DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **33**, Number **1**, January **2013**

doi:10.3934/dcds.2013.33.141

pp. **141–152**

ON THE EXISTENCE AND STABILITY OF PERIODIC SOLUTIONS FOR PENDULUM-LIKE EQUATIONS WITH FRICTION AND ϕ-LAPLACIAN

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Dedicated to Prof. Jean Mawhin on the occasion of his 70th birthday.

Abstract. In this paper we study the existence, multiplicity and stability of T-periodic solutions for the equation $(\phi(x'))' + c\,x' + g(x) = e(t) + s$.

1. **Introduction and preliminaries.** The problem of existence, multiplicity and stability of T-periodic solutions for the classical forced pendulum equation

$$x'' + c\,x' + k \sin x = e(t) + s,$$

with $c,\, k \geq 0$, $\int_0^T e(s)ds = 0$ and $s \in \mathbb{R}$, has a rich and long history, that can be traced back at least to 1922 with the paper by Hamel [12], and it has attracted the attention of many researchers in the last decades (to know more, one can read the excellent review [13] and references therein).

More recently, the forced pendulum with relativistic effects

$$\left(\frac{x'}{\sqrt{1 - \frac{x'^2}{v^2}}} \right)' + c\,x' + k \sin x = e(t), \tag{1}$$

where $v > 0$ is the speed of light in vacuum, has received some attention as a prototype of equation with singular ϕ-laplacian (see the review [14] and also [2, 3, 4, 5, 6, 7]). In particular, for our purposes it is interesting to recall the main result in [20], which assert that for every T-periodic external force $e(t)$ with mean value zero there exists a T-periodic solution provided that $2\,v\,T \leq 1$. As explained in [20], this result means an essential difference between the relativistic and the newtonian $(v = +\infty)$ case.

2010 *Mathematics Subject Classification.* Primary: 34C25.

Key words and phrases. Periodic solution, ϕ-Laplacian, pendulum equation, degree theory, stability.

The first author is partially supported by Ministerio de Educación y Ciencia, Spain, project MTM2010-15314. The second author is partially supported by Ministerio de Educación y Ciencia, Spain, project MTM2011-23652.

Equation (1) can be written as

$$(\phi_r(x'))' + c\,x' + k\sin x = e(t),$$

where $\phi_r :]-v, v[\to \mathbb{R}$ is given by $\phi_r(z) = \frac{z}{\sqrt{1-\frac{z^2}{v^2}}}$, and then it fits into the class of ϕ-Laplacian equations. We say that a ϕ-Laplacian operator satisfying

$(H1)$ $\phi :]-a, a[\longrightarrow]-b, b[$ is an increasing homeomorphism with $\phi(0) = 0$ and $0 < a, b \le +\infty$,

is *singular* when its domain is finite ($a < +\infty$) and on the contrary the operator is said *regular*. On the other hand we say that ϕ is *bounded* if its range is bounded ($b < +\infty$) and *unbounded* in other case (see [3, 4, 5, 11]).

The results in [20] were extended in [21] to the pendulum-type equation

$$(\phi(x'))' + h(x)\,x' + g(x) = e(t) + s, \tag{2}$$

with a Liénard term and a general ϕ-Laplacian. The main goal in [21] is to ensure that the set I_e of mean values s for which (2) has a T-periodic solution is a proper interval, which it is known as the *nondegeneracy problem*. For related results see [2].

In this paper we deal with the ϕ-Laplacian pendulum-type equation

$$(\phi(x'))' + c\,x' + g(x) = e(t) + s. \tag{3}$$

In Section 2 we improve the results of [21] for the particular case $h(x) = c$ and based on the Leray-Schauder degree we give also alternative conditions to ensure the nondegeneracy of (3). In Section 3 we study the stability of the T-periodic solutions previously obtained and finally in Section 4 we give some applications to pendulum equations. We would like to remark that up to our knowledge the stability results presented in this paper are the first ones available in the literature in the framework of ϕ-Laplacian equations, and can be seen as natural extensions of the classical stability conditions [8, 17, 18, 19] for the newtonian equation.

