

then there exists a nonempty compact set

$$\mathcal{I} \subset \mathcal{I}_\infty \subset \mathcal{K}$$

on which the following are fulfilled:

(i) $\mathcal{I}$ *is invariant for* ψ *(i.e.,* $\psi(\mathcal{I}) = \mathcal{I}$*).*

(ii) $\psi|_{\mathcal{I}}$ *is semiconjugate to the Bernoulli shift on two symbols, that is, there exists a continuous map* g *of* $\mathcal{I}$ *onto* $\Sigma_2^+ := \{0,1\}^{\mathbb{N}}$, *endowed with the distance*

$$d(s', s'') := \sum_{i\in\mathbb{N}} \frac{\widetilde{d}(s'_i, s''_i)}{2^{i+1}} \quad \text{for } s' = (s'_i)_{i\in\mathbb{N}},\ s'' = (s''_i)_{i\in\mathbb{N}} \in \Sigma_2^+$$

(where $\widetilde{d}(\cdot,\cdot)$ *is the discrete distance on* $\{0,1\} : \widetilde{d}(s'_i, s''_i) = 0$ *for* $s'_i = s''_i$ *and* $\widetilde{d}(s'_i, s''_i) = 1$ *for* $s'_i \neq s''_i$*), such that the diagram*

$$\begin{array}{ccc} \Lambda & \xrightarrow{\psi} & \Lambda \\ \downarrow g & & \downarrow g \\ \Sigma_2^+ & \xrightarrow[\sigma]{} & \Sigma_2^+ \end{array}$$

commutes, where $\sigma : \Sigma_2^+ \longrightarrow \Sigma_2^+$ *is the Bernoulli shift defined by* $\sigma((s_i)_i) := (s_{i+1})_i$ *for all* $i \in \mathbb{N}$.

(iii) *The set* $\mathcal{P}$ *of the periodic points of* $\psi\,|_{\mathcal{I}_\infty}$ *is dense in* $\mathcal{I}$ *and the preimage* $g^{-1}(s) \subset \mathcal{I}$ *of every* k*-periodic sequence* $s = (s_i)_{i\in\mathbb{N}} \in \sum_2^+$ *contains at least one* k*-periodic point.*

Furthermore, from property (ii) *these follow:*

(iv) $h_{top}(\psi) \geq h_{top}(\psi\,|_{\mathcal{I}}) \geq h_{top}(\sigma) \geq \log(2)$, *where* h_{top} *is the topological entropy.*

(v) *There exists a compact invariant set* $\Lambda \subset \mathcal{I}$ *such that* $\psi\,|_{\Lambda}$ *is semiconjugate to the Bernoulli shift on two symbols, topologically transitive, and has sensitive dependence on initial conditions.*

Our next aim is to derive a method to prove the presence of chaotic dynamics in concrete examples. In this respect, we give the following definitions.

Definition 5.2. *Consider a set* $\mathcal{R}$ *homeomorphic to* $[0,1] \times [0,1]$. *We say that the pair* $\widetilde{\mathcal{R}} = (\mathcal{R}, \mathcal{R}^-)$ *is an "oriented topological rectangle" if* $\mathcal{R}^- = \mathcal{R}_l^- \cup \mathcal{R}_r^-$, *where* $\mathcal{R}_l^-$, $\mathcal{R}_r^-$ *are two disjoint compact arcs contained in the boundary of* $\mathcal{R}$.

Definition 5.3. *Take* (X,d) *with* $X =]0,\infty[\times\mathbb{R}$ *and* d *the Euclidean distance. Given two oriented rectangles* $\widetilde{\mathcal{R}} := (\mathcal{R},\mathcal{R}^-)$, $\widetilde{\mathcal{B}} := (\mathcal{B},\mathcal{B}^-)$ *with* $\mathcal{R},\mathcal{B}\subset X$, *and a compact set* $\mathcal{K}\subset\mathcal{R}$ *we say that a continuous map*

$$\psi : \mathcal{D}\subset X \longrightarrow X$$

$(\mathcal{K},\psi)$ *"stretches"* $\widetilde{\mathcal{R}}$ *to* $\widetilde{\mathcal{B}}$ *along the paths and write*

$$(\mathcal{K},\psi) : \widetilde{\mathcal{R}} \Leftarrow\!\!\!\!\leftarrow\!\!\!\!\rightarrow \widetilde{\mathcal{B}}$$

if the following condition holds:

for every path $\gamma : [0,1] \longrightarrow \mathcal{R}$ *such that* $\gamma(0)\in\mathcal{R}_l^-$ *and* $\gamma(1)\in\mathcal{R}_r^-$, *there exists a subinterval* $[t',t'']\subset[0,1]$ *so that*

$$\gamma(t)\in\mathcal{K}, \quad \psi(\gamma(t))\in\mathcal{B}$$

for all $t\in[t',t'']$ *and, moreover,* $\psi(\gamma(t'))$ *and* $\psi(\gamma(t''))$ *belong to different components of* $\mathcal{B}^-$.

As a next step we link the definition of stretching along paths with the notion of chaotic dynamics on two symbols.

Theorem 5.2 ([27, Theorem 2.3]). *Consider* $\psi : \mathcal{D} \longrightarrow X$ *a continuous map and* $\widetilde{\mathcal{R}} = (\mathcal{R},\mathcal{R}^-)$ *an oriented topological rectangle as above. Assume that there are two disjoint compact sets* $\mathcal{K}_0,\mathcal{K}_1$ *with* $\mathcal{K}_0\cup\mathcal{K}_1\subset\mathcal{R}$ *and such that*

$$(\mathcal{K}_i,\psi) : \widetilde{\mathcal{R}} \Leftarrow\!\!\!\!\leftarrow\!\!\!\!\rightarrow \widetilde{\mathcal{R}} \qquad \textit{for all } i=0,1,$$

then ψ *induces chaotic dynamics on two symbols relative to* $\mathcal{K}_0\cap(\psi^{-1}(\mathcal{K}_0\cup\mathcal{K}_1))$ *and* $\mathcal{K}_1\cap(\psi^{-1}(\mathcal{K}_0\cup\mathcal{K}_1))$. *It follows that the map* ψ *has the properties of Theorem* 5.1.

Remark 5.1. To be precise, it is needed to say that the previous theorem is not exactly Theorem 2.3 in [27] because $\mathcal{R}\not\subset\mathcal{D}$. However, as $\psi(\mathcal{K}_0\cap(\psi^{-1}(\mathcal{K}_0\cup\mathcal{K}_1)))\subset\mathcal{R}$, $\psi(\mathcal{K}_1\cap(\psi^{-1}(\mathcal{K}_0\cup\mathcal{K}_1)))\subset\mathcal{R}$, the proof is exactly the same as in that theorem.

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