Let us introduce some notation. Let C_T be the Banach space of the T-periodic and continuous functions. The space C_T can be decomposed as $C_T = \mathbb{R} \oplus \tilde{C}_T$ where $\tilde{C}_T$ is the space of the T-periodic and continuous functions with zero mean value.

We will reduce the search for T-periodic solutions of (3) to find a fixed point for a suitable operator (see [21, Section 2] for the complete details). We shall assume the existence of two real numbers $R_1 < R_2$ such that

$(H2^+)$ $g \in C^1([R_1, R_2])$ and $g'(x) > 0$ for all $x \in [R_1, R_2]$,

From now on, for $e \in \tilde{C}_T$ we will use the T-periodic function

$$E(t) = \int_0^t e(\tau)d\tau,$$

and we will denote

$$\delta(E) = \max_{t\in[0,T]} E(t) - \min_{t\in[0,T]} E(t).$$

For each $s \in]g(R_1), g(R_2)[$ consider the closed, convex and nonempty set

$$K = \{y \in \tilde{C}_T : y(t) \in [g(R_1) - s, g(R_2) - s]\},$$

the operator

$$F[y](t) = \int_0^t y(\tau)d\tau - E(t) + cg^{-1}(y + s),$$

for each $y \in K$, and the constants

$$A \equiv T(g(R_1) - s) - \max_{t\in[0,T]} E(t) + cR_1,$$

$$B \equiv T(g(R_2) - s) - \min_{t\in[0,T]} E(t) + cR_2.$$

Clearly, for each $y \in K$ we have

$$A \leq \min_{t\in[0,T]} F[y](t) \leq \max_{t\in[0,T]} F[y](t) \leq B.$$

Proposition 1. [21, Section 2] *Suppose that there exist real numbers $R_1 < R_2$ such that $(H2^+)$ holds and $B - A < b$ (this condition is trivially satisfied whenever ϕ is unbounded, i.e. $b = +\infty$). Then for any $y \in K$, there exists a unique choice of C_y, D_y (depending continuously on y) such that*

$$\mathcal{T}[y](t) \equiv \int_0^t g'(g^{-1}(y+s))\phi^{-1}\left(-F[y](t) + C_y\right) ds + D_y \in \tilde{C}_T. \tag{4}$$

Moreover, $C_y \in [A, B]$, the operator $\mathcal{T} : K \to \tilde{C}_T$ is completely continuous and if $y \in K$ is a fixed point of $\mathcal{T}$ then $x = g^{-1}(y+s)$ is a T-periodic solution of (3).

Note that under the reciprocal assumption
$(H2^-)$ $g \in C^1([R_1, R_2])$ and $g'(x) < 0$ for all $x \in [R_1, R_2]$.
an analogous fixed point problem can be formulated, with the evident changes. In this case, the set is

$$K = \{y \in \tilde{C}_T : y(t) \in [g(R_2) - s, g(R_1) - s]\},$$

and the involved constants are

$$\widehat{A} \equiv T(g(R_2) - s) - \max_{t\in[0,T]} E(t) + cR_1,$$

$$\widehat{B} \equiv T(g(R_1) - s) - \min_{t\in[0,T]} E(t) + cR_2.$$

2. **Existence results.**

Theorem 2.1. *Let us assume $(H1)$, $c \geq 0$, $e \in \tilde{C}_T$ and that there exist real numbers $R_1 < R_2$ such that $(H2^+)$ holds and $B - A < b$ (this condition is trivially satisfied whenever ϕ is unbounded, i.e. $b = +\infty$).*

The following conclusions hold:

(I) If

$$L_1 := \frac{T}{2} \max_{x\in[R_1,R_2]} |g'(x)|\phi^{-1}(B - A) \leq |g(R_2) - g(R_1)|,$$

then, for every

$$s \in \left[g(R_1) + \frac{L_1}{2}, g(R_2) - \frac{L_1}{2}\right],$$

there exists at least one T-periodic solution of equation (3) belonging to $[R_1, R_2]$.

(II) If $c > 0$ and

$$L_2 := \frac{2\delta(E) + T|g(R_2) - g(R_1)|}{c} \leq R_2 - R_1,$$

then, for every

$$s \in \left[g\left(R_1 + \frac{L_2}{2}\right), g\left(R_2 - \frac{L_2}{2}\right)\right],$$

there exists at least one T-periodic solution of equation (3) belonging to $[R_1, R_2]$.

Note that $B - A$ does not depend on s, more concretely,

$$B - A = T(g(R_2) - g(R_1)) + \delta(E) + c(R_2 - R_1).$$

Proof. By Proposition 1 it is enough to prove the existence of a fixed point in K for the operator $\mathcal{T}$. The existence of such fixed point is an immediate consequence of the basic properties of the Leray-Schauder degree and the following *a priori* estimates for the solutions of the equation $y = \lambda \mathcal{T} y$, $\lambda \in (0, 1)$.

Case (I).- If $y = \lambda \mathcal{T} y$ then y has mean value zero, $y(0) = y(T)$ and $\|y'\|_\infty < \max\limits_{x \in [R_1, R_2]} |g'(x)| \phi^{-1}(B - A)$. Therefore by [16, Chapter XV, Theorem 3] we obtain that

$$\|y\|_\infty < \frac{T \max\limits_{x \in [R_1, R_2]} |g'(x)| \phi^{-1}(B - A)}{4},$$

and by using the condition over s,

$$g(R_1) - s < y(t) < g(R_2) - s,$$

for all $t \in [0, T]$. Therefore $y \in int(K)$.

Case (II).- If $y = \lambda \mathcal{T} y$ then y is a T-periodic solution of the differential equation

$$\phi \left(\frac{y'}{\lambda g'(g^{-1}(y + s))} \right)' + c \frac{y'}{g'(g^{-1}(y + s))} + y(t) = e(t).$$

Integrating between two points t_1 and t_2 where y attains its maximum and minimum with $0 < t_2 - t_1 \le T/2$, we obtain

$$c \left(g^{-1}(y(t_2) + s) - g^{-1}(y(t_1) + s) \right) = \int_{t_1}^{t_2} e(s) ds - \int_{t_1}^{t_2} y(s) ds,$$

and therefore

$$\max_{t \in [0,T]} g^{-1}(y(t) + s) - \min_{t \in [0,T]} g^{-1}(y(t) + s) < \frac{\delta(E) + \frac{T}{2} |g(R_2) - g(R_1)|}{c}$$

Taking into account that y vanishes at some point (because it has mean value zero) and the condition over s, we deduce from the above inequality that

$$R_1 < g^{-1}(y(t) + s) < R_2,$$

for all $t \in [0, T]$, which imply that $g(R_1) - s < y(t) < g(R_2) - s$ and thus $y \in int(K)$. □

Remark 1. Note that the previous theorem remains true if $e \in \tilde{L}^1[0, T]$. So, part (I) of Theorem 2.1 improves Theorems 1 and 5 in [21] when $h(x) = c$.

On the other hand, we point out that the estimate of part (II) implies that problem (3) has always a T-periodic solution for high values of the friction coefficient c.

If we assume $(H2^-)$ instead of $(H2^+)$ then we obtain a completely analogous theorem with the obvious changes. Combining both type of results will be the key to obtain multiplicity results.

Theorem 2.2. *Let us assume* $(H1)$, $c \geq 0$, $e \in \tilde{C}_T$ *and that there exist real numbers* $R_1 < R_2$ *such that* $(H2^-)$ *holds and* $\widehat{B} - \widehat{A} < b$ *(this condition is trivially satisfied whenever* ϕ *is unbounded, i.e.* $b = +\infty$*).*

The following conclusions hold:

(I) If

$$\widehat{L_1} := \frac{T}{2} \max_{x \in [R_1, R_2]} |g'(x)| \phi^{-1}(\widehat{B} - \widehat{A}) \leq |g(R_2) - g(R_1)|,$$

then, for every

$$s \in \left[g(R_1) + \frac{\widehat{L_1}}{2}, g(R_2) - \frac{\widehat{L_1}}{2} \right],$$

there exists at least one T*-periodic solution of equation* (3) *belonging to* $[R_1, R_2]$.

(II) If $c > 0$ *and*

$$\widehat{L_2} := \frac{2\delta(E) + T|g(R_2) - g(R_1)|\}}{c} < R_2 - R_1,$$

then, for every

$$s \in \left[g\left(R_2 - \frac{\widehat{L_2}}{2} \right), g\left(R_1 + \frac{\widehat{L_2}}{2} \right) \right],$$

there exists at least one T*-periodic solution of equation* (3) *belonging to* $[R_1, R_2]$.

3. **Stability results.** The aim of this section is to investigate the stability of the periodic solutions of (3) found in the previous section. We point out that the stability of a relativistic oscillator (without damping) has been studied in the recent paper [10].

Let us denote $\psi \equiv \phi^{-1}$. We shall assume

$(H3)$ $\psi \in C^1(]-b, b[)$ and $\psi'(x) \neq 0$ for all $x \in]-b, b[$.

Equation (3) is equivalent to the first order system

$$\left\{ \begin{array}{rcl} x' & = & \psi(y), \\ y' & = & -c\,\psi(y) - g(x) + e(t) + s. \end{array} \right. \tag{5}$$

If $x_0(t)$ is a T-periodic solution of (3) we shall understand its stability like the stability of $(x_0(t), \phi(x_0'(t)))$ as a T-periodic solution of (5) (see [1] for the standard notion of stability of first order systems). The corresponding variational system is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & \psi'(\phi(x_0'(t))) \\ -g'(x_0(t)) & -c\,\psi'(\phi(x_0'(t))) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

that can be rewritten into the single equation

$$\left(\frac{x'}{\psi'(\phi(x_0'(t)))} \right)' + c\,x' + g'(x_0(t))x = 0. \tag{6}$$

Now, the time rescaling $t(\tau)$ defined implicitly by

$$\tau = \int_0^t \psi'(\phi(x_0'(s)))ds, \tag{7}$$

leads to the T^*- periodic linear equation

$$x'' + c\,x' + \frac{g'(x_0(t(\tau)))}{\psi'(\phi(x_0'(t(\tau))))}x = 0, \tag{8}$$

where

$$T^* = \int_0^T \psi'(\phi(x_0'(s)))ds.$$

Evidently, the time rescaling does not affect to the stability character of (6). Now, we use a known connection between the index of a periodic solution and its stability developed by Ortega in [19] (see also [17, 18, 9]). Here, γ_{2T*} refers to the associated index of period $2T^*$, that can be computed as $\gamma_{2T*} = \text{sgn}(1-\mu_1^2)(1-\mu_2^2)$, where μ_1, μ_2 are the characteristic multipliers of (8) (see the cited references for further details).

Lemma 3.1. *Let us assume* $(H1)$, $(H3)$, $c > 0$, g *is a continuously differentiable function and* $e \in \tilde{C}_T$. *Suppose moreover that* $x_0(t)$ *is a* T*-periodic solution of* (3). *If the unique solution of* (8) *with period* $2\,T^*$ *is the trivial one, then*

$$x(t) \text{ is asymptotically stable [resp. unstable]} \Longleftrightarrow \gamma_{2T*} = 1 \text{[resp. -1]}$$

Proof. It is a direct consequence of [19, Proposition 2.1]. □

Remark 2. We point out that every solution of (8) with period T^* has also period $2\,T^*$. So the assumption of Lemma 3.1 prevents also the existence of non-trivial T^*-periodic solutions for (8).

The following result provides explicit conditions that avoid the existence of non-trivial $2\,T^*-$periodic solutions. Such results are well-known in this context, so instead of a complete proof we will give the adequate references. From now on, given $f, g \in C_T$ we write $f \prec g$ if $f(t) \leq g(t)$ for all t and the inequality is strict in a set of positive measure.

Lemma 3.2. *Let us assume* $c > 0$, $\alpha \in C_{T^*}$ *and consider the linear differential equation*

$$x'' + c\,x' + \alpha(t)x = 0. \tag{9}$$

If one of the followings conditions holds

(i) $\alpha(t) \prec 0$ *for a.e.* $t \in [0, T^*]$,

(ii) $0 \prec \alpha(t) \prec \left(\frac{\pi}{T^*}\right)^2 + \frac{c^2}{4}$ *for a.e.* $t \in [0, T^*]$,

then the unique solution of (9) *which has period* $2\,T^*$ *is the trivial one.*

Proof. In the case (i), the Sturm comparison Theorem implies that a possible non-trivial $2T^*$-periodic solution has constant sign, then a contradiction is obtained by integrating (9) over a period. Let us consider the case (ii). By contradiction, if $x(t)$ is a $2T^*$-periodic solution then $y(t) = x(t+T) - x(t)$ is a skew-periodic solution. Then, [19, Lemma 2.2] implies that $y(t) \equiv 0$, that is, $x(t)$ should be T^*-periodic. Now, [9, Lemma 2.3] implies that $x(t) \equiv 0$. □

Proposition 2. *Assume* $c > 0$. *Then the periodic solution provided by Theorem 2.2 is always unstable.*

Proof. By $(H2^-)$ the equation (8) satisfies condition (i) in Lemma 3.2. Moreover an homotopy argument shows that the index γ_{2T^*} of the origin as a $2T^*$-periodic solution of equation (8) is $\gamma = -1$. So by Lemma 3.1 the periodic solution is unstable. □

Proposition 3. *Assume $c > 0$. If the periodic solution x_0 provided by Theorem 2.1 satisfies the condition*

$$\frac{g'(x_0(t))}{\psi'(\phi(x_0'(t)))} \prec \left(\frac{\pi}{T^*}\right)^2 + \frac{c^2}{4} \qquad \textit{for a.e. } t \in [0,T], \tag{10}$$

then it is asymptotically stable.

Proof. By $(H2^+)$ and (10) the equation (8) satisfies condition (ii) in Lemma 3.2 and an homotopy argument show that the index of the origin as a $2T^*$ periodic solution of equation (8) is 1. Then its asymptotical stability is a direct consequence of Lemma 3.1. □

In the following results, some concrete conditions for stability are provided.

Corollary 1. *Under the conditions of Theorem 2.1, if R_1, R_2, c satisfy*

$$\frac{\max\limits_{x\in[R_1,R_2]} g'(x)}{\min\limits_{|x|\le B-A} \psi'(x)} < \left(\frac{\pi}{T \max\limits_{|x|\le B-A} \psi'(x)}\right)^2 + \frac{c^2}{4}, \tag{11}$$

then the T-periodic solution is asymptotically stable.

Proof. If x_0 is the T-periodic solution provided by Theorem 2.1 then $x_0 = g^{-1}(y_0 + s)$ where $y_0 \in K$ is a fixed point of operator $\mathcal{T}$ given by (4). Then, one has the bound $\|\phi(x_0')\|_\infty \le B - A$. Hence,

$$T^* \le T \max_{|x|\le B-A} \psi'(x)$$

and (10) follows easily from (11). □

Corollary 2. *Let us define the constant $M = T\max\{|g(R_1)|, |g(R_2)|\} + \|e\|_2 \sqrt{T}$. Under the conditions of Theorem 2.1, if*

$$\frac{\max\limits_{x\in[R_1,R_2]} g'(x)}{\min\limits_{|x|\le M} \psi'(x)} < \left(\frac{\pi}{T \max\limits_{|x|\le M} \psi'(x)}\right)^2 + \frac{c^2}{4}, \tag{12}$$

then the T-periodic solution is asymptotically stable.

Proof. The proof is analogous by using in this case the bound $\|\phi(x_0')\|_\infty \le M$, which we are going to justify in the following. Multiplying (3) by $\phi(x_0')'$, integrating over $[0,T]$ and taking into account Cauchy-Schwarz inequality and $(H2^+)$ we get

$$\|\phi(x_0')'\|_2^2 \le (\sqrt{T}\max\{|g(R_1)|, |g(R_2)|\} + \|e\|_2)\, \|\phi(x_0')'\|_2 .$$

Hence,

$$\|\phi(x_0')'\|_2 \le \sqrt{T}\max\{|g(R_1)|, |g(R_2)|\} + \|e\|_2 .$$

If $\tilde{t}$ is such that $x_0'(\tilde{t}) = 0$, then

$$|\phi(x_0'(t))| = |\int_{\tilde{t}}^{t} \phi(x_0')'| \le \|\phi(x_0')'\|_1 \le \sqrt{T}\,\|\phi(x_0')'\|_2 \le M$$

for all t, as required. □

Remark 3. For the pendulum equation $x''+c\,x'+k\sin\, x = e(t)+s$ both conditions of Corollaries 1 and 2 reduce to

$$0 < k < \left(\frac{\pi}{T}\right)^2 + \frac{c^2}{4}, \tag{13}$$

which it is the classical condition derived by Ortega in [19] for the newtonian case.

4. **Some applications to ϕ-Laplacian pendulum equations.** Since part (I) of Theorems 2.1 and 2.2 provides similar results as in [21] (but with a better estimate) we focus our attention in this section to the results provided by part (II) of Theorems 2.1 and 2.2 on the ϕ-Laplacian pendulum equation

$$(\phi(x'))' + c\,x' + k\sin x = e(t) + s. \tag{14}$$

4.1. **The unbounded case.** In this subsection we deal with an unbounded operator ϕ, that is $b = +\infty$.

Theorem 4.1. *Let us assume $(H1)$, $k > 0$, $e \in \tilde{C}_T$ and*

$$2\delta(E) + 2Tk < c\pi.$$

(Existence and multiplicity) For every

$$s \in \left]k\sin\left(-\frac{\pi}{2} + \frac{\delta(E)+Tk}{c}\right), k\sin\left(\frac{\pi}{2} - \frac{\delta(E)+Tk}{c}\right)\right[,$$

equation (14) possesses at least two T-periodic solutions x_1, x_2 which satisfy $-\frac{\pi}{2} < x_1 < \frac{\pi}{2} < x_2 < \frac{3\pi}{2}$.

(Stability) Under the previous assumptions, if moreover $(H3)$ holds then x_2 is unstable. If in addition condition (11) (or (12)) is fulfilled with $R_1 = -\frac{\pi}{2}, R_2 = \frac{\pi}{2}$, then x_1 is asymptotically stable.

Proof. Since $g(x) = k\sin x$ the existence of x_1 follows from Theorem 2.1 (II) taking $R_1 = -\frac{\pi}{2} + \varepsilon$ and $R_2 = \frac{\pi}{2} - \varepsilon$, while the existence of x_2 is deduced from Theorem 2.2 (II) with $R_1 = \frac{\pi}{2} + \varepsilon$ and $R_2 = \frac{3\pi}{2} - \varepsilon$, where $\varepsilon > 0$ is taken small enough.

On the other hand, the stability results are a direct consequence of Propositions 2 and Corollaries 1 and 2. □

A direct application of the previous result to the relativistic pendulum

$$\left(\frac{x'}{\sqrt{1-\frac{x'^2}{v^2}}}\right)' + c\,x' + k\sin\, x = e(t) + s, \tag{15}$$

gives the following corollary.

Corollary 3. *Let us assume $v, k > 0$, $e \in \tilde{C}_T$ and*

$$2\delta(E) + 2Tk < c\pi.$$

(Existence and multiplicity) For every

$$s \in \left]k\sin\left(-\frac{\pi}{2} + \frac{\delta(E)+Tk}{c}\right), k\sin\left(\frac{\pi}{2} - \frac{\delta(E)+Tk}{c}\right)\right[,$$

equation (15) possesses at least two T-periodic solutions x_1, x_2 which satisfy $-\frac{\pi}{2} < x_1 < \frac{\pi}{2} < x_2 < \frac{3\pi}{2}$.

(Stability) Under the previous assumptions, x_2 is unstable. Moreover, x_1 is asymptotically stable provided that

$$k\left(1+\frac{H^2}{v^2}\right)^{\frac{3}{2}} < \left(\frac{\pi}{T}\right)^2 + \frac{c^2}{4}, \tag{16}$$

being $H = \min\left\{kT + \|e\|_2\sqrt{T},\, 2T + \delta(E) + c\pi\right\}$.

Proof. The existence condition is clear. For the stability, we can use the explicit bounds $R_1 = \frac{\pi}{2}, R_2 = \frac{3\pi}{2}$ of the solution x_2 in order to compute explicitly conditions (11) and (12). For such conditions, the constants M, A, B can be computed and it turns out that

$$M = kT + \|e\|_2\sqrt{T}, \quad B - A = 2T + \delta(E) + c\pi.$$

Then $H = \min\{M, B - A\}$. On the other hand, the inverse of the relativistic laplacian is $\psi(z) = \frac{z}{\sqrt{1+\frac{z^2}{v^2}}}$. Now, it is easy to verify that condition (16) condenses (11) and (12) into a single expression. □

Remark 4. In the newtonian limit case $v = +\infty$ condition (16) reduces to (13).

It is interesting to point out that our results are new even for the newtonian pendulum. Let us review some known results: for the classical pendulum equation

$$x'' + c\,x' + k\sin x = e(t) + s, \tag{17}$$

it is proved in [15], that (17) is non-degenerate and $s = 0 \in I_e$ under the condition

$$T\,\|e\|_2 < c\pi\sqrt{3}, \tag{18}$$

and in [21] assuming

$$T(2T + c\pi + \delta(E)) < 2\sqrt{3}. \tag{19}$$

Note that in the last inequality the $2\sqrt{3}$ can be improved up to 4 by using part (II) of Theorems 2.1 and 2.2.

Finally, the main result of [8] asserts that if (17) is non-degenerate, $s \in \operatorname{int} I_e$ and

$$k < \max\left\{\frac{c^2}{4} + \frac{\pi^2}{T^2}, \frac{\pi}{T}\sqrt{c^2 + \frac{\pi^2}{T^2}}\right\},$$

then (17) has at least one unstable T-periodic solution and one asymptotically stable T-periodic solution.

As a consequence of Theorem 4.1 we obtain the following alternative multiplicity result, which moreover gives us an explicit estimate of the solvability interval, bounds for the solutions as well as information about their stability.

Corollary 4. *If*

$$2\delta(E) + 2Tk < c\pi,$$

then for every

$$s \in \left]k\sin\left(-\frac{\pi}{2} + \frac{\delta(E) + Tk}{c}\right), k\sin\left(\frac{\pi}{2} - \frac{\delta(E) + Tk}{c}\right)\right[,$$

equation (17) possesses two different solutions x_1, x_2 which verify $-\frac{\pi}{2} < x_1 < \frac{\pi}{2} < x_2 < \frac{3\pi}{2}$. Moreover, x_1 is asymptotically stable and x_2 is unstable.

Proof. It follows from Theorem 4.1 taking into account that condition $2\delta(E)+2Tk < c\pi$ implies that

$$k < \frac{c}{2}\frac{\pi}{T} \leq \frac{\pi^2}{T^2} + \frac{c^2}{4} < \frac{2\pi^2}{T^2} + \frac{c^2}{4},$$

and then condition (10) is fulfilled (see Remark 3). □

4.2. **The bounded case.** In the analysis of a bounded ϕ-Laplacian operator $\phi : \mathbb{R} \to]-b, b[$ with $b < +\infty$ we need to impose an additional condition, namely $B - A < b$.

Theorem 4.2. *Let us assume that* $\phi : \mathbb{R} \to]-b, b[$ *with* $b < +\infty$ *and* $k > 0$, $e \in \tilde{C}_T$ *verify*

$$2T + c\pi + \delta(E) < b, \tag{20}$$

and

$$2\delta(E) + 2Tk < c\pi. \tag{21}$$

(Existence and multiplicity) Then for each

$$s \in \left] k\sin\left(-\frac{\pi}{2} + \frac{\delta(E) + Tk}{c}\right), k\sin\left(\frac{\pi}{2} - \frac{\delta(E) + Tk}{c}\right)\right[,$$

equation (14) has at least two T*-periodic solutions* x_1, x_2 *which satisfy* $-\frac{\pi}{2} < x_1 < \frac{\pi}{2} < x_2 < \frac{3\pi}{2}$.
(Stability) If moreover $(H3)$ *holds then* x_2 *is unstable. If in addition condition (11) (or (12)) is fulfilled with* $R_1 = -\frac{\pi}{2}, R_2 = \frac{\pi}{2}$*, then* x_1 *is asymptotically stable.*

Proof. Since $g(x) = k\sin x$ and $b = 1$ condition (20) implies that $B - A < b$ when $R_1 = -\frac{\pi}{2} + \varepsilon$ and $R_2 = \frac{\pi}{2} - \varepsilon$, and $\tilde{B} - \tilde{A} < b$ if $R_1 = \frac{\pi}{2} + \varepsilon$ and $R_2 = \frac{3\pi}{2} - \varepsilon$, for small enough $\varepsilon > 0$. Now the theorem follows from the same arguments than those of the proof of Theorem 4.1. □

Now, a direct corollary can be written for the pendulum equation

$$\left(\frac{x'}{\sqrt{1 + x'^2}}\right)' + c\,x' + k\sin x = e(t) + s, \tag{22}$$

with the one-dimensional mean curvature operator $\phi_m(z) = \frac{z}{\sqrt{1+z^2}}$ for all $z \in \mathbb{R}$. We omit further details.

Corollary 5. *Let us assume* $c > 0$, $k > 0$, $e \in \tilde{C}_T$,

$$2T + c\pi + \delta(E) < 1,$$

and

$$2\delta(E) + 2Tk < c\pi.$$

(Existence and multiplicity) For every

$$s \in \left] k\sin\left(-\frac{\pi}{2} + \frac{\delta(E) + Tk}{c}\right), k\sin\left(\frac{\pi}{2} - \frac{\delta(E) + Tk}{c}\right)\right[,$$

equation (22) possesses at least two T*-periodic solutions* x_1, x_2 *which satisfy* $-\frac{\pi}{2} < x_1 < \frac{\pi}{2} < x_2 < \frac{3\pi}{2}$.

(Stability) Under the previous assumptions, x_2 is unstable. Moreover, x_1 is asymptotically stable provided that

$$k < \left(\frac{\pi}{T}\right)^2 \left(1 - H^2\right)^3 + \frac{c^2}{4}, \tag{23}$$

being $H = \min\left\{kT + \|e\|_2 \sqrt{T},\, 2T + \delta(E) + c\pi\right\}$.

Acknowledgments. We would like to thank professor R. Ortega for his useful comments about this paper.

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Received April 2011; revised January 2012.

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